Combinatorics of Heat Kernel Coefficients

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November 14th, 2004

MSC 2000: 58J50, 58J20

Habilitationsschrift zur Erlangung der *Venia legendi*
an der Rheinischen Friedrich–Wilhelms–Universität Bonn

Contents

Chapter 0

Vorwort

Das Studienobjekt dieser Arbeit sind die Wärmeleitkoeffizienten verallgemeinerter Laplaceoperatoren Δ auf einer Riemannschen Mannigfaltigkeit M, denen eine zentrale Rolle bei der Bestimmung spektraler Invarianten der Operatoren Δ zukommt. Verallgemeinerte Laplaceoperatoren sind elliptische Differentialoperatoren zweiter Ordnung auf Schnitten eines Vektorbündels EM über M, deren Hauptsymbol mit dem Hauptsymbol des klassischen Laplaceoperators $\Delta := -\sum \frac{\partial^2}{\partial x_\mu^2}$ übereinstimmt. Unter relativ schwachen Voraussetzungen an die Mannigfaltigkeit M und den verallgemeinerten Laplaceoperator Δ besitzt die Wärmeleitungsgleichung $\frac{d}{dt}\psi_t + \Delta\psi_t = 0$ eine Fundamentallösung, eine Familie k_t^Δ, $t > 0$, von Schnitten des Bündels $\mathrm{Hom}(EM, EM)$ über $M \times M$ mit Faser $\mathrm{Hom}(E_xM, E_yM)$ im Punkt (x, y), mit der Eigenschaft, dass für jeden Schnitt $\psi \in \Gamma(EM)$ die Familie

$$\psi_t(y) \;:=\; (\,e^{-t\Delta}\psi\,)(y) \;:=\; \int_M k_t^\Delta(x,y)\,\psi(x)\,\mathrm{vol}(x) \qquad t > 0$$

eine Lösung der Wärmeleitungsgleichung mit Anfangswert $\psi_0 := \psi$ ist. Der Ausgangspunkt dieser Arbeit ist die asymptotische Entwicklung des Wärmeleitkernes k_t^Δ für kleine Zeiten

$$k_t^\Delta(\,x,\,y\,) \underset{t\to 0^+}{\sim} \frac{\delta_{x,y}}{\sqrt{4\pi t}^{\dim M}} \sum_{d\geq 0} t^d\, a_d(\,y\,)$$

mit Koeffizienten $a_d \in \Gamma(\mathrm{End}\ EM)$, $d \geq 0$, deren Existenz von Minakshisundaram–Pleijel [MP] bewiesen wurde und die eine entscheidende Rolle im lokalen Indexsatz spielt, dem kritischen Schritt im Wärmeleitungsbeweis des allgemeinen Indexsatzes von Atiyah–Singer.

Das zentrale Ergebnis dieser Arbeit stellt eine geschlossene Formel für die Wärmeleitkoeffizienten $a_d \in \Gamma(\mathrm{End}\ EM)$ verallgemeinerter Laplaceoperatoren Δ dar, die ausschließlich die kovarianten Ableitungen des Riemannschen Krümmungstensors R und der Krümmung R^E des Bündels EM sowie — soweit vorhanden — des Potentials $\mathcal{F} \in \Gamma(\mathrm{End}\ EM)$ involviert. Darüberhinaus werden zwei von einander unabhängige, neue Beweise des lokalen Indexsatzes ausgeführt. Der erste dieser beiden Beweise ist meines Erachtens nach einfacher als Beweise, die die Getzlertransformation benutzen, der zweite beruht auf der geschlossenen Formel für die Wärmeleitkoeffizienten. Beide Schlußweisen benutzen spezielle Eigenschaften von getwisteten Diracoperatoren nur am Rande und sind damit auf eine grössere Klasse von

Laplaceoperatoren anwendbar. Insbesondere wird die $\widehat{A}$–Klasse der Mannigfaltigkeit M in beiden Beweisen zu einem Spiegelbild des Zusammenhangslaplace $\nabla^*\nabla$. Allerdings ist das Cliffordbündel EM in diesem Bild der Spiegel, die Zukunft wird zeigen, ob es auch andere Möglichkeiten gibt, den Zusammenhangslaplace in topologische oder geometrische Invarianten der Mannigfaltigkeit M zu spiegeln.

Die für den lokalen Indexsatz entscheidende Information der Wärmeleitkoeffizienten ist natürlich schon seit dem ersten Beweis des lokalen Indexsatzes durch Patodi [Pa] bekannt. Insofern kann die nunmehr zum Ende gekommene Suche nach einer geschlossenen Formel für die Wärmeleitkoeffizienten vielleicht eher als intellektuelles Problem erscheinen. Andererseits bot die in den letzten 30 Jahren erfolglos betriebene Suche nach dieser Formel Anlaß zur Annahme, dass sich dahinter ein tieferliegendes Problem der Differentialgeometrie verbirgt. Tatsächlich stellt die in dieser Arbeit hergeleitete Formel für die Wärmeleitkoeffizienten nur eine Konsequenz einer allgemeinen Jetformel dar, die es erlaubt, die kovarianten Ableitungen eines Schnittes explizit aus ihren Symmetrisierungen zu rekonstruieren. Andere Anwendungen dieser allgemeinen Jetformel ergeben sich von selbst, zum Beispiel wird in dieser Arbeit eine Formel für den kanonischen bzw. Spencerzusammenhang auf Jetbündeln hergeleitet.

Viele Menschen haben mir bei der Erstellung dieser Arbeit direkt oder indirekt geholfen. Insbesondere danke ich Prof. Brünning, der mir vor einigen Jahren die Frage nach einer geschlossenen Formel für die Wärmeleitkoeffizienten stellte, und Prof. Polterovich, der diese Frage zu mancherlei Gelegenheit mit mir erörterte.

Chapter 1

Introduction

The Local Index Theorem for twisted Dirac operators is a fundamental relationship between local analysis and topology on a compact Riemannian manifold M and relates the index of a twisted Dirac operator which is an object par excellence of Global Analysis to the integral of a locally defined differential form over M. Compared to the general Atiyah–Singer Index Theorem for elliptic operators the strength of the Local Index Theorem resides in the explicit local form form of the index density, and this property compensates the restricted class of operators it applies to. Moreover the Local Index Theorem can be seen as the key step in proving the general Atiyah–Singer Index Theorem using the invariance of the index under homotopies of Fredholm operators [LM].

The squares of the twisted Dirac operators D considered in the Local Index Theorem are rather special examples of generalized Laplace operators Δ, second order elliptic differential operators acting on sections of a vector bundle EM over a Riemannian manifold M with scalar, negative definite symbol. Associated to such a generalized Laplace operator Δ is the heat conduction equation $\frac{d}{dt}\psi_t + \Delta\psi_t = 0$ describing the evolution of an inital heat distribution $\psi_0 \in \Gamma(EM)$ under Newton's law of heat propagation governed by Δ. Under mild assumptions on the manifold M and the operator Δ the fundamental solution $e^{-t\Delta}$ of the heat conduction equation can be defined as an integral operator

$$(e^{-t\Delta}\, \psi)(y) \;\; := \;\; \int_M k_t^{\Delta}(\, x,\, y\,)\, \psi(x)\mathrm{vol}(x)$$

with kernel k_t^{Δ}, which is a section of the vector bundle $\mathrm{Hom}(EM, EM)$ over $M \times M$ with fibre $\mathrm{Hom}(E_xM, E_yM)$ over the point $(x, y) \in M \times M$. Sparking the Local Index Theorem is the observation of McKean–Singer that a twisted Dirac operator D acting on sections of a Clifford bundle EM defines a spectral symmetry for the generalized Laplacian D^2 forcing the trace of the heat conduction operator e^{-tD^2} to be constant in time. By definition the limit of this trace at eternity $t = \infty$ is the index of the Dirac operator D considered while the Local Index Theorem asserts that the limit at time $t = 0$ exists and is given by:

$$\mathrm{index}\, D \;\; = \;\; \int_M \lim_{t \longrightarrow 0^+} \mathrm{tr}\; k_t^{D^2}(\, y,\, y\,)\, \mathrm{vol}(y) \;\; = \;\; \int_M \widehat{A}(\, M\,)\; \mathrm{ch}\; \mathrm{Hom}_{\mathrm{Cl}\,TM}(\, \$M,\; EM\,)$$

Besides the original proof due to Patodi [Pa] there are by now numerous other proofs of the Local Index Theorem including stochastic proofs based on Brownian motion [B] and a particularly influential proof due to Getzler [Ge] [BGV].

Studying the special case of the Laplace–Beltrami operator Δ_g of functions on a Riemannian manifold M Minakshisundaram–Pleijel [MP] exhibited a particularly interesting property of the heat kernels k_t^Δ of generalized Laplacians Δ closely related to the existence of the limit of the trace $\operatorname{tr} k_t^{D^2}(y,y)$ as $t \to 0^+$ for squares of twisted Dirac operators. They described the short time behaviour of the heat kernel k_t^Δ by an asymptotic expansion (2.9)

$$k_t^\Delta(\,x,\,y\,) \underset{t\to 0^+}{\sim} \frac{\delta_{x,y}}{\sqrt{4\pi t}^{\dim M}} \sum_{d\geq 0} t^d\, a_d(\,y\,)$$

with $a_d \in \Gamma(\operatorname{End} EM)$ and $a_0(y) = \operatorname{id}$. In their argument Minakshisundaram–Pleijel extended the heat kernel coefficients a_d to sections of $\operatorname{Hom}(EM, EM)$ over $M \times M$ defined on a neighborhood of the diagonal in $M \times M$ in such a way that the extended coefficient $a_0(x,y) :\ E_xM \longrightarrow E_yM$ is parallel transport along the unique shortest geodesic from x to y while the extensions of the higher heat kernel coefficients a_{d+1}, $d \geq 0$, are defined recursively

$$(\ d\,+\,1\,+\,N_x\ )\,a_{d+1} \ \ := \ \ -\,(\,j^{\frac{1}{2}}\,\Delta_x^*\,j^{-\frac{1}{2}}\,)\,a_d$$

where N_x is Euler's number operator and j is the Jacobian determinant of the exponential map in y considered as a function of x. For the convenience of the reader we will sketch the rationale of this recursion formula in the next section, of course more details can be found in textbooks [BGM] [BGV] [Gi].

The (extended) heat kernel coefficients a_d, $d \geq 0$, associated in this way to a generalized Laplacian Δ are objects of particular interest in analyzing spectral properties of Δ as well as in most proofs of the Local Index Theorem. The main result of the current article is a closed formula for the infinite order jets of these coefficients along the diagonal in $M \times M$ in terms of the covariant derivatives of the curvature R^E of the bundle EM, the Riemannian curvature tensor R of M and the auxiliary potential $\mathcal{F} \in \Gamma(EM)$. In due course we will find two logically independent new proofs of the Local Index Theorem. One of these two proofs uses the closed formula for the heat kernel coefficients derived in this article directly, while the other proof can possibly be seen as a variant of proofs using Getzler's transformation independent of Clifford algebra and valid for a strictly larger class of operators. Seen differently however this latter proof explains immediately the remarkable success of Getzler's transformation and tells us exactly how the limit of the trace $\operatorname{tr} k_t^{D^2}(y,y)$ of the heat kernel is achieved in the process of blowing up the point $y \in M$. In both proofs the $\widehat{A}$–class appears naturally as the mirror image of the connection Laplacian $\nabla^*\nabla$ with the Clifford bundle EM acting as a mirror. In particular the arguments given should easily generalize to families of twisted Dirac operators, it may be more difficult to treat the equivariant case along this line of reasoning.

The central idea of the strategy pursued in calculating the heat kernel coefficients is to study the relationship between iterated covariant derivatives and their symmetrizations. Iterated covariant derivatives are defined inductively as differential operators on sections of a vector bundle EM endowed with a connection ∇ over a manifold M with a torsion–free connection again denoted ∇ on its tangent bundle TM by setting $\nabla^1_X\psi := \nabla_X\psi$ and

$$\nabla^k_{X_1,\ldots,X_k}\psi \;:=\; \nabla_{X_1}(\,\nabla^{k-1}_{X_2,\ldots,X_k}\psi\,) \;-\; \sum_{\mu=2}^{k}\nabla^{k-1}_{X_2,\ldots,\nabla_{X_1}X_\mu,\ldots,X_k}\psi$$

for all $k \geq 2$ and vector fields $X_1, \ldots, X_k$. Although more difficult to define their symmetrizations $\mathrm{jet}^k_{X_1\cdot\ldots\cdot X_k}\psi := \frac{1}{k!}\sum_{\sigma\in S_k}\nabla^k_{X_{\sigma 1},\ldots,X_{\sigma k}}\psi$ have a nice geometric interpretation in terms of Taylor series, their values $\mathrm{jet}^k\psi(y) \in \mathrm{Sym}^k T^*_yM \otimes E_yM$ in a point $y \in M$ are the homogeneous components of the Taylor series of ψ in the trivialization $E_yM \times T_yM \longrightarrow EM$ of the bundle EM given by parallel transport along radial geodesics in normal coordinates centered about $y \in M$. In particular the sections $\mathrm{jet}^k\psi \in \Gamma(\mathrm{Sym}\,T^*M \otimes EM)$ are algebraically independent in striking contrast to the (unsymmetrized) iterated covariant derivatives, which have no straightforward geometric interpretation and involve highly redundant information.

The interpretation of the symmetrized covariant derivatives as Taylor series strikes an important leitmotif in this article, the interaction between explicitly gauge–fixed constructions and their purely covariant interpretations. In the gauge–fixed context introduced in Section 2 we fix a point $y \in M$ and consider exponential coordinates $\exp_y : T_yM \longrightarrow M$ centered about y in order to trivialize the vector bundle $E_yM \times T_yM \longrightarrow EM$ by parallel transport along radial geodesics. Some important constructions in this gauge–fixed context with interesting covariant interpretations besides the Taylor series $\mathrm{jet}\,\psi$ of sections $\psi \in \Gamma(EM)$ are the definition of the forward parallel transport Φ and the modified connection form Ω^E in Section 3.

With the $\mathrm{jet}^k\psi$ being the components of the Taylor series of ψ general nonsense in jet theory tells us that it is possible to reconstruct the iterated covariant derivatives $\nabla^k\psi$, $k \geq 0$, of a section $\psi \in \Gamma(EM)$ in terms of their symmetrizations $\mathrm{jet}^r\psi$, $r = 0, \ldots, k$. All results of this article are more or less direct corollaries of the general Jet Formula 4.2 making precisely this reconstruction principle explicit. In a sense the general Jet Formula solves a universal problem of differential geometry and besides its direct application to heat kernel coefficients of generalized Laplacians studied in this article it will certainly be instrumental in proving other results of similar complexity. With the general Jet Formula the fight against the Hydra of more and more curvature terms cropping up is eventually won!

Instrumental in understanding the general Jet Formula is a considerably simpler special case of the reconstruction principle valid for "almost" symmetrized covariant derivatives, the special Jet Formula 3.1. A convenient reformulation (4.2) of this special Jet Formula reads

$$\begin{aligned}\nabla^{k+1}_{X,\ldots,X,Y}\psi \;&=\; \mathrm{jet}^{k+1}_{X\cdot\ldots\cdot X\cdot Y}\psi \;+\; \sum_{r=2}^{k}\binom{k}{r}\,\mathrm{jet}^{k+1-r}_{X\cdot\ldots\cdot X\cdot\Phi_r(X\cdot\ldots\cdot X)Y}\psi\\ &\quad+\; \sum_{r=1}^{k}\binom{k}{r}\,\Omega^E_r(X\cdot\ldots\cdot X)_Y\,\mathrm{jet}^{k-r}_{X\cdot\ldots\cdot X}\psi\end{aligned}$$

where the forward parallel transport $\Phi \in \Gamma(\mathrm{Sym}\,T^*M \otimes \mathrm{End}\,TM)$ and the modified connection form $\Omega^E \in \Gamma(\mathrm{Sym}\,T^*M \otimes \mathrm{End}\,EM)$ are formal power series in the covariant derivatives of the curvature tensor R and the curvature R^E of the connections on TM and EM.

