Subvarieties of hypercomplex manifolds with holonomy in $SL(n, \mathbb{H})$

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A B S T R A C T

A hypercomplex manifold M is a manifold with a triple I, J, K of complex structure operators satisfying quaternionic relations. For each quaternion $L = aI + bJ + cK, L^2 = -1$, L is also a complex structure operator on M, called *an induced complex structure*. We study compact complex subvarieties of (M, L), for L a generic induced complex structure. Under additional assumptions (Obata holonomy contained in $SL(n, \mathbb{H})$, the existence of an HKT-metric), we prove that (M, L) contains no divisors, and all complex subvarieties of codimension 2 are trianalytic (that is, also hypercomplex).

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1. Introduction

1.1. Hypercomplex manifolds: an introduction

Definition 1.1. A manifold M is called *hypercomplex* if M is equipped with a triple of complex structures I, J, K, satisfying the quaternionic relations $I \circ J = -J \circ I = K$. If, in addition, M is equipped with a Riemannian metric g which is Kähler with respect to I, J, K, it is called *hyper-Kähler* [1,2].

The term "hypercomplex manifold" is due to Boyer [2], who classified compact hypercomplex manifolds of quaternionic dimension 1, though the notion was considered as early as in 1955, by Obata [3].

The first interesting non-hyper-Kähler examples of hypercomplex manifolds were found by physicists in [4], and independently by Joyce in [5]. In the same paper, Joyce classified all homogeneous hypercomplex structures on simply connected compact manifolds, using Wang's classification of homogeneous spaces [6].

Next, we recall the definition of an HKT-metric.

Definition 1.2. Let (M, I, J, K) be a hypercomplex manifold and g a quaternionic Hermitian metric. Consider the Hermitian forms

$$\omega_I(X, Y) = g(IX, Y), \qquad \omega_J(X, Y) = g(JX, Y), \qquad \omega_K(X, Y) = g(KX, Y).$$

If any two of these forms are closed, the manifold is hyper-Kähler. Define $\Omega_I = \omega_J + \sqrt{-1}\omega_K$. It is easy to check that $\Omega_I \in \Lambda_I^{2,0}M$. The metric g is called HKT ("hyper-Kähler with torsion") if $\partial \Omega_I = 0$, where $\partial: \Lambda_I^{p,q}M \to \Lambda_I^{p+1,q}M$ is the (1, 0)-part of the de Rham differential. In this case, the form Ω_I is called an *HKT-form*, and (M, I, J, K, g) *an HKT-manifold*.

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HKT-metrics were introduced by Howe and Papadopoulos [7] (see also [8]) and have been much studied since then. The existence of an HKT-metric puts a significant constraint on a global geometry of a hypercomplex manifold [9,10].

Since the advent of string theory, hypercomplex manifolds have become an important object in physics, because the corresponding σ-models exhibit interesting supersymmetries [11]. After Strominger's paper [12] appeared, supersymmetric σ-models associated with non-Kähler target spaces became a popular object of study. Strominger proposed using the antisymmetric torsion connections on the target spaces. In mathematics, such structures were studied by Bismut [13] in connection with the local index formula. In the hypercomplex setting, Bismut connections were studied by Howe and Papadopoulos in the 1990s in a series of papers starting with [7]. This research led them to the discovery of HKT-metrics.

Since [8], HKT-metrics have become an important ingredient in the mathematical study of hypercomplex geometry. The HKT-metrics have much in common with the Kähler structures. Like Kähler metrics, they are locally defined by a potential [14], but can be used to obtain Hodge-theoretic restrictions on the cohomology [15].

In the present paper, we use the HKT-geometry to study complex subvarieties in hypercomplex manifolds.

Any hypercomplex manifold admits a torsion-free connection preserving I, J and K, which is necessarily unique. This connection is called *the Obata connection*, after Obata, who discovered it in [3]. Any almost complex structure which is preserved by a torsion-free connection is necessarily integrable. Therefore, for any $a, b, c \in \mathbb{R}$, with $a^2 + b^2 + c^2 = 1$, the almost complex structure $L = aI + bJ + cK$ is in fact integrable. By the Newlander–Nirenberg theorem, L defines a complex structure on M. We denote by (M, L) the complex manifold corresponding to this complex structure.