Trying to exponentiate this formulation of the special Jet Formula in a suitable sense to the general Jet Formula leads us quite naturally to consider special rooted forests. Rooted forests have been used for a long time in the theory of operads [MSS] to study abstract calculation flow charts. Rooted forests with colored vertices will serve similar purposes throughout this article using various minor modifications of the basic notion of jet forests introduced in Definition 4.1. Jet forests are rooted forests with a labelling of their leaves and a coloring of their vertices by colors white and black and later on transparent and red satisfying some additional constraints. Particularly important will be the budding condition: the bud of every vertex V, the leaf of maximal label in the subtree rooted at V, is connected directly to V. Moreover the colors black and later on red are only allowed for roots in a jet forest, all leaves and all interior vertices are either white or later on transparent.

Thinking of rooted forests as abstract calculation flow charts the white and black vertices correspond to two different multilinear operations given by the homogeneous components of the power series Φ and Ω^E respectively, while red and transparent vertices introduced later on correspond similarly to multilinear operations given by the homogeneous components of the infinite order jet $\mathcal{F} \in \Gamma(\mathrm{Sym}\, T^*M \otimes \mathrm{End}\, EM)$ of the auxiliary potential and the power series $\Omega \in \Gamma(\mathrm{Sym}\, T^*M \otimes \mathrm{End}\, TM)$ arising from Ω^E by replacing the curvature R^E of EM by the curvature tensor R of TM. In this way every tree $\mathfrak{T} \subset \mathfrak{F}$ with white or black root in a jet forest $\mathfrak{F}$ corresponds to a multilinear form $\Phi(\mathfrak{T})$ or $\Omega^E(\mathfrak{T})$ on TM with values in TM or $\mathrm{End}\, EM$ respectively, which is eventually a polynomial in the curvature R^E of EM and the curvature tensor R of TM. With this much of the notation introduced we can formulate the general Jet Formula in the following way:

Lemma 4.2 (General Jet Formula)
For all vector fields $X_1, \ldots, X_k$, $k \geq 1$, on an affine manifold M with torsion free connection and all sections ψ of a vector bundle EM over M endowed with a connection ∇ we have

$$\nabla^k_{X_1,\ldots,X_k}\psi \;=\; \sum_{\substack{\text{jet forest}\\ |\mathfrak{F}|=k}} \; \sum_{\substack{\text{feedback}\\ f}} \Big(\overrightarrow{\prod_{\substack{\mathfrak{T}\subset\mathfrak{F}\ \text{black}\\ f(\mathfrak{T})=\infty}}} \Omega^E(\mathfrak{T})_{X^{f}(\mathfrak{T})} \Big)\, \mathrm{jet}\,\psi \Big(\prod_{\mathfrak{T}\subset\mathfrak{F}\ \text{white}} \Phi(\mathfrak{T})_{X^{f}(\mathfrak{T})} \Big)$$

where $X^f(\mathfrak{T})$ are the argument vector fields associated to the leaves of a tree $\mathfrak{T} \subset \mathfrak{F}$.

In order to limit the level of technicality in this introduction we refrain from saying what the summation over feedback is about. Suffice it to say that a feedback map is a map from the black trees in $\mathfrak{F}$ to $\{1, \ldots, k, \infty\}$ used to modify the argument vector fields to $X^f_1, \ldots, X^f_k$. A direct application of the general Jet Formula provides the formula (4.8) for the canonical or Spencer connection in the model of jet theory based on symmetrized covariant derivatives

$$\begin{aligned}\nabla^{k+1}_{Y,X,\ldots,X}\psi \;=\; \mathrm{jet}^{k+1}_{X\cdot\ldots\cdot X\cdot Y}\psi \;&-\; \sum_{r=2}^{k}\frac{1}{r}\binom{k}{r}\,\mathrm{jet}^{k+1-r}_{X\cdot\ldots\cdot X\cdot[(\Phi-\mathrm{id})\Phi^*]_r(X\cdot\ldots\cdot X)Y}\psi\\ &+\; \sum_{r=1}^{k}\frac{1}{r}\binom{k}{r+1}\,\mathrm{jet}^{k-r}_{X\cdot\ldots\cdot X\cdot[\Omega_{\Phi^*}]_r(X\cdot\ldots\cdot X)_Y X}\psi\\ &-\; \sum_{r=1}^{k}\frac{1}{r}\binom{k}{r}\,[\,\Omega^E_{\Phi^*}\,]_r(X\cdot\ldots\cdot X)_Y\,\mathrm{jet}^{k-r}_{X\cdot\ldots\cdot X}\psi\end{aligned}$$

where $\Phi^* \in \Gamma(\mathrm{Sym}\, T^*M \otimes \mathrm{End}\, TM)$ is a formal power series reducing to the adjoint of Φ on a Riemannian manifold M. Interestingly this formula seems to be considerably harder to prove than the version of the special Jet Formula 3.1 formulated above and to our knowledge it has not appeared in the literature before. In any case the argument given in Section 4 depends crucially on the power of the general Jet Formula. In light of this closed formula for the Spencer connection a prospective application of the general Jet Formula is the calculation of the Fedosov connection [F] on the Weyl algebra $\mathrm{Sym}\, T^*M$ on a Poisson manifold M endowed with a torsion–free Poisson connection allowing us to write down the Moyal–Vey product of deformation quantization and the corresponding trace functional explicitly.

The relevance of the general Jet Formula 4.2 for calculating heat kernel coefficients of generalized Laplacians is most easily seen for the particularly interesting subclass of twisted Laplacians introduced in Section 7, which contains most of the well–known Laplace operators in differential geometry. Twisted Laplace operators $\Delta_\diamond$ are characterized among all generalized Laplacians by the form of the auxiliary potential $\mathcal{F} \in \Gamma(\mathrm{End}\, EM)$ in the decomposition $\Delta = \nabla^*\nabla + \mathcal{F}$, for twisted Laplacians $\Delta_\diamond$ this potential is a curvature term arising from a Weitzenböck formula. In other words twisted Laplacians can be written

$$\Delta_\diamond\psi \;\; := \;\; \nabla^*\nabla\psi \; + \; \frac{1}{2}\sum_{\mu<\nu}(dX_\mu \wedge dX_\nu) \diamond R^E_{X_\mu, X_\nu}\psi$$

with a suitable parallel map $\diamond : \Lambda^2 T^*M \longrightarrow \mathrm{End}\, EM$ and a local base $\{X_\mu\}$ of TM with dual base $\{dX_\mu\}$. Introducing the application map $\mathrm{app} : \mathrm{End}\, EM \otimes EM \longrightarrow EM$ and the twisted trace $\mathrm{tr}_\diamond : T^*M \otimes T^*M \longrightarrow \mathrm{End}\, EM$ defined by $\mathrm{tr}_\diamond(\alpha \otimes \beta) := \langle \alpha, \beta \rangle \mathrm{id}_E - (\alpha \wedge \beta)\diamond$ we can rewrite the definition of a twisted Laplacian $\Delta_\diamond := \mathrm{app} \circ (-\mathrm{tr}_\diamond \otimes \mathrm{id}_E) \circ \nabla^2$. In this characterization of twisted Laplace operators both maps app and $\mathrm{tr}_\diamond$ are parallel so that it is possible to express the powers $\Delta^k_\diamond$, $k \geq 0$, in terms of iterated covariant derivatives:

$$\Delta^k_\diamond\,\psi \;\; = \;\; \overrightarrow{\mathrm{app}} \circ (\, -\, \mathrm{tr}_\diamond\,)^k\, (\, \nabla^{2k}\psi\,)$$

This equation is the key idea of applying the general Jet Formula to the calculation of the heat kernel of twisted Laplacians $\Delta_\diamond$, because it allows us to express the total symbol of all $\Delta^k_\diamond$ explicitly in the trivialization $E_yM \times T_yM \longrightarrow EM$ of the vector bundle EM using parallel transport along radial geodesics in normal coordinates centered about $y \in M$. According to Theorem 2.4 however knowing the total symbol of the formal power series $e^{-t\Delta}$ of differential operators in this particular trivialization of the bundle EM is equivalent to knowing the infinite order jet of the heat kernel coefficients along the diagonal in $M \times M$.

In order to describe the resulting closed formula for the heat kernel coefficients of a twisted Laplace operator $\Delta_\diamond$ we introduce a minor variant of jet forests called Laplace forests in Definition 7.3. Associated to the trees $\mathfrak{T}$ in a Laplace forests $\mathfrak{F}$ are multilinear forms $\Phi(\mathfrak{T}) \in \Gamma(\bigotimes^{|\mathfrak{T}|} T^*M \otimes TM)$ and $\Omega^E(\mathfrak{T}) \in \Gamma(\bigotimes^{|\mathfrak{T}|} T^*M \otimes \mathrm{End}\, EM)$, which can be assembled into a multilinear form associated to the Laplace forest $\mathfrak{F}$ itself

$$\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ black}} \Omega^E(\mathfrak{T}) \otimes \prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} \Phi(\mathfrak{T})^\sharp \;\; \in \;\; \Gamma(\, \bigotimes^{|\mathfrak{F}|} T^*M \, \otimes \bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ black}} \mathrm{End}\, EM \, \otimes \, \mathrm{Sym}^{\,\#\mathrm{White}\,\mathfrak{F}} T^*M\,)$$

using the labelling of the leaves in $\mathfrak{F}$, where $\mathrm{White}\,\mathfrak{F}$ denotes the set of white trees in $\mathfrak{F}$. Eventually we need to reduce this multilinear form to a section of $\mathrm{End}\, EM \otimes \mathrm{Sym}\, T^*M$

depending not only on the Laplace forest $\mathfrak{F}$ but also on the choice of a subset $L \subset \operatorname{Leaf}\mathfrak{F}$ of marked leaves. In a first step we apply a linear operator m_L, which multiplies the form factors in $\bigotimes T^*M$ associated to the leaves in L to the polynomial factor $\operatorname{Sym}^{\#\mathrm{White}\,\mathfrak{F}}T^*M$. The remaining form factors in $\bigotimes T^*M$ are contracted in pairs to endomorphisms on EM in a second step using powers of the twisted trace $\mathrm{tr}_\diamond$. Eventually we multiply the original endomorphisms $\Omega^E(\mathfrak{T})$ and the endomorphisms arising from the twisted trace together in a delicately chosen order encoded in a sorted multiplication operator m^*. The sections of $\operatorname{End} EM \otimes \operatorname{Sym}^{\#\mathrm{White}\,\mathfrak{F}+\#L}T^*M$ associated in this way to a Laplace forest $\mathfrak{F}$ with marked leaves L form the basic building blocks of the heat kernel coefficients:

Theorem 7.5 (Heat Kernel Coefficients of Twisted Laplacians)
The generating series $\sum_{d\geq 0} t^d a_d$ of the heat kernel coefficients of a twisted Laplacian satisfies:

$$j^{\frac{1}{2}}\,a(t) = e^{t\Delta}\left[\sum_{\substack{\mathfrak{F}\ \textit{Laplace forest}\\ L\ \textit{marked leaves}}} \frac{t^{\frac{|\mathfrak{F}|-\#L}{2}-\#\mathrm{White}\,\mathfrak{F}}}{2^{\#\mathrm{White}\,\mathfrak{F}}\,\left(\frac{|\mathfrak{F}|+\#L}{2}\right)!}\, m^*\circ \mathrm{tr}_\diamond^{\frac{|\mathfrak{F}|-\#L}{2}}\circ m_L\left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \textit{black}}\Omega^E(\mathfrak{T})\otimes\prod_{\mathfrak{T}\subset\mathfrak{F}\ \textit{white}}\Phi(\mathfrak{T})^\sharp\right)\right]$$

where Δ is the formal Laplace operator on $\operatorname{Sym} T^*M$ *defined by the Riemannian metric.*

Evidently this formula is a closed formula for the heat kernel coefficients involving only the covariant derivatives of the Riemannian curvature tensor R of M and the curvature R^E of the bundle EM. Instead of thinking of Theorem 7.5 as a closed formula for the heat kernel coefficients of twisted Laplacians in terms of the iterated covariant derivatives of curvature however it seems more natural to use the homogeneous components of the power series Φ and Ω^E as well as Ω directly, which do not suffer from the illness of containing redundant information. It is particularly intriguing that the extension of this formula to twisted Laplace operators $\Delta_{\diamond,\mathcal{F}} = \Delta_\diamond + \mathcal{F}$ with auxiliary potential $\mathcal{F} \in \Gamma(EM)$ given in Theorem 7.8 is almost identical to the formula in the untwisted case up to the appearance of red roots, whose associated multilinear operations depend only on $\operatorname{jet}\mathcal{F} \in \Gamma(\operatorname{Sym} T^*M \otimes \operatorname{End} EM)$.

Let us now make a few more detailed comments on the different sections and the results proved in this article. In Section 2 we recall the notion of a generalized Laplacian Δ acting on sections of a vector bundle EM and discuss the decomposition $\Delta = \nabla^*\nabla + \mathcal{F}$ into the connection Laplacian for an associated connection on EM and the potential $\mathcal{F} \in \Gamma(\operatorname{End} EM)$. Having sketched the argument of Minakshisundaram–Pleijel leading to the recursive definition (2.8) of the extended heat kernel coefficients a_d, $d \geq 0$, we turn to an alternative characterization of these coefficients in the second half of Section 2, which is a modified and reedited version of the unpublished preprint [W]. The unique solution of the Intertwining Property 2.1 thought of as an equation for the unknown infinite order jet of the generating series $a(t) := \sum_{d\geq 0} t^d a_d$ for the heat kernel coefficients is derived in Theorem 2.4 and reads

$$\operatorname{jet} j^{\frac{1}{2}} a(t) \;=\; e^{t\Delta}\,(2t)^{-N}\,e^{-t|\,\cdot\,|^2}\,\operatorname{ev}[\,e^{-t\Delta}\,]^\sharp$$

in terms of the total symbol $\operatorname{ev}[\,e^{-t\Delta}\,]^\sharp$ of the generating series $e^{-t\Delta} := \sum_{k\geq 0}\frac{t^k}{k!}\Delta^k$ of the powers of the Laplace operator Δ in the trivialization $T_yM \times E_yM \longrightarrow EM$ given by parallel transport along radial geodesics in normal coordinates centered about $y \in M$.

In Section 3 we discuss the special Jet Formula 3.1 and its consequences. In particular we will identify the Taylor series $\mathrm{jet}\,\psi \in \mathrm{Sym}\,T_y^*M \otimes E_yM$ of a section $\psi \in \Gamma(EM)$ in the origina $0 \in T_yM$ with respect to the trivialization $E_yM \times T_yM \longrightarrow EM$ given by parallel transport along radial geodesics with the value of the symmetrized iterated covariant derivatives $\mathrm{jet}\,\psi \in \Gamma(\mathrm{Sym}\,T^*M \otimes EM)$ of ψ in $y \in M$. Moreover the special Jet Formula provides a covariant interpretation of the backward parallel transport Φ^{-1} and the connection form ω^E defined originally in the gauge–fixed context and thus allows us to calculate their Taylor series in $0 \in T_yM$ in Theorem 3.2 covariantly as formal power series in the covariant derivatives of the Riemannian curvature tensor R of M and the curvature R^E of EM. In consequence we can identify the forward parallel transport Φ and the modified connection form $\Omega^E := \omega^E_\Phi$ with such formal power series, too.