Definition 1.3. A complex structure $L = aI + bJ + cK$, with $a^2 + b^2 + c^2 = 1$, is called *induced by quaternions*, and the corresponding family, parameterized by $\mathbb{C}P^1 \cong S^2$, is called *the twistor family*.

Let (V, I, J, K) be a quaternionic vector space of real dimension $4n$. The group $GL(n, \mathbb{H})$ consists of linear transformations of V that preserve the complex structures I, J and K. Consider the Hodge decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = V_I^{1,0} \oplus V_I^{0,1}$, where $V_I^{1,0}$ and $V_I^{0,1}$ are eigenspaces of I with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $\Lambda_I^{2n,0}V$ be the top exterior power of $V_I^{1,0}$. Recall that $SL(n, \mathbb{H})$ is a subgroup in $GL(n, \mathbb{H})$ consisting of those elements that act trivially on $\Lambda_I^{2n,0}V$.

Let (M, I, J, K) be a hypercomplex manifold and ∇ the corresponding Obata connection. Denote by $\mathrm{Hol}(\nabla)$ the holonomy group of ∇. Since the Obata connection preserves the quaternionic structure, we have $\mathrm{Hol}(\nabla) \subset GL(n, \mathbb{H})$.

Definition 1.4. If the holonomy group $\mathrm{Hol}(\nabla)$ of the Obata connection on a hypercomplex manifold M is contained in $SL(n, \mathbb{H})$, we call M an $SL(n, \mathbb{H})$-manifold.

Remark 1.5. It is easy to see that an $SL(n, \mathbb{H})$-manifold has a trivial canonical bundle (in fact, the canonical bundle of such a manifold has a canonical flat connection with trivial monodromy). The converse is also true, for compact manifolds admitting an HKT-metric, as follows from the Hodge theory of HKT-manifolds [16].

Example 1.6. Let G be a connected, simply connected nilpotent Lie group, and $\Gamma \subset G$ a discrete, co-compact subgroup. The quotient $N := \Gamma \backslash G$ is called *a nilmanifold*. Suppose that I, J, K are left-invariant complex structures on G that satisfy quaternionic relations. Then the hypercomplex structure descends to N and we call N *a hypercomplex nilmanifold*. It was shown in [10] that any hypercomplex nilmanifold is in fact an $SL(n, \mathbb{H})$-manifold.

Example 1.7. One more example of an $SL(n, \mathbb{H})$-manifold, due to Swann, is a torus fibration over a hyper-Kähler base [17]. Let (X, I, J, K) be a hyper-Kähler manifold. A 2-form $\alpha \in \Lambda^2 X$ is called anti-self-dual if it is of type $(1, 1)$ with respect to any induced complex structure. If α represents an integral cohomology class, then it defines a principal $U(1)$-bundle over X. Given $4k$ such forms, $\alpha_1, \ldots, \alpha_{4k}$, we obtain a principal T^{4k}-bundle $\pi: M \to X$. This bundle admits an instanton connection A, given by 1-forms $\theta_i \in \Lambda^1 M$, such that $d\theta_i = \pi^*(\alpha_i)$. The hypercomplex structure on M is defined as follows: on horizontal subspaces of A the quaternionic action is lifted from X, and on vertical subspaces it is given by a flat hypercomplex structure of $4k$-dimensional torus. From this construction it is easy to see that M is an $SL(n, \mathbb{H})$-manifold.

The Hopf manifold $H = (\mathbb{H}^n \backslash 0)/\langle A \rangle$ equipped with a standard hypercomplex structure is not an $SL(n, \mathbb{H})$-manifold. Indeed, the holonomy of the Obata connection on H is $\mathbb{Z}$ acting on TM as a matrix A with all eigenvalues $|\alpha_i| > 1$.

It follows from the adjunction formula that none of the homogeneous hypercomplex manifolds constructed by Joyce in [5] has holonomy in $SL(n, \mathbb{H})$. Indeed, such a manifold is fibered over a homogeneous Fano manifold with toric fibers; hence its canonical bundle is non-trivial. However, an $SL(n, \mathbb{H})$-manifold has trivial (even flat) canonical bundle. It was shown in [18] that the manifold $SU(3)$ with its Joyce hypercomplex structure has holonomy $GL(2, \mathbb{H})$; a similar conjecture is stated, but not proven, for all homogeneous hypercomplex manifolds.

1.2. Trianalytic subvarieties

Definition 1.8. Let M be a hypercomplex manifold. A subset $Z \subset M$ is called *trianalytic* if it is complex analytic in (M, L) for all induced complex structures L.

The geometry of trianalytic subvarieties was studied at some length in [19]. It was shown that any trianalytic subvariety can be desingularized by taking a normalization, and this desingularization is smooth and hypercomplex.

The following theorem was proved in [20] (see also [21]).

Theorem 1.9. *Let M be a hyper-Kähler manifold, not necessarily compact. Then there exists a countable subset $S \subset \mathbb{C}P^1$ of induced complex structures, such that for all compact complex subvarieties $Z \subset (M, L)$, $L \notin S$, the subset $Z \subset M$ is trianalytic.*

Remark 1.10. We call an induced complex structure L generic if $L \in \mathbb{C}P^1 \setminus S$. If L is a generic induced complex structure on a hyper-Kähler manifold M, then (M, L) has no compact complex subvarieties except trianalytic subvarieties. Since trianalytic subvarieties are hypercomplex in their smooth points, their complex codimension is even. Therefore, such an (M, L) has no compact odd-dimensional subvarieties. This implies that (M, L) is not algebraic.

It is interesting to note that this result is manifestly false for a general hypercomplex manifold. For example, consider a Hopf surface $H := (\mathbb{H} \setminus 0)/(x \sim 2x)$. For each induced complex structure $L = aI + bJ + cK$, the manifold (H, L) is fibered over $\mathbb{C}P^1$ with the fibers elliptic curves, isomorphic to $(\mathbb{C} \setminus 0)/(x \sim 2x)$. Therefore, (H, L) contains divisors for each induced complex structure L.

However, for $SL(n, \mathbb{H})$-manifolds we still retain some control over subvarieties. In [22], the results of [21,20] were interpreted in terms of calibrations on hyper-Kähler manifolds (Definition 2.5). It turns out that some of the calibrations constructed in hyper-Kähler geometry survive in a more general hypercomplex setting (Theorem 2.6). This is used to obtain a weaker version of Theorem 1.9:

Theorem 1.11. *Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold admitting an HKT-metric. Then there exists a countable subset $S \subset \mathbb{C}P^1$ such that for any induced complex structure $L \in \mathbb{C}P^1 \setminus S$, the manifold (M, L) has no compact divisors, and all compact complex subvarieties $Z \subset (M, L)$ of complex codimension 2 are trianalytic.*

Proof. See the paragraph after the proof of Theorem 3.2. □

Without an HKT assumption, one can prove non-existence of holomorphic Lagrangian subvarieties (for a definition of holomorphic Lagrangian subvarieties in $SL(n, \mathbb{H})$-manifolds, please see Definition 2.11).

Theorem 1.12. *Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold. Then there exists a countable subset $S \subset \mathbb{C}P^1$ such that for any induced complex structure $L \in \mathbb{C}P^1 \setminus S$, the manifold (M, L) has no compact holomorphic Lagrangian subvarieties.*

Proof. For any holomorphic Lagrangian subvariety $X \subset (M, I)$, one has $TX \cap J(TX) = 0$, because $TX \subset TM$ is a Lagrangian subspace, for any quaternionic Hermitian metric. Therefore, Theorem 1.12 is immediately implied by Theorem 3.2 (see also Remark 2.8). □

In the following section we recall some facts about $SL(n, \mathbb{H})$-manifolds and calibrations. We prove Theorem 1.11 in Section 3.

2. Introduction to the geometry of $SL(n, \mathbb{H})$-manifolds

This section is an introduction to the HKT-geometry of $SL(n, \mathbb{H})$-manifolds and their calibrations. We follow [22,23].

2.1. The quaternionic Dolbeault complex on $SL(n, \mathbb{H})$-manifolds

In this subsection, we recall the definition of a quaternionic Dolbeault algebra of a hypercomplex manifold. We follow [23], though this complex is essentially due to [24].