In order to extend the special Jet Formula 3.1 to the more powerful general Jet Formula 4.2 we introduce jet forests in Section 4, describe the rules turning a tree $\mathfrak{T}$ in a jet forest $\mathfrak{F}$ into a multilinear form $\Phi(\mathfrak{T})$ or $\Omega^E(\mathfrak{T})$ in detail and use the inductive nature on the set of jet forests to prove the general Jet Formula. As a direct application we derive the closed formula (4.8) for the canonical or Spencer connection.

Resuming the gauge–fixed context in Section 5 we replace the heat kernel coefficients $a_d \in C^\infty(T_yM, \mathrm{End}\,E_yM)$ for a trivial vector bundle $E_yM \times T_yM$ over a euclidian vector space by universal heat kernel coefficients $a_d \in \mathrm{Sym}\,T_y^*M \otimes \mathcal{U}\mathfrak{hol}_y^E M$, $d \geq 0$, associated to the underlying principal connection, which map to the infinite order Taylor series of the actual heat kernel coefficients under the representation homomorphism $\mathcal{U}\mathfrak{hol}_y^E M \longrightarrow \mathrm{End}\,E_yM$. With this replacement we can introduce an idea instrumental to both proofs of the Local Index Theorem presented in this article, the use of filtrations as algebraic analogues of Getzler's transformation. More precisely the natural filtration on the universal enveloping algebra $\mathcal{U}\mathfrak{hol}_y^E M$ can be extended to a filtration

$$\mathbb{F}^r(\,\mathrm{Sym}\,T_y^*M \otimes \mathcal{U}\mathfrak{hol}_y^E M\,) \;:=\; \bigoplus_{l \geq 0} \mathrm{Sym}^{\,l}T_y^*M \otimes \mathcal{U}^{\leq \lfloor \frac{l+r}{2} \rfloor}\mathfrak{hol}_y^E M$$

on the space $\mathrm{Sym}\,T_y^*M \otimes \mathcal{U}\mathfrak{hol}_y^E M$ of universal heat kernel coefficients with the remarkable property that the operator $j^{\frac{1}{2}}\nabla^*\nabla j^{-\frac{1}{2}}$ is filtered degree 2. Twisting the definition of the filtration slightly for a Clifford bundle the same is true for the operator $-j^{\frac{1}{2}}\,D^2\,j^{-\frac{1}{2}}$ and we conclude $a_d \in \mathbb{F}^{2d}(\mathrm{Sym}\,T_y^*M \otimes \mathcal{U}\mathfrak{hol}_y^E M)$ from the recursion formula (2.8) of Minakshisundaram–Pleijel starting with the value $a_0 = 1$ in $\mathbb{F}^0(\mathrm{Sym}\,T_y^*M \otimes \mathcal{U}\mathfrak{hol}_y^E M)$. In light of the simple Lemma 5.7 about the character of the spinor representation this result implies the regularity of the trace of the heat kernel of squares of twisted Dirac operators. In the subsequent Section 6 we calculate in Theorem 6.2 the symbol class of the universal heat kernel coefficients using an ansatz motivated by Mehler's formula and recall the standard argument leading to the Local Index Theorem.

Presumably the most interesting and perhaps most difficult section for the reader will be Section 7, in which we calculate the formulas for the infinite order jets of the heat kernel coefficients of twisted Laplacians and twisted Laplacians with potential along the diagonal. Pursuing the strategy described above we use the general Jet Formula to interprete the total symbol $\mathrm{ev}[\,e^{-t\Delta_\diamond}\,]^\sharp$ of the powers of a twisted Laplace operator $\Delta_\diamond$ defined in the gauge–fixed context of Section 2 covariantly and apply Theorem 2.4 to calculate the heat kernel coefficients. Most of the subsequent work in this section is spent on eliminating the summation over feedback maps by reordering the factors in a product over endomorphismus or more succinctly in defining the sorted multiplication m^*. With Lemma 7.7 it is possible to extend the strategy to twisted Laplacians with potential resulting in Theorem 7.8.

In the final Section 8 we prove the Local Index Theorem without reference to Mehler's formula and take the opportunity to discuss Theorems 7.5 and 7.8 in greater detail. In particular we study some interesting aspects of the abstract summation over Laplace forests using a filtration extending the filtration on the space of universal heat kernel coefficients discussed in Section 5. Eventually we restrict the summation in Theorem 7.5 to a small subclass of Laplace forests carrying the index information and show that for these forests the sum turns into the expected exponential. The hyperbolic tangens emerges in the process of summation by the combinatorial identity (8.2).

Numerous people have supported my work on heat kernel coefficients during the years. In particular I vividly remember a discussion with Prof. Brüning in June 1999 arousing my interest in the intellectual problem of calculating heat kernel coefficients. Moreover I would like to thank Prof. Polterovich for many interesting hours of discussion on this topic and Prof. Ballmann for constant encouragement and support.

Chapter 2

Characterizations of Heat Kernel Coefficients

Essentially there are two different ways to calculate the extended heat kernel coefficients of a generalized Laplacian acting on sections of a vector bundle EM over a manifold M. Besides the straightforward approach using the original recursion formula (2.8) of Minakshisundaram–Pleijel for the extended coefficients a_d, $d \geq 0$, in turn it is also possible to use an interesting intertwining property of the generating power series $a(t) := \sum_{d\geq 0} t^d a_d$ for these coefficients formulated in Theorem 2.1. The first method leads more or less directly to a proof of the Local Index Theorem whereas the second method will give us an explicit formula for all heat kernel coefficients for a large class of generalized Laplacians called twisted Laplacians. We will discuss both methods in some detail in this section taking [BGM] [BGV] as general references to the topics covered.

Consider a connection ∇ on a vector bundle EM over a Riemannian manifold M with metric g. The Levi–Civita connection on the tangent bundle TM associated to g allows us to define the iterated covariant derivative $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ as a differential operator on sections of EM and straightforward calculation proves the operator identity

$$\nabla^2_{X,Y} \circ f \;=\; f \circ \nabla^2_{X,Y} \;+\; Xf \circ \nabla_Y \;+\; Yf \circ \nabla_X \;+\; \mathrm{Hess}\, f(\,X,\,Y\,) \tag{2.1}$$

valid for all $f \in C^\infty(M)$ where $\mathrm{Hess}\, f(X,Y) := XYf - \nabla_X Y\, f$ is the Hessian of f and functions are thought of as multiplication operators on sections. The connection Laplacian is the negative trace $-\mathrm{tr}_g \nabla^2 := -\sum \nabla^2_{X_\mu, X_\mu}$ of the iterated covariant derivative ∇^2 over a local orthonormal base $\{X_\mu\}$. Inspecting (2.1) we readily find its principal symbol:

$$\sigma_{-\mathrm{tr}_g \nabla^2}(\,d_y f,\, d_y \tilde{f}\,) \;:=\; \tfrac{1}{2}\,[\,[\,-\mathrm{tr}_g \nabla^2 f\,],\, \tilde{f}\,]\Big|_{E_y M} \;=\; -\,g(\,d_y f,\, d_y \tilde{f}\,)\,\mathrm{id}_{E_y M}$$

Similarly a generalized Laplacian Δ is a second order differential operator acting on sections of EM with scalar, negative definite principal symbol

$$\sigma_\Delta(\,d_y f,\, d_y \tilde{f}\,) \;:=\; \tfrac{1}{2}\,[\,[\,\Delta,\, f\,],\, \tilde{f}\,]\Big|_{E_y M} \;=\; -\,g(\,d_y f,\, d_y \tilde{f}\,)\,\mathrm{id}_{E_y M} \tag{2.2}$$

for all $f, \tilde{f} \in C^\infty(M)$ and $y \in M$ for some positive definite symmetric bilinear form g on $T^*_y M$. Being positive definite g defines a scalar product on $T_y M$ as well and thus a

Riemannian metric on M and its Levi–Civita connection ∇ on the tangent bundle TM. Taking the operator identity (2.1) as a clue we can even associate a connection

$$\nabla_{\mathrm{grad}_g f} \;:=\; \tfrac{1}{2}\big(\, (\Delta_g f) \,-\, [\,\Delta,\, f\,]\,\big)$$

on EM to a generalized Laplacian Δ, where $\Delta_g f \,:=\, -\mathrm{tr}_g \mathrm{Hess}\, f$ is the Laplace–Beltrami operator on functions. In fact the right hand side is easily seen to be a derivation (sic!) in f and thus its value at a point only depends on the gradient of f, whereas the Leibniz rule for this connection is a reformulation of the definition (2.2) of the symbol. Given the Riemannian metric g on M and the connection ∇ on EM associated to a generalized Laplacian Δ the difference $\mathcal{F} \,:=\, \Delta + \mathrm{tr}_g\nabla^2$ is a fortiori a differential operator of order zero or equivalently a section $\mathcal{F} \in \Gamma(\mathrm{End}\, EM)$ of the endomorphism bundle of EM called the potential, so that the generalized Laplacian we started with can be written in the form $\Delta \,=\, -\mathrm{tr}_g\nabla^2 + \mathcal{F}$. Using this presentation and the operator identity (2.1) we verify the commutator relation

$$e^f \circ \Delta \circ e^{-f} \;=\; \Delta \,+\, 2\,\nabla_{\mathrm{grad}_g f} \,+\, \mathrm{div}_g\,\mathrm{grad}_g\, f \,-\, g(\, df,\, df\,) \tag{2.3}$$

for $f \in C^\infty(M)$, which can actually be taken as a characterization for generalized Laplacians equivalent to the original definition. A formal consequence of this characterization is that the formally adjoint operator Δ^* of a generalized Laplacian Δ on EM is the generalized Laplacian acting on sections of E^*M defined by the same symbol metric and the dual connection. It is slightly more work to check that the potential of the generalized Laplacian Δ^* is the adjoint $\mathcal{F}^*$ of the potential $\mathcal{F}$ of Δ or equivalently that the formal adjoint of $-\mathrm{tr}_g\nabla^2$ is simply $-\mathrm{tr}_g\nabla^{*2}$ with the dual connection ∇^* on E^*M. Starting with the formal adjoint $-(\,\nabla^*_X + \mathrm{div}_g X\,)$ of the covariant derivative ∇_X along some vector field $X \in \Gamma(TM)$ and using the relation $\mathrm{div}_g(\mathrm{div}_g X\, Y) \,=\, Y\mathrm{div}_g X + \mathrm{div}_g X \mathrm{div}_g Y$ we easily arrive at

$$\begin{aligned}\big(\,\nabla^2_{X,Y}\,\big)^* \;&=\; (\,\nabla^*_Y + \mathrm{div}_g Y\,)(\,\nabla^*_X + \mathrm{div}_g X\,) \,+\, \big(\,\nabla^*_{\nabla_X Y} + \mathrm{div}_g(\nabla_X Y)\,\big) \\ &=\; \nabla^{*2}_{Y,X} \,+\, \nabla^*_{(\nabla_X Y + \mathrm{div}_g X\, Y) + (\nabla_Y X + \mathrm{div}_g Y\, X)} \,+\, \mathrm{div}_g(\nabla_X Y + \mathrm{div}_g X\, Y)\end{aligned} \tag{2.4}$$

but only the first summand on the right survives taking the trace over a local orthonormal base $\{X_\mu\}$ due to the classical identity $\sum \nabla_{X_\mu} X_\mu \,=\, -\sum \mathrm{div}_g X_\mu\, X_\mu$. Of course the distinction between the vector bundle EM and its dual E^*M is artificial in most if not all interesting situations, because there exists a parallel isomorphism $\sharp :\; EM \longrightarrow E^*M$ arising say from a parallel metric or a parallel symplectic form. The formally adjoint operator of the connection ∇ can thus be thought of as a differential operator $\nabla^* : \Gamma(T^*M \otimes EM) \longrightarrow \Gamma(EM)$ and the Laplacian $\nabla^*\nabla$ is defined and agrees with $-\mathrm{tr}_g\nabla^2$.

Let us now describe the considerations made by Minakshisundaram–Pleijel culminating in the construction of the extended heat kernel coefficients a_d, $d \geq 0$, in somewhat more detail. Consider the heat conduction equation $\frac{d}{dt}\psi_t + \Delta\psi_t = 0$ associated to a generalized Laplacian Δ on M, which describes the time evolution of an initial heat distribution $\psi_0 \in \Gamma(EM)$ under Newton's law of heat propagation governed by Δ. Under mild assumptions on the generalized Laplacian Δ and the manifold M the heat conduction equation has a well–defined fundamental solution in terms of an integral operator

$$(\, e^{-t\Delta}\,\psi\,)(y) \;:=\; \int_M k^\Delta_t(\, x,\, y\,)\,\psi(x)\mathrm{vol}(x)$$

whose kernel k_t^Δ is called the heat kernel associated to Δ. In other words the heat kernel k_t^Δ is a family of sections of the vector bundle $\mathrm{Hom}(EM, EM)$ over $M \times M$ with fiber $\mathrm{Hom}(E_xM, E_yM)$ over (x, y) defined for positive times $t > 0$, such that $k_t^\Delta(\cdot, y)$ converges to $\delta_y(\cdot)$ in the sense of distributions as time goes to zero and $\frac{d}{dt}k_t^\Delta + \Delta_y\, k_t^\Delta = 0$, where Δ_y denotes Δ acting on the target variable y. Whenever defined the integral operator $e^{-t\Delta}$ will commute with Δ or equivalently $\Delta_y k_t^\Delta = \Delta_x^* k_t^\Delta$, where Δ_x^* denotes the operator formally adjoint to Δ acting on the source variable x. Hence the heat kernel satisfies the dual heat conduction equation as well, which is more convenient to use once we fix the target $y \in M$:

$$\frac{d}{dt}\, k_t^\Delta \;+\; \Delta_x^*\, k_t^\Delta \;\;=\;\; 0 \tag{2.5}$$

Following Minakshisundaram–Pleijel we will from now on restrict ourselves to the special case of a generalized Laplacian Δ acting on sections of a trivial vector bundle $E \times T$ over a euclidian vector space T such that the symbol metric g agrees with the scalar product $\langle\,,\rangle$ of T in the origin and the exponential map $\exp_0 :\; T_0T \cong T \longrightarrow T$ is the identity. In addition we will assume that parallel transport for the connection ∇ on $E \times T$ along the radial geodesics $t \longmapsto tX$ is the identity $E \times \{0\} \longrightarrow E \times \{X\}$. The general case of a generalized Laplacian Δ acting on sections of a vector bundle EM over a Riemannian manifold M can essentially be reduced to this special case by fixing the target $y \in M$ and setting $T := T_yM$ and $E := E_yM$ while pulling back all the geometric data to the tangent space. Evidently the volume forms vol and $\mathrm{vol}_g = j(X)\mathrm{vol}$ of the flat scalar product $\langle\,,\rangle$ and the symbol metric g respectively will differ by the determinant $j(X)$ of the Jacobian of the exponential map $\exp_0$ in $X \in T$.