Let (M, I, J, K) be a hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$. There is a natural multiplicative action of $SU(2) \subset \mathbb{H}^*$ on $\Lambda^*(M)$, associated with the hypercomplex structure.

It is well-known that any irreducible complex representation of $SU(2)$ is a symmetric power $S^i(W_1)$, where W_1 is a fundamental two-dimensional representation. We say that a representation U has weight i if it is isomorphic to $S^i(W_1)$. It follows from the Clebsch–Gordan formula that the weight is multiplicative in the following sense: if $i \leqslant j$, then

$$W_i \otimes W_j = \bigoplus_{k=0}^{i} W_{i+j-2k},$$

where $W_i = S^i(W_1)$ denotes the irreducible representation of weight i.

Let $V^i \subset \Lambda^i(M)$ be a sum of all irreducible subrepresentations $W \subset \Lambda^i(M)$ of weight $< i$. Since the weight is multiplicative, $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$.

It is easy to see that the de Rham differential d increases the weight by 1 at most: $dV^i \subset V^{i+1}$. So $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

Definition 2.1. Denote by $(\Lambda_+^*(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$. It is called *the quaternionic Dolbeault algebra of M*, or *the quaternionic Dolbeault complex* (qD-algebra or qD-complex for short).

The Hodge bigrading is compatible with the weight decomposition of $\Lambda^*(M)$, and gives a Hodge decomposition of $\Lambda_+^*(M)$ [15]:

$$\Lambda_+^i(M) = \bigoplus_{p+q=i} \Lambda_{+,I}^{p,q}(M).$$

The spaces $\Lambda_{+,I}^{p,q}(M)$ are the weight spaces for a particular choice of a Cartan subalgebra in $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of the weight spaces within an irreducible representation. This gives the following result [15]:

Proposition 2.2. *Let (M, I, J, K) be a hypercomplex manifold and*

$$\Lambda_+^i(M) = \bigoplus_{p+q=i} \Lambda_{+,I}^{p,q}(M)$$

the Hodge decomposition of the qD-*complex defined above. Then there is a natural isomorphism*

$$\mathcal{R}_{p,q} : \Lambda_I^{p+q,0}(M) \longrightarrow \Lambda_{I,+}^{p,q}(M). \tag{2.1}$$

Consider the projection $\Pi_+^{p,q} : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_{I,+}^{p,q}(M)$ and let

$$R : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{p+q,0}(M)$$

denote the composition $\mathcal{R}_{p,q}^{-1} \circ \Pi_+^{p,q}$.

Now, let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, $\dim_{\mathbb{R}} M = 4n$. Let Φ_I be a nowhere degenerate holomorphic section of $\Lambda_I^{2n,0}(M)$. Assume that Φ_I is real, that is, $J(\Phi_I) = \overline{\Phi}_I$. The existence of such a form is equivalent to $\mathrm{Hol}(\nabla) \subset SL(n, \mathbb{H})$, where ∇ is the Obata connection (see [25]). It is often convenient to define the $SL(n, \mathbb{H})$-structure by fixing the quaternionic action and the holomorphic form Φ_I.

Define the map

$$\mathcal{V}_{p,q} : \Lambda_I^{p+q,0}(M) \longrightarrow \Lambda_I^{n+p,n+q}(M)$$

by the relation

$$\mathcal{V}_{p,q}(\eta) \wedge \alpha = \eta \wedge R(\alpha) \wedge \overline{\Phi}_I, \tag{2.2}$$

for any test form $\alpha \in \Lambda_I^{n-p,n-q}(M)$.

The following proposition establishes some important properties of $\mathcal{V}_{p,q}$ (for the proof, see [23, Proposition 4.2] or [26, Theorem 3.6]):

Proposition 2.3. *Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, and*

$$\mathcal{V}_{p,q} : \Lambda_I^{p+q,0}(M) \longrightarrow \Lambda_I^{n+p,n+q}(M)$$

the map defined above. Then:

  (i) $\mathcal{V}_{p,q}(\eta) = \mathcal{R}_{p,q}(\eta) \wedge \mathcal{V}_{0,0}(1)$.
 (ii) *The map $\mathcal{V}_{p,q}$ is injective, for all p, q.*
(iii) $(\sqrt{-1})^{(n-p)^2} \mathcal{V}_{p,p}(\eta)$ *is real if and only if $\eta \in \Lambda_I^{2p,0}(M)$ is real, and weakly positive if and only if η is weakly positive.*
(iv) $\mathcal{V}_{p,q}(\partial \eta) = \partial \mathcal{V}_{p-1,q}(\eta)$, *and* $\mathcal{V}_{p,q}(\partial_J \eta) = \overline{\partial} \mathcal{V}_{p,q-1}(\eta)$.
 (v) $\mathcal{V}_{0,0}(1) = \lambda \mathcal{R}_{n,n}(\Phi_I)$, *where λ is a positive rational number, depending only on the dimension n.*

2.2. Calibrations on $SL(n, \mathbb{H})$-manifolds

In this subsection, we recall the construction of the sequence of calibrations on $SL(n, \mathbb{H})$-manifolds, following [22]. These calibrations will play the central role in the proof of the main theorem.

Definition 2.4 (*[27]*)**.** Let (V, g) be a Euclidean space. For any p-form $\eta \in \Lambda^p(V^*)$, let $\mathrm{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, \ldots, v_p)}{|v_1||v_2|\ldots|v_p|}$, for all p-tuples $(v_1, \ldots, v_p)$ of vectors in V.

Definition 2.5 (*[27]*)**.** A *precalibration* on a Riemannian manifold is a differential form η with $\mathrm{comass}(\eta) \leqslant 1$ everywhere. A *calibration* is a precalibration which is closed.

Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, with Φ_I a holomorphic volume form on (M, I) preserved by the Obata connection. We will assume that Φ_I is real, that is $J(\Phi_I) = \overline{\Phi}_I$. A number of interesting calibrations can be constructed in this situation. The following theorem was proved in [22, Theorem 5.4]:

Theorem 2.6. *Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, and $(\Phi_I)_J^{n,n}$ the (n, n)-part of Φ_I taken with respect to J, and g an HKT-metric. Then there exists a function $c_i(m)$ on M such that $V_{n+i,n+i}^J := (\Phi_I)_J^{n,n} \wedge \omega_J^i$ is a calibration with respect to the conformal metric $\widetilde{g} = c_i g$, calibrating complex subvarieties of (M, J) which are coisotropic with respect to the $(2, 0)$-form $\widetilde{\omega}_K + \sqrt{-1}\widetilde{\omega}_I$.*

Note, that $V_{n+i,n+i}^J \in \Lambda_J^{n+i,n+i}M$, but using the same construction we can obtain a similar calibration $V_{n+i,n+i}^L \in \Lambda_L^{n+i,n+i}M$ for any induced complex structure L.

We will need the following characterization of the form $V_{n+i,n+i}^I$ (for the proof, see [22, Remark 3.8 and Proposition 3.9]):

Proposition 2.7. *Let $V_{n+i,n+i}^I \in \Lambda^{n+i,n+i}(M, I)$ be a calibration from Theorem 2.6. Then $V_{n+i,n+i}^I$ is proportional to $\mathcal{V}_{i,i}(\Omega_I^i)$ and to $\Pi_+^{n+i,n+i}(\omega_I^{n+i})$ with some positive coefficients that do not depend on the complex structure I (here Ω_I is an HKT-form). In particular, the form $V_{n+i,n+i}^I$ is of maximal weight and for any $\alpha \in \Lambda^{n-i,n-i}$ we have*

$$V_{n+i,n+i}^I \wedge \alpha = a_i \Omega_I^i \wedge R(\alpha) \wedge \overline{\Phi}_I, \tag{2.3}$$

where a_i are some positive functions on M.

Remark 2.8. Note that the calibrations $V_{n+i,n+i}^I$ are constructed for the case when the metric is HKT. However, this assumption is not necessary for $i = 0$. Since by Proposition 2.3 the form $\mathcal{V}_{0,0}(1)$ is always closed, Proposition 2.7 is true for $i = 0$ even if the metric is not HKT. This remark makes it possible to prove Theorem 1.12 without the HKT assumption.