Thinking of sections of the trivial vector bundle $E \times T$ as functions on T with values in E we can define the flat Laplacian $\Delta = -\sum \frac{\partial^2}{\partial x_\mu^2}$ with an orthonormal base $x_1, \ldots, x_n$ in addition to the generalized Laplacian Δ. A closed formula for the heat kernel of the flat Laplacian has been known since the seminal work of Fourier on the propagation of heat

$$k_t^\Delta(\,X,\,0\,) \;\;:=\;\; \frac{1}{\sqrt{4\pi t}^{\dim T}}\, e^{-\frac{|X|^2}{4t}}\,\mathrm{id}_E$$

so it is prudent to take it as a clue on the general case and make an ansatz of the form

$$k_t^\Delta(\,X,\,0\,) \;\;:=\;\; \frac{1}{\sqrt{4\pi t}^{\dim T}}\, e^{-\frac{|X|^2}{4t}}\, j^{-\frac{1}{2}}(\,X\,)\, a(t,\,X\,) \tag{2.6}$$

for the heat kernel k_t^Δ with $a \in C^\infty(\mathbb{R}_+ \times T, \mathrm{End}\,E)$. Recalling that the exponential map is a radial isometry we observe that the metric and flat gradient of the distance squared coincide $\mathrm{grad}_g|\cdot|^2 = 2N = \mathrm{grad}|\cdot|^2$ with twice the Euler field $N := \sum x_\mu \frac{\partial}{\partial x_\mu}$, which has length squared $|\cdot|^2$ with respect to both the scalar product $\langle\,,\rangle$ and the metric g. However the metric divergence $\mathrm{div}_g N = N(\log j) + \mathrm{div}\, N$ of the Euler field will differ from its divergence $\mathrm{div}\, N = \dim T$ and the factor $j^{-\frac{1}{2}}$ is included in the ansatz in order to cancel the additional term $N(\log j)$. With parallel transport along radial geodesics being the identity or $\nabla_N = N$

en nuce with a slight abuse of notation the commutation relation (2.3) becomes:

$$\begin{aligned}\Delta^*_X\, k^\Delta_t(X,0) &= \frac{1}{\sqrt{4\pi t}^{\dim T}}\, e^{-\frac{|X|^2}{4t}} \left[\, \Delta^* \,+\, \frac{1}{t}\nabla_N \,+\, \frac{1}{2t}\mathrm{div}_g\, N \,-\, \frac{|\cdot|^2}{4t^2}\,\right] j^{-\frac{1}{2}}(X)\, a(t,X) \\ &= \frac{1}{\sqrt{4\pi t}^{\dim T}}\, e^{-\frac{|X|^2}{4t}}\, j^{-\frac{1}{2}}(X) \left[\, (j^{\frac{1}{2}}\Delta^* j^{-\frac{1}{2}}) \,+\, \frac{N}{t} \,+\, \frac{\dim\, T}{2t} \,-\, \frac{|\cdot|^2}{4t^2}\,\right] a(t,X)\end{aligned}$$

Using this result turns the dual heat equation (2.5) for k^Δ_t into the equation

$$\frac{d}{dt}\,a \,+\, \frac{N}{t}\,a \;=\; -\,(\,j^{\frac{1}{2}}\,\Delta^*\,j^{-\frac{1}{2}}\,)\,a \tag{2.7}$$

for the unknown function $a \in C^\infty(\mathbb{R}_+ \times T, \mathrm{End}\,E)$. Although this equation does not look like a simplification the important point is that the Dirac like singularity of k^Δ_t as time goes to zero is due to the factor $t^{-\frac{\dim T}{2}}$ of the heat kernel of the flat Laplacian Δ whereas the function a may stay finite at $t = 0$. In fact the fundamental idea of Minakshisundaram–Pleijel was to treat the transformed equation (2.7) as a recursion formula

$$(\,d \,+\, 1 \,+\, N\,)\,a_{d+1} \;=\; -\,(\,j^{\frac{1}{2}}\,\Delta^*\,j^{-\frac{1}{2}}\,)\,a_d \tag{2.8}$$

for the would–be Taylor expansion $a(t,X) \sim \sum_{d\geq 0} t^d\, a_d(X)$ of the family a at time $t = 0$ with sections $a_d \in C^\infty(T, \mathrm{End}\,E)$ subject to the initial condition $a_0(X) = \mathrm{id}$. Interestingly this initial condition is the only (sic!) choice consistent with both the recursion formula and the fact $a_0(0) = \mathrm{id}$ already known to Weyl, although this condition simply does not make sense globally unless the bundle EM is trivial.

Despite the fact that the recursion formula (2.8) is ill–motivated unless we know a priori that the function $a \in C^\infty(\mathbb{R}_+ \times T, \mathrm{End}\,E)$ has a well defined Taylor series at $t = 0$ we can head on and solve the recursion formula (2.8), which in essence is an ordinary differential equation, to all orders $d \geq 0$. Representing the formal power series $a(t,X) = \sum t^d a_d(X)$ we get this way by some C^∞–function $\tilde{a}(t,X)$ we get an integral kernel $\tilde{k}^\Delta_t$ close enough to the real heat kernel k^Δ_t to serve as a parametrix, the initial value for a standard iteration procedure to approximate the real heat kernel k^Δ_t to arbitrary precision. Known estimates on the convergence of this iteration procedure [BGV] then prove that the values $a_d(0)$ are the coefficients in the asymptotic expansion

$$k^\Delta_t(\,X,\,0\,) \underset{t\to 0^+}{\sim} \frac{\delta_0(X)}{\sqrt{4\pi t}^{\dim T}} \sum_{d\geq 0} t^d\, a_d(0) \tag{2.9}$$

we looked for. In order to settle this discussion for the general case of a Laplacian Δ acting on sections of a vector bundle EM over M, too, we need to pull back all the geometric data via the exponential map and parallel transport to $T := T_yM$ and $E := E_yM$. Clearly under this identification the value $a_d(0)$ of the d–th heat kernel coefficient in the origin becomes the coefficient $a_d(y)$ in the asymptotic expansion mentioned in the introduction.

In the second part of this section we want to discuss a completely different interpretation of the extended heat kernel coefficients a_d, $d \geq 0$, which is definitely less constructive but can be used to give a closed formula for the Taylor series of all these coefficients in the origin in terms of the powers Δ^k, $k \geq 1$, of the generalized Laplacian Δ. In this alternative characterization the heat kernel coefficients show up intertwining in a sense the generalized Laplacian Δ with the flat Laplacian Δ. In particular it will become apparent that it is just about as complicated to write down a closed formula for the heat kernel coefficients a_d or their values $a_d(0)$ in the origin as it is to calculate the powers Δ^k at a given point. This observation provides ample justification to consider the class of twisted Laplacians $\Delta_\diamond$ in more detail whose powers can be calculated in principle using the general Jet Formula 4.2.

Consider as before the case of a generalized Laplace operator Δ acting on sections of a trivial vector bundle $E \times T$ over a euclidian vector space T with scalar product $\langle\,,\rangle$ and a auxiliary Riemannian metric g such that the metric g agrees with $\langle\,,\rangle$ in the origin while its exponential map $\exp_0 : T_0T \cong T \longrightarrow T$ there is the identity. In addition we will assume that parallel transport for the non–trivial connection ∇ on $E \times T$ along radial geodesics $t \longmapsto tX$ is the identity $E \times \{0\} \longrightarrow E \times \{X\}$. Let $\mathrm{ev} : C^\infty(T, E) \longrightarrow E$, $\psi \longmapsto \psi(0)$ be the evaluation at the origin and let $j \in C^\infty T$ denote the Jacobian determinant $\mathrm{vol}_g =: j\,\mathrm{vol}$ of the exponential map $\exp_0$ at $X \in T$ equivalently $j := \det^{\frac{1}{2}} g$ is the square root of the determinant of the symbol metric g with respect to the flat scalar product $\langle\,,\rangle$. By the remarks following [BGV, Theorem 2.30] there exists an asymptotic expansion

$$\mathrm{ev}\left[\, e^{-t\Delta}\psi \,\right] \underset{t\to 0^+}{\sim} \sum_{k\geq 0} \frac{(-t)^k}{k!}\, \mathrm{ev}\left[\, \Delta^k \psi \,\right]$$

for the heat operator applied to sections $\psi \in C^\infty(T, E)$ of compact support, which complements the asymptotic expansion (2.9) of its integral kernel k_t^Δ. Using both asymptotic expansions and observing that the factor $\frac{1}{\sqrt{4\pi t}^{\dim T}} \exp -\frac{|X|^2}{4t}$ is the heat kernel of the flat Laplacian Δ acting on sections of $E \times T$ as well we find for a smooth section $\psi \in C^\infty(T, E)$ with support in a sufficiently small compact neighborhood of the origin:

$$\begin{aligned}
\int_T k_t^\Delta(X)\psi(X)\,\mathrm{vol}_g(X) \quad &\underset{t\to 0^+}{\sim} \quad \sum_{k\geq 0} \frac{(-t)^k}{k!}\, \mathrm{ev}\left[\, \Delta^k\psi \,\right] \\
&\underset{t\to 0^+}{\sim} \quad \int_T \frac{1}{\sqrt{4\pi t}^{\dim T}}\, e^{-\frac{|X|^2}{4t}}\, j^{-\frac{1}{2}}(X) \sum_{d\geq 0} t^d\, a_d(X)\,\psi(X)\,\mathrm{vol}_g(X) \\
&\underset{t\to 0^+}{\sim} \quad \sum_{d\geq 0} t^d \int_T \frac{1}{\sqrt{4\pi t}^{\dim T}}\, e^{-\frac{|X|^2}{4t}}\, (\, j^{\frac{1}{2}}(X)\, a_d(X)\,\psi(X)\,)\,\mathrm{vol}(X) \\
&\underset{t\to 0^+}{\sim} \quad \sum_{d\geq 0} t^d \sum_{l\geq 0} \frac{(-t)^l}{l!}\, \mathrm{ev}\Big[\, \Delta^l(\, j^{\frac{1}{2}}\, a_d\, \psi\,)\,\Big]
\end{aligned}$$

Note that the asymptotic expansion of the heat kernel k_t^Δ is locally uniform in $X \in T$ according to [BGV, Theorem 2.30] and hence we may integrate it over the compact support of ψ to obtain an asymptotic expansion of the integral. Existence of asymptotic expansions implies uniqueness and so we need only sort out the different powers of t to prove the Intertwining Property of the heat kernel coefficients in the form:

Theorem 2.1 (Intertwining Property of Heat Kernel Coefficients [W])
Let Δ be a generalized Laplacian acting on sections of the trivial vector bundle $E \times T$, such that the exponential map for its symbol metric g and the parallel transport along radial geodesics for its connection ∇ are the identities $T \longrightarrow T$ and $E \times \{0\} \longrightarrow E \times \{X\}$ respectively. The coefficients a_d, $d \geq 0$, of the asymptotic expansion of the heat kernel k_t^Δ intertwine the powers of Δ and Δ in the following sense:

$$\frac{(-1)^k}{k!} \operatorname{ev}\left[\, \Delta^k \psi \,\right] \quad = \quad \sum_{l=0}^{k} \frac{(-1)^l}{l!} \operatorname{ev}\left[\, \Delta^l \,(\, j^{\frac{1}{2}}\, a_{k-l}\, \psi \,)\, \right]$$

Philosophically the Intertwining Property 2.1 provides a precise geometric interpretation for the coefficients a_d, $d \geq 0$, of the heat kernel expansion in terms of Δ and the flat model operator Δ for generalized Laplacians. Similar considerations should apply to other model operators arising e. g. in Heisenberg calculus or in other parabolic calculi. A convenient reformulation of the intertwining property can be given with the help of the generating series $a(t) := \sum t^d a_d$ of the heat kernel coefficients a_d, $d \geq 0$, and the formal power series $e^{-t\Delta}$ and $e^{-t\Delta}$ of differential operators acting on sections of $E \times T$, namely the Intertwining Property is just another way to write down the equality $\operatorname{ev}[e^{-t\Delta}\psi] = \operatorname{ev}[e^{-t\Delta}(j^{\frac{1}{2}}a(t)\psi)]$ of formal power series in t. Of course only the infinite order jet of the functions $a_d \in C^\infty(T, \operatorname{End} E)$ in the origin can ever be sensed by evaluating formal power series of differential operators on a_d. Strikingly however it will turn out in Theorem 2.4 below that the infinite order jet of the generating series $a(t)$ in the origin is the unique solution to the Intertwining Property thought of as a set of equations in (the jets of) the unknowns a_d, $d \geq 0$.

At this point we digress a little bit on a very interesting property of the flat Laplacian Δ acting on functions $C^\infty T$ on a euclidian vector space T. More general we can consider the flat Laplacian Δ acting on sections $C^\infty(T, E)$ of the trivial E–bundle $E \times T$ over T, but the auxiliary vector space E never enters into the argument directly and so we will stick to the case $E = \mathbb{R}$ in order to simplify notation. Along with the flat Laplacian Δ comes the operator $|\cdot|^2$ of multiplication by the square of the distance to the origin, in orthogonal coordinates $\{x_\mu\}$ on T these two operators can be written:

$$\Delta \quad := \quad -\sum_\mu \frac{\partial^2}{\partial x_\mu^2} \qquad\qquad |\cdot|^2 \quad := \quad \sum_\mu x_\mu^2\,.$$

A simple calculation shows that the commutator $[\,\Delta, |\cdot|^2\,] = (-4)(N + \frac{\dim T}{2})$ of these two operators is essentially Euler's number operator $N = \sum x_\mu \frac{\partial}{\partial x_\mu}$ shifted by half the dimension of T. All three operators Δ, $|\cdot|^2$ and N restrict to operators on the subspace $\operatorname{Sym} T^* \subset C^\infty T$ of polynomials on T, moreover N acts as multiplication by k on $\operatorname{Sym}^k T^*$. Consequently the operators $X := \frac{1}{2}|\cdot|^2$ and $Y := \frac{1}{2}\Delta$ close with $H := N + \frac{\dim T}{2}$ to an algebra of operators on $\operatorname{Sym} T^*$ isomorphic to $\mathfrak{sl}_2$. Iterated commutators in $\mathfrak{sl}_2$–representations can quite generally be evaluated using the factorial polynomials $[\lambda]_k := \lambda(\lambda-1)\ldots(\lambda-k+1)$ or binomial coefficients $\binom{\lambda}{k} := \frac{1}{k!}[\lambda]_k$. The standard relation $Y^k X^k \psi = k![-\lambda]_k \psi$ in this context is valid for lowest weight vectors ψ of weight λ with $Y\psi = 0$ and $H\psi = \lambda\psi$ and becomes for the constant polynomial 1 the classical formula:

$$\Delta^k\,|\cdot|^{2k} \quad = \quad 4^k\, k!\, [\,-\tfrac{\dim T}{2}\,]_k \tag{2.10}$$

Slightly more useful for our purposes is the following derived identity:

Lemma 2.2 *For all smooth functions $\psi \in C^\infty T$ and all $k,\, l \,\geq\, 0$:*

$$\mathrm{ev}\left[\,\frac{(-1)^{k+l}}{(k+l)!}\,\Delta^{k+l}\,(\,\frac{1}{k!}\,|\cdot|^{2k}\,\psi\,)\,\right] \quad=\quad (-4)^k\begin{pmatrix}-\frac{\dim T}{2}-l\\ k\end{pmatrix}\,\mathrm{ev}\left[\,\frac{(-1)^l}{l!}\Delta^l\,\psi\,\right]$$

Proof: Only a finite number of partial derivates of ψ in the origin 0 are actually involved in this identity and hence we may assume that ψ is a polynomial without loss of generality. Moreover only the homogeneous component of ψ of degree $2l$ contributes to left and right hand side, which are both evidently $\mathbf{SO}\,T$–invariant linear functionals in $\psi \in \mathrm{Sym}^{2l}T^*$. However there is but one $\mathbf{SO}\,T$–invariant linear functional on $\mathrm{Sym}^{2l}T^*$ up to scale, so that it is sufficient to check the identity in question, which can be rewritten as

$$\mathrm{ev}\,[\,\Delta^{k+l}\,(\,|\cdot|^{2k}\,\psi\,)\,] \quad=\quad \frac{4^{k+l}\,(k+l)!\,[-\frac{\dim T}{2}]_{k+l}}{4^l\quad l!\quad[-\frac{\dim T}{2}]_l}\,\mathrm{ev}\,[\,\Delta^l\,\psi\,]$$

on the single polynomial $\psi := |\cdot|^{2l}$, for which it is true by the classical formula (2.10). $\square$

Returning to the general case of a Laplacian Δ acting on sections of the trivial vector bundle $E\times T$ we recall that the generating series $a(t)$ for the coefficients in the asymptotic expansion of the heat kernel intertwines the formal power series $e^{-t\Delta}$ and $e^{-t\Delta}$ of differential operators. Using this Intertwining Property 2.1 together with Lemma 2.2 we calculate:

$$\begin{aligned}\mathrm{ev}\left[\,\frac{(-1)^{d+k}}{(d+k)!}\,\Delta^{d+k}\,(\,\frac{1}{k!}\,|\cdot|^{2k}\,\psi\,)\,\right] &= \sum_{l=0}^{d+k}\mathrm{ev}\left[\,\frac{(-1)^l}{l!}\,\Delta^l\,(\,\frac{1}{k!}\,|\cdot|^{2k}\,j^{\frac12}\,a_{d+k-l}\,\psi\,)\,\right]\\ &= \sum_{l=0}^{d}\mathrm{ev}\left[\,\frac{(-1)^{k+l}}{(k+l)!}\,\Delta^{k+l}\,(\,\frac{1}{k!}\,|\cdot|^{2k}\,j^{\frac12}\,a_{d-l}\,\psi\,)\,\right]\\ &= (-4)^k\sum_{l=0}^{d}\begin{pmatrix}-\frac{\dim T}{2}-l\\ k\end{pmatrix}\mathrm{ev}\left[\,\frac{(-1)^l}{l!}\Delta^l\,(\,j^{\frac12}\,a_{d-l}\,\psi\,)\,\right]\end{aligned}$$

Combining this equation with the binomial inversion formula

$$\sum_{k=0}^{r}\begin{pmatrix}r+\frac{\dim T}{2}\\ r-k\end{pmatrix}\begin{pmatrix}-\frac{\dim T}{2}-l\\ k\end{pmatrix} \quad=\quad \begin{pmatrix}r-l\\ r\end{pmatrix} \quad=\quad \delta_{l,\,0}$$

valid as soon as $r\geq l\geq 0$ we eventually arrive at the following inversion formula for the heat kernel coefficients a_d, $d\geq 0$, of a generalized Laplacian Δ:

Theorem 2.3 (Polterovich's Inversion Formula [Po])
Consider a generalized Laplacian Δ acting on sections of a trivial vector bundle $E\times T$ over a euclidian vector space T such that the exponential map for the symbol metric is the identity $T\longrightarrow T$ while parallel transport along radial geodesics is the natural identification $E\times\{0\}\longrightarrow E\times\{X\}$. Thinking of the distance $|\cdot|$ from the origin as a function on T

and noting $j^{\frac{1}{2}}(0) = 1$ we can compute the action of the endomorphism $a_d(0) \in \operatorname{End} E$ on $\psi(0) \in E$ extended arbitrarily to a section $\psi \in C^\infty(T, E)$ by means of the inversion formula:

$$\operatorname{ev}[\, a_d\,\psi\,] \quad = \quad \sum_{k=0}^{r} (\,-\frac{1}{4}\,)^k \binom{r + \frac{\dim T}{2}}{r-k} \operatorname{ev}\Big[\, \frac{(-1)^{d+k}}{(d+k)!}\, \Delta^{d+k}\,(\,\frac{1}{k!}\,|\cdot|^{2k}\,\psi\,)\,\Big] \qquad\qquad r \geq d$$

Our normalization of the coefficients a_d, $d \geq 0$, drops the factor $(4\pi)^{-\frac{\dim T}{2}}$ arising from the value of the euclidian heat kernel for the flat Laplacian Δ on T at the origin to have the Intertwining Property 2.1 in as simple a form as possible. In the original formulation of Polterovich this factor is part of a_d and of course in every conceivable application this factor has to be reinserted by hand.

In conclusion the Intertwining Property 2.1 alone is sufficient to determine the value of the coefficients a_d, $d \geq 0$, of the heat kernel expansion at the origin. Before we proceed to show that in fact the jets of infinite order of the coefficients a_d in the origin are determined by the Intertwining Property we want to make a few general remarks concerning our guiding philosphy in the calculations to come. We want to avoid formulas involving compositions of differential operators, because such formulas are hardly if ever of any use in explicit calculations. In favourable situations it may still be possible to calculate the values $\operatorname{ev}\mathcal{D}^k$, $k \geq 1$, of powers of a differential operator $\mathcal{D}$ in the origin without knowing the partial derivatives of their coefficients. Note that the usual arguments of symbol calculus become meaningless if we have no control over the partial derivatives of the differential operators in a singular point like the origin.

In general the value of a differential operator $\mathcal{D}$ acting on sections of a vector bundle EM at a point y will be an element of $\operatorname{Hom}(\,\operatorname{Jet}^\infty_y EM, E_yM)$. For a trivial bundle $E \times T$ over a vector space T however we may identify the $\operatorname{Jet}^\infty_0(E \times T)$ with the formal power series completion of $\operatorname{Sym} T^* \otimes E$ and hence $\operatorname{Hom}(\,\operatorname{Jet}^\infty_0(E \times T), E\,)$ with $\operatorname{Sym} T \otimes \operatorname{End} E$ using Taylor's theorem in the usual way. The scalar product of the euclidian vector space T extends to a scalar product on $\operatorname{Sym} T^*$ defined via Gram's permanent and characterized by $\langle e^\alpha, e^\beta\rangle_{\operatorname{Sym} T^*} = e^{\langle \alpha, \beta\rangle}$ for all α, $\beta \in T^*$ with a slight abuse of notation. Alternatively we may choose orthonormal coordinates $\{x_\mu\}$ on T and write down the scalar product directly

$$\begin{aligned}\langle \psi, \tilde{\psi}\rangle_{\operatorname{Sym} T^*} \quad &:= \quad \sum_{r \geq 0} \frac{1}{r!} \sum_{\mu_1,\ldots,\mu_r} \operatorname{ev}\Big[\, \frac{\partial^r}{\partial x_{\mu_1}\ldots \partial x_{\mu_r}} \psi\,\Big] \operatorname{ev}\Big[\, \frac{\partial^r}{\partial x_{\mu_1}\ldots \partial x_{\mu_r}} \tilde{\psi}\,\Big] \\ &= \quad \big(\,(\operatorname{ev} \otimes \operatorname{ev}) \circ e^{\langle \nabla, \nabla\rangle}\,\big)\;(\,\psi \otimes \tilde{\psi}\,)\end{aligned}$$

where $\langle \nabla, \nabla\rangle$ is the bidifferential operator $\psi \otimes \tilde{\psi} \longmapsto \sum_\mu \frac{\partial}{\partial x_\mu}\psi \otimes \frac{\partial}{\partial x_\mu}\tilde{\psi}$. Written in this form it is clear that the scalar product extends to the formal power series completion of $\operatorname{Sym} T^*$ or even to smooth functions provided that the defining sum converges. The musical isomorphism $\sharp : \operatorname{Sym} T \longrightarrow \operatorname{Sym} T^*$ associated to this scalar product is the algebra homomorphism extending the musical isomorphism of T and extends to the formal power series completion of $\operatorname{Sym} T$, too. Together with the evaluation map for differential operators it provides us with a unique polynomial $\operatorname{ev}[\mathcal{D}]^\sharp$ in $\operatorname{Sym} T^* \otimes \operatorname{End} E$ satisfying $\operatorname{ev}[\mathcal{D}\psi] = \langle\, \operatorname{ev}[\mathcal{D}]^\sharp, \psi\,\rangle_{\operatorname{Sym} T^*}$ for every section $\psi \in C^\infty(T, E)$ of $E \times T$. In the formulas below the scalar product $\langle\,,\rangle$

on $\mathrm{Sym}\,T^*$ will enter through the bidifferential operator $\langle\nabla,\nabla\rangle$, which in turn makes its appearance via Green's identity for the Laplacian Δ and the multiplication map m reading

$$(\,\Delta\circ m\,)\,(\,\psi\otimes\tilde{\psi}\,)\quad=\quad(\,m\circ(\,\Delta\otimes\mathrm{id}\;-\;2\langle\nabla,\nabla\rangle\;+\;\mathrm{id}\otimes\Delta\,)\,)\,(\,\psi\otimes\tilde{\psi}\,)$$

which features three commuting operators $\Delta\otimes\mathrm{id}$, $\mathrm{id}\otimes\Delta$ and $\langle\nabla,\nabla\rangle$. Hence we are free to put these operators in arbitrary order upon exponentiation. With this in mind we find

$$\begin{aligned}\mathrm{ev}\Big[\,e^{-t\Delta}\,(\,j^{\frac{1}{2}}\,a(t)\,\psi\,)\,\Big]\;&=\;(\,\mathrm{ev}\,\circ\,e^{-t\Delta}\,\circ\,m\,)\,(\,j^{\frac{1}{2}}\,a(t)\otimes\psi\,)\\&=\;\Big(\,(\mathrm{ev}\otimes\mathrm{ev})\,\circ\,e^{2t\langle\nabla,\nabla\rangle}\,\Big)\;(\,e^{-t\Delta}(\,j^{\frac{1}{2}}\,a(t)\,)\otimes e^{-t\Delta}\psi\,)\\&=\;\langle\;(2t)^N\;e^{-t\Delta}\,(\,j^{\frac{1}{2}}\,a(t)\,),\;e^{-t\Delta}\,\psi\,\rangle_{\mathrm{Sym}\,T^*}\\&=\;\langle\;e^{t|\cdot|^2}\,(2t)^N\;e^{-t\Delta}\,(\,j^{\frac{1}{2}}\,a(t)\,),\;\psi\,\rangle_{\mathrm{Sym}\,T^*}\end{aligned}$$

because the operator $e^{t|\cdot|^2}$ is adjoint to $e^{-t\Delta}$ with respect to the scalar product $\langle\,,\rangle$. On the other hand the very definition of the symbol map $\sharp$ for differential operators reads $\mathrm{ev}\left[\,e^{-t\Delta}\psi\,\right]\;=\;\langle\,\mathrm{ev}[\,e^{-t\Delta}\,]^\sharp,\,\psi\,\rangle_{\mathrm{Sym}\,T^*}$ and as ψ can be chosen arbitrarily we conclude:

$$\mathrm{ev}[\,e^{-t\Delta}\,]^\sharp\quad=\quad e^{t|\cdot|^2}\;(2t)^N\;e^{-t\Delta}(\,j^{\frac{1}{2}}a(t)\,)\tag{2.11}$$

Evidently the operators $e^{t|\cdot|^2}$ and $e^{-t\Delta}$ are invertible with inverses $e^{-t|\cdot|^2}$ and $e^{t\Delta}$ respectively, so that we can solve equation (2.11) for the infinite order jet of $j^{\frac{1}{2}}a(t)$ in the origin:

Theorem 2.4 (Explicit Formula for Heat Kernel Coefficients [W])
The infinite order $\mathrm{jet}(\,j^{\frac{1}{2}}\,a(t)\,)\;\in\;\mathrm{Sym}\,T^*M\otimes\mathrm{End}\,EM[\![\,t\,]\!]$ *of the generating series* $a(t)$ *for the heat kernel coefficients of a generalized Laplacian* Δ *acting on sections of a vector bundle* EM *over a manifold* M *is characterized by its Intertwining Property 2.1. More precisely the jet of any solution* $a(t)$ *to the equation* $\mathrm{ev}\left[e^{-t\Delta}\psi\right]\;=\;\mathrm{ev}\left[e^{-t\Delta}(j^{\frac{1}{2}}a(t)\psi)\right]$ *is given by:*

$$\mathrm{jet}\;j^{\frac{1}{2}}a(t)\quad=\quad e^{t\Delta}\;(2t)^{-N}\;e^{-t|\cdot|^2}\;\mathrm{ev}[\,e^{-t\Delta}\,]^\sharp$$

Needless to say the polynomial $e^{-t|\cdot|^2}$ is the total symbol of the differential operator $e^{t\Delta}$. Thus it is possible to replace $e^{-t|\cdot|^2}\,\mathrm{ev}[\,e^{-t\Delta}\,]^\sharp$ by $\mathrm{ev}[\,e^{-t\Delta}e^{t\Delta}\,]^\sharp$ but not by $\mathrm{ev}[\,e^{t\Delta}e^{-t\Delta}\,]^\sharp$, because the latter would involve partial derivatives of the coefficients of $e^{-t\Delta}$ in the origin. Nevertheless we cheated a little bit, because while certainly injective the operator $(2t)^N$ is definitely not surjective. It is quite remarkable in itself that the value of the differential operator $e^{-t\Delta}e^{t\Delta}$ in the origin lies in the image of $(2t)^N$ by equation (2.11), because its coefficients for the different powers t^k, $k\geq 0$, of t must be differential operators of order less than or equal to k in the origin to have this true!

Coming to a full circle one can show by direct if tedious calculation that the unique solution to the Intertwining Property 2.1 derived in Theorem 2.4 satisfies the recursion formula (2.8) of Minakshisundaram–Pleijel we started with. It was in fact this very calculation which made us realize a couple of interesting subtleties of this recursion formula. For example the operator $j^{\frac{1}{2}}\Delta j^{-\frac{1}{2}}$ is formally selfadjoint with respect to the flat volume form provided Δ is formally selfadjoint and thus points out fascinating symmetries in the rather complicated explicit coefficients of Δ in the exponential trivialization we used. Perhaps more important is the point that in the recursion formula (2.8) the coefficients of $j^{\frac{1}{2}}\Delta j^{-\frac{1}{2}}$ are to act by multiplication with their adjoint from the right on the endomorphism valued function a. We will provide a direct argument for this behaviour at the time we solve the recursion formula up to terms of lower order in Section 6 in order to prove the Local Index Theorem.

Chapter 3

Iterated Covariant Derivatives and Taylor Series

Iterated covariant derivatives ∇^k are differential operators of order $k \geq 0$ acting on sections of a vector bundle EM endowed with a connection ∇ over an affine manifold M. Understanding these differential operators in detail is of particular importance in differential geometry. Considered as a model of the jet operator however the iterated covariant derivatives ∇^k contain redundant information and so people are inclined to take their symmetrizations in order to get a model of jet theory suitable for introducing concepts from representation theory of the holonomy groups Hol^E and Hol of the connection on EM and the affine connection on TM. On the other hand it is known that precisely this redundant information comes in myriad seemingly unrelated curvature terms which make prominent appearance in Weitzenböck type formulas. In this section we will define iterated covariant derivatives and prove the special Jet Formula 3.1 which provides a covariant interpretation for the Taylor series of the connection form ω^E and the backward parallel transport Φ^{-1} in the origin of normal coordinates. A more powerful generalization of this formula taming all the different curvature terms is formulated and proved in the next section.