Remark 2.9. In general, the form $V_{n+i,n+i}^J$ is *not* parallel with respect to the Obata connection. Otherwise, since Φ_I is parallel, ω_J would also be parallel. Then the manifold (M, I, J, K, g) would necessarily be hyper-Kähler. In fact, $V_{n+i,n+i}^J$ is not parallel with respect to any torsion-free connection on M (see [8, Claim 6.6]).

2.3. Holomorphic Lagrangian subvarieties in $SL(n, \mathbb{H})$-manifolds

Let (M, I, J, K) be a $SL(n, \mathbb{H})$-manifold, and $\Phi_J \in \Lambda^{2n,0}(M, J)$ a section of the canonical bundle of (M, J) parallel with respect to the Obata connection. Since I and J anticommute, $I(\Phi_J)$ is a section of $\Lambda^{0,2n}(M, J)$, and hence satisfies $I(\Phi_J) = \alpha \overline{\Phi}_J$, for α a complex number such that $|\alpha| = 1$. Rescaling Φ_J, we can always assume that $I(\Phi_J) = \overline{\Phi}_J$. Denote by $\widetilde{V}_{n,n} := \frac{1}{n!}(\operatorname{Re}\Phi_J)_I^{n,n}$ the (n, n)-part of Φ_J, taken with respect to I. In [22] it was shown that $\widetilde{V}_{n,n}$ is a calibration for any quaternionic Hermitian metric which satisfies $|\Phi_J| = 1$. The corresponding calibrated subvarieties were described [22, Proposition 5.1] as follows.

Theorem 2.10. *Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, $X \subset M$ a subvariety, and $\widetilde{V}_{n,n} \in \Lambda^{n,n}(M, I)$ the calibration defined above. Consider a quaternionic Hermitian metric h on (M, I, J, K), and let $\Omega := \omega_J + \sqrt{-1}\omega_K$ be a $(2, 0)$-form constructed from h as in Definition 1.2. Then the following conditions are equivalent.*

(i) *$\widetilde{V}_{n,n}$ calibrates X.*
(i) *$X \subset (M, I)$ is a complex subvariety which is Lagrangian with respect to Ω.*

Proof. [22, Proposition 5.1]. □

Definition 2.11. Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold, and $X \subset (M, I)$ a complex subvariety. We say that X is *holomorphic Lagrangian* if it is calibrated by $\widetilde{V}_{n,n}$.

Remark 2.12. It is remarkable that one is able to define holomorphic Lagrangian subvarieties in the absence of a holomorphic symplectic form. More precisely, the property of being holomorphic Lagrangian is independent from the choice of a quaternionic Hermitian structure which determines the $(2, 0)$-form $\Omega := \omega_J + \sqrt{-1}\omega_K$.

3. Subvarieties in $SL(n, \mathbb{H})$-manifolds

In this subsection, we prove the main result of this paper (Theorem 3.2), which is used to prove Theorem 1.11.

Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold equipped with an HKT-metric g. In the previous subsection we have constructed a sequence of closed positive forms $V_{n+i,n+i}^I \in \Lambda_I^{n+i,n+i}M$, $i = 0, 1, \ldots, n$. We will use these forms to prove Theorem 1.11.

The proof of Theorem 1.11 is based on an observation which is essentially linear algebraic. Let (U, I, J, K) be a quaternionic vector space of real dimension $4n$, $\Phi_I \in \Lambda_I^{2n,0}(U^*)$ a complex volume form and $V_{n+i,n+i}^I$ the element of $\Lambda_I^{n+i,n+i}(U^*)$

constructed in Theorem 2.6. Consider an I-invariant subspace $W \subset U$, of complex dimension $n + i$. Note that $\dim_\mathbb{C}(W \cap J(W)) \geqslant 2i$. Let $\xi_W \in \Lambda_I^{n+i,n+i}U$ be a volume polyvector of W (it is well defined up to a scalar multiplier). Consider a function $\psi : SU(2) \rightarrow \mathbb{R}$ mapping $g \in SU(2)$ to $\langle V_{n+i,n+i}^I, g(\xi_W)\rangle$. Since W is I-invariant, ψ is constant on the $U(1)$-subgroup of $SU(2)$ associated with the complex structure I. This allows one to consider ψ as a function on $SU(2)/U(1) = \mathbb{C}P^1$.