Recall that an affine manifold M is a manifold endowed with a connection ∇ on its tangent bundle TM. Consider a vector bundle EM over an affine manifold M endowed with its own connection for convenience again denoted by ∇. The iterated covariant derivatives ∇^k, $k \geq 0$, are defined as differential operators of order k from sections of EM to $\bigotimes^k T^*M \otimes EM$ via:

$$\Gamma(EM) \ \xrightarrow{\nabla}\ \Gamma(T^*M \otimes EM) \ \xrightarrow{\nabla}\ \Gamma(\textstyle\bigotimes^2 T^*M \otimes EM) \ \xrightarrow{\nabla}\ \ldots \ \xrightarrow{\nabla}\ \Gamma(\textstyle\bigotimes^k T^*M \otimes EM)$$

Alternatively we may set $\nabla^0\psi := \psi$ and define ∇^k recursively for all $k \geq 1$ by

$$\nabla^k_{X_1,\ldots,X_k}\psi \ := \ \nabla_{X_1}(\,\nabla^{k-1}_{X_2,\ldots,X_k}\psi\,) \ - \ \sum_{\mu=2}^{k} \nabla^{k-1}_{X_2,\ldots,\nabla_{X_1}X_\mu,\ldots,X_k}\psi$$

for all vector fields $X_1, \ldots, X_k$. A formal consequence of either definition is the coassociativity identity $\nabla^{k+l} = \nabla^k \circ \nabla^l$ of differential operators acting on sections of EM with values in $\bigotimes^{k+l} T^*M \otimes EM$. The iterated covariant derivatives satisfy Leibniz' rule

$$\nabla^k_{X_1,\ldots,X_k}(B(\psi_1 \otimes \psi_2)) \;=\; \sum_{r=0}^{k} \sum_{\mu_1<\ldots<\mu_r} B(\,\nabla^r_{X_{\mu_1},\ldots,X_{\mu_r}}\psi_1 \otimes \nabla^{k-r}_{X_1,\ldots,\widehat{X}_{\mu_1},\ldots,\widehat{X}_{\mu_r},\ldots,X_k}\psi_2\,)$$

for all parallel bilinear maps $B:\; E_1M \otimes E_2M \longrightarrow EM$ and its generalizations for multilinear maps. Coassociativity and Leibniz rule conspire in the fundamental curvature identity

$$\begin{aligned}
&\nabla^{k+l+2}_{X_1,\ldots,X_k,A,B,Y_1,\ldots,Y_l} - \nabla^{k+l+2}_{X_1,\ldots,X_k,B,A,Y_1,\ldots,Y_l} \\
&\quad= \sum_{r=0}^{k}\sum_{\mu_1<\ldots<\mu_r}\Big((\,\nabla^r_{X_{\mu_1},\ldots,X_{\mu_r}}R^E\,)_{A,B}\,\nabla^{k+l-r}_{X_1,\ldots,\widehat{X}_{\mu_1},\ldots,\widehat{X}_{\mu_r},\ldots,X_k,Y_1,\ldots,Y_l} \\
&\qquad\qquad - \sum_{\nu=1}^{l}\nabla^{k+l-r}_{X_1,\ldots,\widehat{X}_{\mu_1},\ldots,\widehat{X}_{\mu_r},\ldots,X_k,Y_1,\ldots,(\,\nabla^r_{X_{\mu_1},\ldots,X_{\mu_r}}R\,)_{A,B}Y_\nu,\ldots,Y_l}\Big)
\end{aligned} \tag{3.1}$$

whenever the connection on TM is torsion free, where R^E and R are the curvature tensors of the connections on EM and TM respectively. Without the assumption of torsion freeness additional terms depending on the torsion need to be added of course, otherwise there is no essential difference in the arguments presented below. Although we will only make very limited use of this curvature identity it seems to be the key to a better understanding of all the calculations in this and the next Section 4.

In order to keep in line with our general philosophy and to simplify notation we will essentially reduce to the case where the affine manifold T is a vector space and the vector bundle $E \times T$ is trivial such that the exponential map for the affine connection on the tangent bundle of T in the origin $\exp_0:\; T_0T \cong T \longrightarrow T$ is the identity while the parallel transport along radial geodesics $t \longmapsto tX$, $X \in T$, for the connection on $E \times T$ is given by the identity $E \times \{0\} \longrightarrow E \times \{tX\}$. Fixing a point y in a manifold M we can always reduce locally to this case by choosing $T := T_yM$ and $E := E_yM$ while pulling back all the geometric data to the tangent space via the exponential map in y. Of course the geometric data may cease to be well–defined off a neighborhood of the origin, for simplicity we will nevertheless assume it is.

Reducing to this special case amounts to fixing a gauge for the action of the diffeomorphism group of M and the group of automorphisms of the vector bundle EM. Therefore it shouldn't be too surprising that this setup allows for special constructions, which are not evidently covariant, although we will see in a moment that they have a neat covariant interpretation. In particular we can define the forward and backward parallel transport

$$\Phi(\,X\,):\quad T_0T \cong T \longrightarrow T_XT \cong T \qquad\qquad \Phi^{-1}(\,X\,):\quad T_XT \cong T \longrightarrow T_0T \cong T$$

as functions on T with values in $\operatorname{End} T$ via parallel transport of tangent vectors along the radial geodesic $t \longmapsto tX$ through $X \in T$. Moreover we can define the connection form ω^E for the connection ∇ on $E \times T$ as a function on T with values in $T^* \otimes \operatorname{End} E$ by

$$(\,\nabla_Y\psi\,)(\,X\,) \;=\; (\,\frac{\partial}{\partial Y}\psi\,)(\,X\,) \;+\; \omega^E(\,X\,)_Y\;\psi(\,X\,)$$

for every section ψ and every constant vector field Y. The reason we restrict to constant vector fields here is that the notation $\frac{\partial}{\partial Y}$ tends to become ambiguous for general vector fields in expressions like $\frac{\partial^2}{\partial Y \partial \tilde{Y}}$. We will try to get away using only constant vector fields as the alternative of introducing a special notation for the trivial connection on $E \times T$ seems worse.

In general the connection form ω^E of some connection ∇ on $E \times T$ takes values in the holonomy algebra $\mathfrak{hol}^E \subset \mathrm{End}\, E$ in the origin whenever parallel transport for ∇ is the identity along all radial geodesics $t \longmapsto tX$. Omitting the details of this argument we add that in fact the holonomy algebra $\mathfrak{hol}^E$ of a connection ω^E defined on a star–shaped neighborhood of the origin is generated by the values of ω^E in this neighhorhood. For an analytic connection these statements are easily verified by looking at the infinite order Taylor series of ω^E in the origin given explicitly below.

In case the vector bundle is the tangent bundle $T \times T$ the trivialization given by parallel transport along radial geodesics clearly differs from the standard trivialization of the tangent bundle of T and it is exactly this difference which is reflected in the parallel transport Φ. Instead of defining a connection form ω with values in the holonomy algebra $\mathfrak{hol}$ of the affine connection it is more natural to use the standard trivialization in the case of the tangent bundle of T leading to the Christoffel symbols $\Gamma \in C^\infty(\,T,\, T^* \otimes \mathrm{End}\, T\,)$ characterized by:

$$(\,\nabla_Y Z\,)(\,X\,) \;\;=\;\; (\,\frac{\partial}{\partial Y} Z\,)(\,X\,) \;+\; \Gamma(\,X\,)_Y\; Z(\,X\,)$$

Relating the trivialization given by parallel transport along radial geodesics to the standard trivialization via backward parallel transport Φ^{-1} one easily derives the following relation

$$\Gamma(\,X\,)_Y \;\;=\;\; \Phi(\,X\,)\,(\,\frac{\partial}{\partial Y}\Phi^{-1}\,)(\,X\,) \;+\; \Phi(\,X\,)\,\omega(\,X\,)_Y\,\Phi^{-1}(\,X\,)$$

which allows us to find the Taylor series of Γ in the origin once we know the Taylor series of the Levi–Civita connection ω and the parallel transport Φ^{-1}. Let us start calculating these Taylor series with a genuinely covariant interpretation of Φ^{-1} and the connection form ω^E:

Lemma 3.1 (Special Jet Formula)
Let M be an affine manifold with a torsion free connection ∇ and EM a vector bundle over M endowed with a connection again denoted by ∇ so that the iterated covariant derivatives ∇^k are defined. There exist unique formal power series $\Phi^{-1} \in \Gamma(\mathrm{Sym}\, T^*M \otimes \mathrm{End}\, TM)$ and $\omega^E \in \Gamma(\mathrm{Sym}\, T^*M \otimes T^*M \otimes \mathrm{End}\, EM)$ such that for all vector fields X, Y we have an equality

$$\frac{d}{dt}\Big|_0 \sum_{k\geq 0} \frac{1}{k!}\nabla^k_{X+tY,\ldots,X+tY} \;\;=\;\; \sum_{k\geq 0}\frac{1}{k!}\nabla^{k+1}_{X,X,\ldots,X,\Phi^{-1}(X)Y} \;-\; \omega^E(X)_Y \sum_{k\geq 0}\frac{1}{k!}\nabla^k_{X,\ldots,X}$$

between formal power series of differential operators on sections of EM. Moreover the values of Φ^{-1} and ω^E at a point $y \in M$ are the Taylor series of the backward parallel transport $\Phi^{-1}(X) :\; T_XT_yM \longrightarrow T_0T_yM$ and the connection form ω^E for the trivialization of EM given by parallel transport along radial geodesics in normal coordinates centered about y.

Proof: Of course the geometric interpretation proves existence and uniqueness of these power series once we have shown that the Taylor series of Φ^{-1} and ω^E make this identity true. For this purpose we pull back the geometric data to a neighborhood of $0 \in T_yM$ via the exponential map and thus reduce to the special case of a non–trivial connection on a trivial vector bundle $E_yM \times T_yM$ over the vector space T_yM discussed before. With parallel transport along radial geodesics being the identity we have for every section ψ of $E_yM \times T_yM$

$$\frac{d}{dt}\,\psi(\,tX\,) \;\;=\;\; (\,\nabla_X\psi\,)(\,tX\,)$$

along the radial geodesic $t \longmapsto tX$. Continuing this way we find an asymptotic expansion

$$\psi(X) \underset{X\to 0}{\sim} \sum_{k\geq 0}\frac{1}{k!}\,\frac{d^k}{dt^k}\Big|_0\,\psi(\,tX\,) \;\;=\;\; \sum_{k\geq 0}\frac{1}{k!}\,(\,\nabla^k_{X,\ldots,X}\psi\,)(\,0\,) \tag{3.2}$$

using that $t \longmapsto tX$ is a geodesic curve. In the same vein the forward parallel transport $\Phi(X):\; T_0T_yM \longrightarrow T_XT_yM$ allows us to associate a vector field $\Phi(X)Y$ to every $Y \in T_yM$ which is parallel along every radial geodesic $t \longmapsto tX$. Arguing as before we conclude:

$$(\,\nabla_{\Phi(X)Y}\psi\,)(\,X\,) \underset{X\to 0}{\sim} \sum_{k\geq 0}\frac{1}{k!}\,\frac{d^k}{dt^k}\Big|_0\,(\,\nabla_{\Phi(tX)Y}\psi\,)(\,tX\,) \;\;=\;\; \sum_{k\geq 0}\frac{1}{k!}\,(\,\nabla^{k+1}_{X,\ldots,X,Y}\psi\,)(\,0\,)$$

If we replace $\Phi(X)$ by its Taylor series and reinterprete this asymptotic expansion as a formal power series we can invert the series Φ and find the asymptotic expansion of $\nabla_Y\psi$ for the constant vector field on T_yM associated to Y:

$$(\,\nabla_Y\psi\,)(\,X\,) \underset{X\to 0}{\sim} \sum_{k\geq 0}\frac{1}{k!}\,(\,\nabla^{k+1}_{X,\ldots,X,\Phi^{-1}(X)Y}\psi\,)(\,0\,)$$

On the other hand we know that for constant vector fields we have the asymptotic expansion

$$\begin{aligned}(\,\nabla_Y\psi\,)(\,X\,) \;\;&=\;\; \frac{d}{dt}\Big|_0\,\psi(\,X+tY\,) \;+\; \omega^E(\,X\,)_Y\,\psi(\,X\,)\\ &\underset{X\to 0}{\sim}\;\; \frac{d}{dt}\Big|_0\,\sum_{k\geq 0}\frac{1}{k!}(\,\nabla^k_{X+tY,\ldots,X+tY}\psi\,)(\,0\,) \;+\; \omega^E(X)_Y\sum_{k\geq 0}\frac{1}{k!}(\,\nabla^k_{X,\ldots,X}\psi\,)(\,0\,)\end{aligned}$$

where ω^E has to be reinterpreted as a Taylor series in 0 in the second line. □

With this lemma at hand it is possible to calculate the Taylor series for Φ^{-1} and ω^E as well as for Φ to arbitrary order in a completely covariant fashion. In fact the simple calculation

$$\frac{d}{dt}\Big|_0\,\tfrac{1}{6}\,\nabla^3_{X+tY,X+tY,X+tY} \;\;=\;\; \tfrac{1}{2}\,\nabla^3_{X,X,Y} \;-\; \tfrac{1}{2}\,R^E_{X,Y}\nabla_X \;+\; \tfrac{1}{6}\,\nabla_{R_{X,Y}X} \;-\; \tfrac{1}{3}\,(\,\nabla_XR^E\,)_{X,Y}$$

already provides the Taylor expansions of Φ^{-1} and ω^E up to third order

$$\begin{aligned}\Phi^{-1}(\,X\,)Y \;\;&=\;\; Y \;+\; \tfrac{1}{6}\,R_{X,Y}X \;+\; O(\,X^3\,)\\ \omega^E(\,X\,)_Y \;\;&=\;\; \tfrac{1}{2}\,R^E_{X,Y} \;+\; \tfrac{1}{3}\,(\,\nabla_XR^E\,)_{X,Y} \;+\; O(\,X^3\,)\end{aligned}$$

with the classical consequences:

$$\begin{array}{rcccccl} j(X)Y &=& \det\Phi^{-1}(X) &=& 1 &-& \frac{1}{6}\,\mathrm{Ric}(X,X) + O(X^3) \\ g(X)(Y,Z) &=& \langle\Phi^{-1}(X)Y,\Phi^{-1}(X)Z\rangle &=& \langle Y,Z\rangle &+& \frac{1}{3}\langle R_{X,Y}X,Z\rangle + O(X^3) \end{array}$$

In order to derive the complete Taylor series for both Φ^{-1} and ω^E we generalize the term $\frac{d}{dt}\big|_0 \nabla^k_{X+tY,\ldots,X+tY}$ we are eventually interested in by looking at a telescope sum of the form

$$\sum_{\mu=0}^{k-1}\binom{\mu+K}{K}\nabla^k_{\underbrace{X,\ldots,X}_{\mu},Y,\underbrace{X,\ldots,X}_{k-1-\mu}} = \binom{k+K}{K+1}\nabla^k_{X,\ldots,X,Y} - \sum_{\mu=0}^{k-2}\binom{\mu+K+1}{K+1}\Big(\nabla^k_{\underbrace{X,\ldots,X}_{\mu},X,Y,\underbrace{X,\ldots,X}_{k-2-\mu}} - \nabla^k_{\underbrace{X,\ldots,X}_{\mu},Y,X,\underbrace{X,\ldots,X}_{k-2-\mu}}\Big)$$

for given $K \geq 0$. Using the general curvature identity (3.1) in order to expand the right hand side and sorting out terms we readily arrive at the key relation

$$\begin{aligned} \sum_{\mu=0}^{k-1}\binom{\mu+K}{K}\nabla^k_{\underbrace{X,\ldots,X}_{\mu},Y,\underbrace{X,\ldots,X}_{k-1-\mu}} \;=\;& \binom{k+K}{K+1}\nabla^k_{X,\ldots,X,Y} \\ & - \sum_{\nu=0}^{k-2}\binom{\nu+K+1}{\nu}\binom{k+K}{\nu+K+2}(\nabla^\nu_{X,\ldots,X}R^E)_{X,Y}\,\nabla^{k-2-\nu}_{X,\ldots,X} \\ & + \sum_{\nu=0}^{k-3}\binom{\nu+K+1}{\nu}\sum_{\rho=0}^{k-3-\nu}\binom{\rho+\nu+K+2}{\nu+K+2}\nabla^{k-2-\nu}_{\underbrace{X,\ldots,X}_{\rho},(\nabla^\nu_{X,\ldots,X}R)_{X,Y}X,\underbrace{X,\ldots,X}_{k-3-\nu-\rho}} \end{aligned}$$

valid for all $k,\,K \geq 0$. The disturbing third summand is of exactly the same form as the left hand side and hence we can keep on expanding this term recursively. Starting with $K = 0$ and turning this crank once or twice to see what happens the reader will have no difficulties to guess the Taylor series for Φ^{-1} and ω^E in the origin as formal power series in the curvature tensors R and R^E and their covariant derivatives. Rather surprisingly after this laborious calculation the resulting Taylor series turn out to be governed by remarkably simple differential equations:

Theorem 3.2 (Taylor Series of Backward Parallel Transport)
The infinite Taylor series of the backward parallel transport Φ^{-1} and the connection form ω^E in the origin of normal coordinates are the unique solutions of the differential equations [1]

$$N(N+1)\,\Phi^{-1} = \mathcal{R}\,\Phi^{-1} \qquad\qquad (N+1)\,\omega^E = \rho^E_{\Phi^{-1}}$$

subject to the initial condition $\Phi^{-1}(X) = \mathrm{id} + O(X^2)$ with parameters $\mathcal{R}$ and ρ^E given by:

$$\mathcal{R}(X)Y := \sum_{k\geq 0}\frac{1}{k!}(\nabla^k_{X,\ldots,X}R)_{X,Y}X \qquad\qquad \rho^E(X)_Y := \sum_{k\geq 0}\frac{1}{k!}(\nabla^k_{X,\ldots,X}R^E)_{X,Y}$$

[1] Theorem 3.2 has presumably appeared a couple of times in different formulations in the mathematical literature. In particular the author is aware of an article proving Theorem 3.2 in the language of differential forms, but could not track down this reference.