Proposition 3.1. *In the above assumptions, let* $\dim_\mathbb{C}(W \cap J(W)) = 2k$. *Then*

(i) *If* $k = i$, *then* ψ *considered as a function on* $\mathbb{C}P^1$ *has a strict extremum at the point corresponding to the complex structure* I.
(ii) *If* $k > i$, *then* ψ *is identically zero*.

Proof. Let us fix a quaternionic hermitian metric in U, such that $\Phi_I \wedge \overline{\Phi_I}$ is its volume form. Denote by $\eta_W \in \Lambda^{n-i,n-i}(U^*)$ the form dual to ξ_W, that is $\eta_W = *(\xi_W^\sharp)$, where $\sharp$ denotes the duality with respect to the metric and $*$ is the Hodge star operator.

Then we have $\psi(g) = \langle V_{n+i,n+i}^I, g(\xi_W)\rangle = *(V_{n+i,n+i}^I \wedge g(\eta_W))$ and by (2.3) we obtain

$$V_{n+i,n+i}^I \wedge g(\eta_W) = a_i \Omega_I^i \wedge R(g(\eta_W)) \wedge \overline{\Phi_I}.$$

We can choose an orthonormal basis in $U^{1,0}$ of the form

$$\langle e_1, J\overline{e_1}, \ldots, e_n, J\overline{e_n}\rangle,$$

such that

$$W^{1,0} = \langle e_1, J\overline{e_1}, \ldots, e_k, J\overline{e_k}, e_{k+1}, e_{k+2}, \ldots, e_{n+i-k}\rangle,$$

If $k > i$ then for any $g \in SU(2)$ we see that $R(g(\eta_W))$ has to belong to the $(2n - 2i)$th exterior power of the subspace in $(U^*)^{1,0}$ spanned by $e_{k+1}^*, J\overline{e_{k+1}^*}, \ldots, e_n^*, J\overline{e_n^*}$. But this exterior power vanishes, so $R(g(\eta_W)) = 0$, which proves the second part of the proposition.

If $k = i$ then

$$\eta_W = J\overline{e_{i+1}^*} \wedge \cdots \wedge J\overline{e_n^*} \wedge Je_{i+1}^* \wedge \cdots \wedge Je_n^*$$

and

$$R(\eta_W) = e_{i+1}^* \wedge J\overline{e_{i+1}^*} \wedge \cdots \wedge e_n^* \wedge J\overline{e_n^*}.$$

Since $\Omega_I = \sum e_j^* \wedge J\overline{e_j^*}$, we see that in this case $\Omega_I^i \wedge R(\eta_W)$ does not vanish, that is $\psi(1) \neq 0$. We claim that the function ψ is non-constant in this case. Otherwise, the function would be equal to its average over $SU(2)$, which equals $\langle \mathrm{Av}_{SU(2)} V_{n+i,n+i}^I, \xi_W\rangle$. But in the last expression, $\mathrm{Av}_{SU(2)} V_{n+i,n+i}^I = 0$ because $V_{n+i,n+i}^I$ is of maximal weight and belongs to the non-trivial irreducible representation of $SU(2)$. Therefore, the average of ψ is zero.

Now, consider the action of $U(1)$ on the two-dimensional sphere by rotations around the axis that passes through the two points corresponding to the complex structures I and $-I$. We claim that the function ψ is invariant under this action. This follows from the definition of ψ: observe that $V_{n+i,n+i}^I$ and ξ_W are I-invariant. Therefore, $g \mapsto \langle V_{n+i,n+i}^I, g(\xi_W)\rangle$ is invariant (as a function on $SU(2)$) under the adjoint action of the $U(1)$-subgroup corresponding to I.

Since ψ is an analytic non-constant function on the sphere, and it is invariant under the $U(1)$-action considered above, it must have a strict extremum at I. This proves the proposition. $\square$

Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold equipped with an HKT-metric, and $[Z] \in H_{2n+2i}(M, \mathbb{Z})$ an integer homology class. Consider a function $\varphi_Z : \mathbb{C}P^1 \longrightarrow \mathbb{R}$ associating with each $L \in \mathbb{C}P^1$ a number $\int_Z V_{n+i,n+i}^L$, where $V_{n+i,n+i}^L \in \Lambda^{n+i,n+i}(M, L)$ is the corresponding calibration form.

Note that if $L = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$, then $\omega_L = a\omega_I + b\omega_J + c\omega_K$. Since $V_{n+i,n+i}^L$ is proportional to $\Pi_+^{n+i,n+i}(\omega_L^{n+i})$ with the coefficient that does not depend on L (see Proposition 2.7), we see that the function φ_Z is a restriction to S^2 of a homogeneous polynomial in $\mathbb{R}^3$. Such a function can have only a finite number of strict extrema (this follows from the fact that a real algebraic variety can have only finitely many connected components; see [28]). Let $S \subset \mathbb{C}P^1$ be the set of all strict extrema of φ_Z for all integer homology classes. Since for each fixed $[Z] \in H_{2n+2i}(M, \mathbb{Z})$ the set of strict extrema of φ_Z is finite, the set S is countable.

Theorem 3.2. *For each* $L \in \mathbb{C}P^1 \setminus S$, *and any compact complex subvariety* $Z \subset (M, L)$ *of complex dimension* $n + i$, *one has* $\dim_\mathbb{C}(TZ \cap J(TZ)) > 2i$.

Proof. Fix a volume form dv on Z and assume that $\dim_\mathbb{C}(TZ \cap J(TZ)) = 2i$. Then

$$\varphi_Z(L_1) = \int_Z \langle V_{n+i,n+i}^{L_1}, \xi_{TZ}\rangle dv.$$

Fix an arbitrary smooth point $x \in Z$. Note that for any $g \in SU(2)$ we have $\langle V_{n+i,n+i}^L, g(\xi_{T_xZ})\rangle = \langle V_{n+i,n+i}^{\mathrm{Ad}_g L}, \xi_{T_xZ}\rangle$. Thus, Proposition 3.1 implies that the function $L_1 \mapsto \langle V_{n+i,n+i}^{L_1}, \xi_{T_xZ}\rangle$ has a strict extremum at $L_1 = L$. Thus, φ_Z also has a strict extremum at this point, which contradicts our assumption that $L \in \mathbb{C}P^1 \setminus S$. $\square$

Proof of Theorem 1.11. Let $L \in \mathbb{C}P^1 \setminus S$ and Z be a divisor in (M, L), that is a compact L-complex subvariety of complex dimension $2n - 1$. Then Theorem 3.2 implies that $\dim_{\mathbb{C}}(TZ \cap J(TZ)) > 2n - 2$. Since $TZ \cap J(TZ)$ is $\mathbb{H}$-invariant, the last inequality would imply that the dimension equals $2n$, which is impossible. So there exist no divisors in (M, L).

Analogously, if $\dim_{\mathbb{C}} Z = 2n - 2$, then we have $\dim_{\mathbb{C}}(TZ \cap J(TZ)) > 2n - 4$. This implies that $\dim_{\mathbb{C}}(TZ \cap J(TZ)) = 2n - 2$, that is, TZ is $\mathbb{H}$-invariant and Z trianalytic. This completes the proof of the theorem. $\square$

Remark 3.3. We should note that the existence of an HKT-metric was essential for the proof of the main theorem. It still remains unclear whether this condition could be removed.

On the other hand, the condition that the holonomy of the Obata connection is contained in $SL(n, \mathbb{H})$ is known to be necessary. There exist examples of HKT-manifolds with odd-dimensional complex subvarieties for each induced complex structure. An HKT-structure on compact Lie groups due to Joyce [5] gives such an example: it is well-known (see e.g. [29]) that these manifolds admit a toric fibration over a rational base, and hence they always contain divisors.

Remark 3.4. Let T be a compact hyper-Kähler torus, and L a generic induced complex structure. Then all complex subvarieties of T are again tori [30]. We conjecture that something similar would happen for nilmanifolds, and for flat hypercomplex manifolds.

Question 3.5. Let M be a compact $SL(n, \mathbb{H})$-manifold with flat Obata connection, and L a generic induced complex structure. Is it true that all the complex subvarieties of (M, L) are also flat?

Question 3.6. Let M be a hypercomplex nilmanifold (Example 1.6), and L a generic induced complex structure. Is it true that all complex subvarieties of (M, L) are also nilmanifolds?

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