Intriguingly the proof of this theorem sketched above works backwards deducing the differential equations obeyed by Φ^{-1} and ω^E from the explicit form of their solutions in terms of formal power series. Of course this is a rather roundabout way to prove a result as important as this and it would be interesting to know whether there is a direct argument to the same end. In any case it is advantageous to use the differential equations instead of the Taylor series for Φ^{-1} and ω^E, not only is it easier to remember the differential equations, but it is sometimes possible to solve the equations explicitly in a different way. Say on a symmetric space the power series $\mathcal{R}$ is particularly simple $\mathcal{R}(X)Y = (\mathrm{ad}\,X)^2 Y$ suggesting an ansatz $\Phi^{-1}(X) = \phi^{-1}(\mathrm{ad}\,X)$ for Φ^{-1} with some power series $\phi^{-1}(z)$, which turns the differential equation above into the equation

$$z\,\frac{d}{dz}\,(\,z\,\frac{d}{dz}\;+\;1\,)\,\phi^{-1}(z) \;\;=\;\; z^2\,\phi^{-1}(z)$$

with unique solution $\phi^{-1}(z) = \frac{\sinh z}{z}$ subject to the initial condition $\phi^{-1}(z) = 1 + O(z^2)$. Similarly the power series $\rho^E(X)_Y = -[\,X,\,Y\,]\star$ encodes the infinitesimal representation $\star$ of the isotropy subalgebra on the fiber E of a homogeneous vector bundle on a symmetric space and under the ansatz $\omega^E(X)_Y = -w(\mathrm{ad}\,X)Y\star$ we get the differential equation

$$(\,z\,\frac{d}{dz}\;+\;1\,)\,w(z) \;\;=\;\; z\,\phi^{-1}(z) \;\;=\;\; \sinh z$$

with the unique solution $w(z) = \frac{\cosh z - 1}{z}$. In particular the backward parallel transport on a symmetric space $\Phi^{-1}(X) = \frac{\sinh \mathrm{ad}X}{\mathrm{ad}X}$ is symmetric with respect to the flat scalar product $\langle\,,\rangle$, whereas it will differ from its adjoint $\Phi^{-*}(X)$ by terms of rather unexpectedly high order

$$\Phi^{-*}(\,X\,) \;\;=\;\; \Phi^{-1}(\,X\,) \;+\; \tfrac{1}{360}\,[\,R_{X,\cdot}X,\;(\,\nabla_X R\,)_{X,\cdot}X\,] \;+\; O(\,X^6\,)$$

on a general Riemannian manifold M. The primary obstruction $[\,R_{X,\cdot}X,\,(\nabla_X R)_{X,\cdot}X\,]$ can be thought of as a section of $\mathrm{Sym}\,^5 T^*M \otimes \Lambda^2 TM$. What manifolds besides symmetric spaces have this section vanishing everywhere?

Remark 3.3 (Gauß Lemma)
A characteristic property of Riemannian geometry called the Gauß Lemma is that the exponential map is a radial isometry in the sense $g(X)(X,Y) = \langle X,Y\rangle$ for all X in its domain. Noting the identities $\Phi^{-1}(X)X = X$ and $\omega^E(X)_X = 0$, which express the gauge fixing conditions in terms of formal power series, we can rewrite the Gauß Lemma in the form:

$$\langle X,Y\rangle \;\;=\;\; g(\,X\,)(\,X,\,Y\,) \;\;=\;\; \langle X,\Phi^{-1}(\,X\,)Y\rangle$$

Eventually replacing Y by $\Phi(X)Y$ we find $\langle X,\Phi(X)Y\rangle = \langle X,Y\rangle$, which is readily verified without alluding to the Gauß Lemma using $\langle(\nabla^k_{X,\ldots,X}R)_{X,Z}X,X\rangle = 0$ for all $k \geq 0$.

Even without writing down the formal power series solution for the Taylor series of Φ^{-1} and ω^E we can derive additional information from the differential equations of Theorem 3.2. By definition we have say for the form ρ describing the affine connection on the tangent bundle the equality $\rho(X)_Y X = \mathcal{R}(X)Y$ for all $X, Y \in TM$ or more succinctly $\rho(\cdot)\cdot = \mathcal{R}$. With the differential equations for the connection form ω and Φ^{-1} we observe

$$N[\,\omega(\cdot)\cdot\,] \;=\; [\,(N+1)\omega\,](\cdot)\cdot \;=\; \mathcal{R}\,\Phi^{-1} \;=\; N\,(N+1)\,\Phi^{-1}$$

and thus find the identity $\omega(\cdot)\cdot = (N+1)\Phi^{-1} - \mathrm{id}$ valid to all orders without ever having to write down a Taylor series. We can make similar use of the differential equations to show that the homogeneous components of the Taylor series of Φ^{-1} and ω^E as polynomials in X are polynomials in R and R^E and their covariant derivatives homogeneous with respect to a suitable definition of weight. Of course this is not too surprising in itself simply because there are not that many ways to write down a polynomial in R and R^E and their covariant derivatives producing a vector or an endomorphism of E from a fixed number of arguments. This reasoning does not specify the weight for R^E and its covariant derivatives relative to the weight of R however, because the curvature of E will only enter ω^E linearly. Considerations done later in this article fix this problem in favor of the following definition:

Definition 3.4 (Weight of Curvature Tensors)
Define the weight operator W as a derivation on the space of formal power series in the curvature tensors R and R^E as well as their covariant derivatives by specifying the weights:

$$W\,\nabla^k R \;:=\; (k+2)\,\nabla^k R \qquad\qquad W\,\nabla^k R^E \;:=\; k\,\nabla^k R^E$$

Not only the power series Φ^{-1} and ω^E but all the important derived formal power series describing the geometry of the vector bundle EM over the affine manifold M have a nice relation between their weight W as a polynomial in the curvature tensors and their degree N as a polynomial on the tangent space TM. Besides the Levi–Civita connection ω and the Christoffel symbols Γ we will need in particular the power series Ω^E and $j^{\frac{1}{2}}$ defined by:

$$\Omega^E \;:=\; \omega^E_\Phi \qquad\qquad j^{\frac{1}{2}} \;:=\; \det{}^{\frac{1}{2}}\,\Phi^{-1} \;=\; \exp(\,\tfrac{1}{2}\,\mathrm{tr}\,\log\Phi^{-1}\,)$$

Noting the relations $W\mathcal{R} = N\mathcal{R}$ and $W\rho^E = (N-1)\rho^E$ as well as $W\rho = (N+1)\rho$ we find:

$$\begin{array}{lllllllll} W\,\Phi^{-1} & = & N\,\Phi^{-1} & W\,\omega^E & = & (N-1)\,\omega^E & W\,\Omega^E & = & (N-1)\,\Omega^E \\ W\,\Phi & = & N\,\Phi & W\,\omega & = & (N+1)\,\omega & W\,\Omega & = & (N+1)\,\Omega \\ W\,j^{\frac{1}{2}} & = & N\,j^{\frac{1}{2}} & W\,\Gamma & = & (N+1)\,\Gamma & & & \end{array} \tag{3.3}$$

Before closing this section we want to illustrate our considerations with an argument deriving a non–trivial consistency relation between the Taylor series for ω^E and Φ^{-1}. According to the central idea (3.2) of the special Jet Formula 3.1 the value of the symmetrized covariant derivatives $\nabla^k R^E$ at a point $y \in M$ are the components of the Taylor series of the curvature R^E of EM in exponential coordinates centered about y considered as a section of the bundle $\Lambda^2 T^*M \otimes \mathrm{End}\,EM$ trivialized by parallel transport along radial geodesics. On the other hand the very definition of curvature reads $d\omega^E + \omega^E \wedge \omega^E$ when considered as a 2–form in exponential coordinates with values in End EM. The two different trivializations of T^*M involved are related by backward parallel transport Φ^{-1} and so we must have the equality

$$\sum_{k\geq 0}\frac{1}{k!}\,(\,\nabla^k_{X,\ldots,X}R^E\,)_{\Phi^{-1}(X)Y,\Phi^{-1}(X)Z} \;=\; (\,d\omega^E \;+\; \omega^E\wedge\omega^E\,)(X)_{Y,Z} \tag{3.4}$$

between sections of $\mathrm{Sym}\, T^*M \otimes \Lambda^2 T^*M \otimes \mathrm{End}\, EM$. To second order in X it reads

$$\begin{aligned} R^E_{Y,Z} \;+\; (\nabla_X R^E)_{Y,Z} \;+\; \tfrac{1}{2}(\nabla^2_{X,X} R^E)_{Y,Z} \;+\; R^E_{\frac{1}{6}R_{X,Y}X,Z} \;+\; R^E_{Y,\frac{1}{6}R_{X,Z}X} \;+\; \ldots \\ = \; \Big(\tfrac{1}{2}R^E_{Y,Z} \;+\; \tfrac{1}{3}(\nabla_Y R^E)_{X,Z} \;+\; \tfrac{1}{3}(\nabla_X R^E)_{Y,Z} \;+\; \tfrac{1}{8}(\nabla^2_{Y,X} R^E)_{X,Z} \;+\; \tfrac{1}{8}(\nabla^2_{X,Y} R^E)_{X,Z} \\ +\; \tfrac{1}{8}(\nabla^2_{X,X} R^E)_{Y,Z} \;+\; \tfrac{1}{24}R^E_{Y,R_{X,Z}X} \;+\; \tfrac{1}{24}R^E_{X,R_{Y,Z}X} \;+\; \tfrac{1}{24}R^E_{X,R_{X,Z}Y}\Big) \\ -\Big(\text{ all terms repeated with } Y \leftrightarrow Z \Big) \;+\; \tfrac{1}{4}R^E_{X,Y}R^E_{X,Z} \;-\; \tfrac{1}{4}R^R_{X,Z}R^E_{X,Y} \;+\; \ldots \end{aligned}$$

which can hardly be called obvious. Actually this relation boils down to the first and second Bianchi identity for R and R^E respectively, if we take the implicit integrability condition

$$(\,\nabla^2_{Y,X} R^E\,)_{X,Z} \;\;=\;\; (\,\nabla^2_{X,Y} R^E\,)_{X,Z} \;-\; [\,R^E_{X,Y},\, R^E_{X,Z}\,] \;+\; R^E_{R_{X,Y}X,Z} \;+\; R^E_{X,R_{X,Y}Z}$$

for granted. Identities like the two identities above do come out of the blue, but are manifestations of rather stringent inner consistency conditions showing that the coefficients in the Taylor series for Φ^{-1} and ω^E are not at all random. Whatever else this example may be good for we hope that at least it will make the reader understand that trying to guess curvature identities without guiding principle is not really an option.

Changing the point of view it is possible to show that the Taylor series of ω^E is the unique solution to the consistency identity (3.4) subject to the initial condition $\omega^E(X) \,=\, O(X)$ and the constraint $\omega^E(X)_X \,=\, 0$, because on the space of solutions to the constraint the formal exterior derivative $d:\;\mathrm{Sym}\, T^*M \otimes T^*M \otimes \mathrm{End}\, EM \longrightarrow \mathrm{Sym}\, T^*M \otimes \Lambda^2 T^*M \otimes \mathrm{End}\, EM$ is injective by an elementary version of Hodge theory. Moreover it can be shown that the Taylor series for Φ^{-1} and ω^E are both uniquely determined by the consistency identity (3.4) provided we impose an additional constraint on Φ^{-1} equivalent to the vanishing of torsion. The natural context for formulating this peculiar characterization of the curvature is the calculus of difference elements. A detailed exposition of this calculus however is beyond the scope of this article.

Chapter 4

From Special to General Jet Formula

With the special Jet Formula 3.1 proved in the previous section determining all from itself the Taylor series of some interesting geometric objects like the connection form ω^E and the backward parallel transport Φ^{-1} in the origin of normal coordinates one wonders what use a general formula expressing the iterated covariant derivatives in terms of their symmetrizations could be. It turns out that such a formula indeed exists and can be proved by straightforward induction once the right data structure keeping track of all the different curvature terms is found. It turns out the right data structure is a jet forest, whose precise Definition 4.1 is the key point of this section. The inductive nature of the set of all jet forests corresponds directly to the induction step in the proof of the general Jet Formula 4.2. Besides other applications the general Jet Formula will allow us to give a closed formula for all powers of the class of twisted generalized Laplacians, which can then be used to calculate the heat kernel coefficients of all generalized Laplacians explicitly.

Before we start it is perhaps appropriate to make a few general remarks about the identification of Taylor series of functions and symmetric multilinear forms in $\mathrm{Sym}\, T^*$. Associated to a function or formal power series F like Ω or Φ considered above is a sequence of symmetric multilinear forms $F_k \in \mathrm{Sym}^k T^*$, $k \geq 0$, defined as the iterated directional derivatives

$$F_k(\, X_1 \cdot X_2 \cdot \,\ldots\, \cdot X_k \,) \;\; := \;\; \left(\frac{\partial^k}{\partial X_1\, \partial X_2\, \ldots\, \partial X_k} \, F \right)(0) \tag{4.1}$$

in such a way that Taylor's formula reads $F(X) = \sum_{k\geq 0} \frac{1}{k!} F_k(X \cdot X \cdot \ldots \cdot X)$. Conversely we will take this formula as the definition of the formal power series F associated to a sequence $F_k \in \mathrm{Sym}^k T^*$, $k \geq 0$, of symmetric multilinear forms or in turn to a function F. The advantage of this convention is that the definition of the shuffle product on $\mathrm{Sym}\, T^*$ turns verbatim into the classical Leibniz formula for iterated derivatives of products, whereas the only advantage of an alternative convention seen quite often in literature is that it prevents people using it from doing any more sophisticated calculations. Usually we will simplify notation and write $F(X_1 \cdot \ldots \cdot X_k)$ instead of $F_k(X_1 \cdot \ldots \cdot X_k)$. Although the reader should be aware that the expression $F(X)$ becomes ambiguous this way refering alternatively to $F_1(X)$, this ambiguity is of little practical importance.

In order to illustrate the identification of formal power series with sequences of symmetric multilinear forms we consider the sequence of symmetrized iterated covariant derivatives

$\operatorname{jet}^k\psi(X_1\cdot\ldots\cdot X_k) := \frac{1}{k!}\sum_{\sigma\in S_k}\nabla^k_{X_{\sigma 1},\ldots,X_{\sigma k}}\psi$ for $k\geq 0$, whose associated formal power series $\operatorname{jet}\psi \in \Gamma(\operatorname{Sym}T^*M\otimes EM)$ will be called the infinite order jet of ψ. Sometimes we will use the typographical variation $\operatorname{jet}^k_{X_1\cdot\ldots\cdot X_k}\psi$ in notation. According to the central argument (3.2) in the proof of the special Jet Formula 3.1 the value of $\operatorname{jet}\psi$ in a point $y\in M$ is the infinite order Taylor series of the section ψ in the origin of normal coordinates centered about y with the bundle EM trivialized by parallel transport along radial geodesics, because

$$\psi(X) \underset{X\to 0}{\sim} \operatorname{jet}\psi(\,X\,) \;:=\; \sum_{k\geq 0}\frac{1}{k!}\nabla^k_{X,\ldots,X}\psi$$

for all vector fields X by the preceeding remarks about symmetric forms. In consequence the differential operator $\operatorname{jet}:\ \Gamma(EM)\longrightarrow\Gamma(\operatorname{Sym}T^*M\otimes EM)$ is a model for the jet operator and enjoys its universal property. In particular it is possible to express the iterated covariant derivatives $\nabla^k\psi$ in terms of their symmetrizations $\operatorname{jet}^r\psi$, $r=0,\ldots,k$, alone. The general Jet Formula 4.2 will make precisely this reconstruction principle explicit.

The key step in the proof of the general Jet Formula is the special Jet Formula 3.1 proved in the previous section. Interpreted the right way it expresses the covariant derivatives $\nabla^{k+1}\psi$ symmetrized over all but the last entry in terms of their complete symmetrizations $\operatorname{jet}^r\psi$, $r=0,\ldots,k+1$. In order to see this more clearly we cast the special Jet Formula

$$\Big(\frac{\partial}{\partial Y}\operatorname{jet}\psi\Big)(\,X\,) \;=\; [\,(\operatorname{jet}\nabla\psi\,)(\,X\,)\,]_{\Phi^{-1}(X)Y} \;-\; \omega^E(\,X\,)_Y\;(\operatorname{jet}\psi\,)(\,X\,)$$

into jet notation, replace Y by $\Phi(\,X\,)Y$ and ω^E by the formal power series $\Omega^E := \omega^E_\Phi$. Now $\Phi(X)Y = Y - \frac{1}{6}R_{X,Y}X + O(X^3)$ agrees with the identity up to at least quadratic terms while $\Omega^E(X)_Y = \frac{1}{2}R^E_{X,Y} + O(X^2)$ has no constant term so that we eventually arrive at:

$$\begin{aligned}\nabla^{k+1}_{X,\ldots,X,Y}\psi \;=\; [\operatorname{jet}^k_{X\cdot\ldots\cdot X}\nabla\psi\,]_Y \;=\; \operatorname{jet}^{k+1}_{X\cdot\ldots\cdot X\cdot Y}\psi \;&+\; \sum_{r=2}^{k}\binom{k}{r}\operatorname{jet}^{k+1-r}_{X\cdot\ldots\cdot X\cdot\Phi_r(X\cdot\ldots\cdot X)Y}\psi \\ &+\; \sum_{r=1}^{k}\binom{k}{r}\,\Omega^E_r(X\cdot\ldots\cdot X)_Y\operatorname{jet}^{k-r}_{X\cdot\ldots\cdot X}\psi\end{aligned}\qquad(4.2)$$

In a sense the general Jet Formula is the "functorial" extension of this version (4.2) of the special Jet Formula 3.1. Book keeping however is a feat in itself in the process and to simplify this task we need to introduce jet forests. Recall that a tree is a connected graph without loops while a general graph without loops is called a forest. In this way every forest is the union of its connected components and thus a union of trees. A rooted forest is a forest together with an orientation for each of its edges such that every vertex is adjacent to at most one outgoing edge. In a rooted forest a vertex without incoming edge is called a leaf, similarly a vertex without outgoing edges is called a root. Clearly every rooted forest is a union of rooted trees and every rooted tree has a unique root. Note however that the sets $\operatorname{Leaf}\mathfrak{F}$ and $\operatorname{Root}\mathfrak{F}$ of leaves and roots of $\mathfrak{F}$ need not be disjoint, the rooted forest $\mathfrak{F}$ may have solitary vertices adjacent to no edge at all. A variant of a well–known property of forests asserts that every automorphism of a rooted forest is uniquely determined by the induced permutation of its leaves. One way to break the action of the automorphism group is to label the leaves of a forest $\mathfrak{F}$ of order $|\mathfrak{F}| := \#\mathrm{Leaf}\,\mathfrak{F}$ with numbers $1, \ldots, |\mathfrak{F}|$.

In the considerations to come the rooted forests are used as abstract flow charts for doing calculations. The leaves correspond to arguments and the vertices different from the leaves to some as yet unspecified operations while the roots are associated to results. With this interpretation rooted forests are particularly important in the theory of operads [MSS]. However the rooted forests we will use have some characteristic feature quite annoying from the more general point of view of operads, every vertex besides a leaf has a distinguished incoming edge linking it directly to some leaf called the bud of the vertex in question. Rooted forests with this property are appropriately called budding forests, moreover a labelled budding forest will be assumed to have the labelling chosen in such a way that the bud of a vertex V is the leaf of maximal label in the subtree rooted at V. In particular the bud label of a tree $\mathfrak{T}$, the label taken by the bud of the root, is the maximal label of all leaves of $\mathfrak{T}$. Later on we will need the notion of a twin bud, too, a vertex V in a labelled budding forest is said to have a twin bud if the two leaves with maximal labels in the subtree rooted at V carry consecutive labels and are both connected directly to V.

Definition 4.1 (Jet Forest and Feedback)
A jet forest $\mathfrak{F}$ of order $k \geq 1$ is a rooted forest with $|\mathfrak{F}| = k$ leaves together with a labelling $\mathrm{Leaf}\,\mathfrak{F} \xrightarrow{\cong} \{\,1,\,\ldots,\,k\,\}$ of its leaves with numbers $1, \ldots, k$ and a coloring of its vertices by colors black and white meeting the following conditions:

- *All leaves are white while all black vertices are roots. Moreover every black vertex has at least two incoming edges, every white vertex at least three unless it is a leaf.*
- *For all vertices V besides leaves the leaf with maximal label in the subtree rooted at V is connected directly to V and will be called the bud of V.*

Depending on the color of their roots the trees in a jet forest $\mathfrak{F}$ will be called black or white trees. A feedback map for a jet forest $\mathfrak{F}$ of order k is a not necessarily injective map f from the set of black trees to $\{1, \ldots, k, \infty\}$ such that $f(\mathfrak{T})$ exceeds the bud label of $\mathfrak{T}$.

In the interpretation of rooted forests as abstract flow charts a feedback map should be thought of as a modification of the arguments of $\mathfrak{F}$ by the results of the calculations in the black trees. In this context the condition imposed on $f(\mathfrak{T})$ ensures that the result of a calculation in a black tree $\mathfrak{T}$ is never even implicitly used to modify its own arguments. A different method to encode the information provided by a feedback map would be to introduce new "transparent" vertices in a jet forest $\mathfrak{F}$ sitting on the outgoing edges of its leaves. The necessary reformulation of the budding condition makes the resulting alternative definition of a jet forest rather contrieved as many transparent vertices could cascade on the budding edge. This alternative definition of a jet forest has its merits however and we will return to this concept in due time. At the moment splitting the information into a jet forest $\mathfrak{F}$ and a feedback map f as we have done is quite convenient.

Perhaps the most intriguing aspect of the definition of jet forests is the budding condition imposed on every vertex besides leaves. In fact it is precisely this condition which gives the set of all jet forests with feedback a very simply inductive structure. Namely every jet forest

$\mathfrak{F}^{\text{new}}$ of order $k+1$, $k \geq 0$, arises from a unique jet forest $\mathfrak{F}$ of order k by either adding a solitary white vertex labelled $k+1$ or by adding a white or black vertex with bud labelled $k+1$ and the other incoming edges linking to a subset of the white roots of $\mathfrak{F}$. Similarly every feedback map for $\mathfrak{F}^{\text{new}}$ extends a unique feedback f for $\mathfrak{F}$ by choosing a subset of the black trees $\mathfrak{T}$ of $\mathfrak{F}$ with $f(\mathfrak{T}) = \infty$ to have feedback $k+1$. Evidently the presence of this inductive structure allows us to generate the set of all jet forests with feedback efficiently, however the most important aspect of the inductive structure is the way it reflects the induction step in the proof of the general Jet Formula.

In order to fill the abstract flow chart interpretation of jet forests with life we need to specify rules turning a tree $\mathfrak{T}$ in a jet forest $\mathfrak{F}$ into a multilinear expression in the arguments associated to the leaves. Inductively let us decorate the black and white vertices in a jet forest $\mathfrak{F}$ of order k by endomorphisms and vector fields respectively starting with k argument vector fields $X_1, \ldots, X_k$ on M decorating the correspondingly labelled leaves. Consider now the source vertices of the incoming edges of a vertex V in $\mathfrak{F}$. These vertices are all necessarily white and we may assume by induction that they are decorated with vector fields Y and $X_1, \ldots, X_r$, where Y is the decoration of the bud. In turn we will decorate the vertex V with either the vector field $\Phi_r(X_1 \cdot \ldots \cdot X_r)Y$ or with one of the endomorphisms $\Omega_r(X_1 \cdot \ldots \cdot X_r)_Y$ and $\Omega^E_r(X_1 \cdot \ldots \cdot X_r)_Y$ respectively depending on the color of V. Evidently the decoration of the root of a white tree $\mathfrak{T}$ in $\mathfrak{F}$ we get this way is a multilinear expression $\Phi(\mathfrak{T})_{X(\mathfrak{T})}$ in the arguments $X(\mathfrak{T})$ associated to the leaves of $\mathfrak{T}$, which depends essentially only on the isomorphism class of $\mathfrak{T}$ as a rooted tree with buds. Similarly we get multilinear expressions $\Omega(\mathfrak{T})_{X(\mathfrak{T})}$ or $\Omega^E(\mathfrak{T})_{X(\mathfrak{T})}$ respectively for the black trees in $\mathfrak{F}$.

Lemma 4.2 (General Jet Formula)
For all vector fields $X_1, \ldots, X_k$, $k \geq 1$, on an affine manifold M with torsion free connection and all sections ψ of a vector bundle EM over M endowed with a connection ∇ we have

$$\nabla^k_{X_1,\ldots,X_k}\psi \;=\; \sum_{\substack{\text{jet forest}\\ |\mathfrak{F}|=k}} \; \sum_{\substack{\text{feedback}\\ f}} \Big(\overrightarrow{\prod_{\substack{\mathfrak{T}\subset\mathfrak{F}\ \text{black}\\ f(\mathfrak{T})=\infty}}} \Omega^E(\mathfrak{T})_{X^f(\mathfrak{T})} \Big)\, \mathrm{jet}\,\psi \Big(\prod_{\mathfrak{T}\subset\mathfrak{F}\ \text{white}} \Phi(\mathfrak{T})_{X^f(\mathfrak{T})} \Big)$$

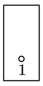
where the black trees $\mathfrak{T}$ with $f(\mathfrak{T}) < \infty$ are used to modify the vector fields $X_1, \ldots, X_k$ to:

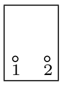
$$X^f_r \;:=\; \Big(\overrightarrow{\prod_{\substack{\mathfrak{T}\subset\mathfrak{F}\ \text{black}\\ f(\mathfrak{T})=r}}} \Omega(\mathfrak{T})_{X^f(\mathfrak{T})} \Big)^* X_r$$

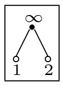
In this formula $*$ denotes the canonical adjoint antiautomorphism of $\mathcal{U}\,\mathrm{End}\,TM$ multiplying all factors of the product with -1 and reversing their order. Because $\Omega^E(\mathfrak{T})$ and $\Omega(\mathfrak{T})$ are endomorphisms we need to specify that they are multiplied in increasing order of the maximal labels of leaves in $\mathrm{Leaf}\,\mathfrak{T}$ *or equivalently bud labels as indicated by the arrows over $\prod$.*

In order to follow the reasoning of the proof it is perhaps better to have a pictorial idea of the statement and concrete non–trivial examples in mind. For this reason we want to write out explicitly a couple of jet forests and the associated curvature terms. In the diagrams representing the jet forests the edges are oriented upwards and the leaves on the bottom are labelled left to right, while the black roots are shown together with their feedback. To first and second order of course the general Jet Formula 4.2 agrees with the self–evident formulas

$$\nabla^1_{X_1}\psi \;=\; \mathrm{jet}^1_{X_1}\psi \qquad\qquad \nabla^2_{X_1,X_2}\psi \;=\; \mathrm{jet}^2_{X_1\cdot X_2}\psi \;+\; \tfrac{1}{2}R_{X_1,X_2}\,\mathrm{jet}^0\psi$$

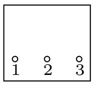


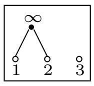


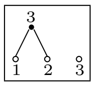


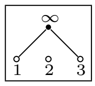
Already to third order however the general Jet Formula features some rather strange terms

$$\nabla^3_{X_1,X_2,X_3}\psi \;=\; \mathrm{jet}^3_{X_1\cdot X_2\cdot X_3}\psi + \tfrac{1}{2}R^E_{X_1,X_2}\,\mathrm{jet}^1_{X_3}\psi - \mathrm{jet}^1_{\frac{1}{2}R_{X_1,X_2}X_3}\psi + \tfrac{1}{2}R^E_{X_1,X_3}\,\mathrm{jet}^1_{X_2}\psi \tag{4.3}$$

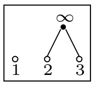


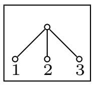


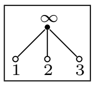




$$+\tfrac{1}{2}R^E_{X_2,X_3}\,\mathrm{jet}^1_{X_1}\psi - \mathrm{jet}^1_{\frac{1}{6}[R_{X_1,X_3}X_2+R_{X_2,X_3}X_1]}\psi + \tfrac{1}{3}\,[(\nabla_{X_1}R^E)_{X_2,X_3} + (\nabla_{X_2}R^E)_{X_1,X_3}]\,\mathrm{jet}^0\psi$$







where the sums in brackets expand the symmetric bilinear forms associated to the quadratic pieces of the Taylor series of the forward parallel transport $\Phi(X) = Y - \frac{1}{6}R_{X,Y}X + O(X^3)$ and the connection form $\omega^E(X)_Y = \frac{1}{2}R_{X,Y} + \frac{1}{3}(\nabla_X R^E)_{X,Y} + O(X^3)$. There are 21 jet forests of order 4 and 30 different jet forests with feedback in total so that we refrain from writing out the general Jet Formula to this order.

Proof: As there is only one jet forest $\mathfrak{F}$ of order 1 having a single solitary white vertex and no possible feedback the formula is certainly true for $k = 1$ for every vector bundle EM over M. Assume by induction hypothesis that the formula is correct for some $k \geq 1$ and every vector bundle over M, say for the vector bundle $T^*M \otimes EM$ endowed with the product connection. Applying the formula to the section $\nabla\psi \in \Gamma(T^*M \otimes EM)$ we get:

$$\nabla^k_{X_1,\ldots,X_k}\nabla\psi \;=\; \sum_{\substack{\text{jet forest}\\ |\mathfrak{F}|=k}}\;\sum_{\substack{\text{feedback}\\ f}}\Big(\overrightarrow{\prod_{\substack{\mathfrak{T}\subset\mathfrak{F}\text{ black}\\ f(\mathfrak{T})=\infty}}}\Omega^{T^*\otimes E}(\mathfrak{T})_{X^{f(\mathfrak{T})}}\Big)\,\mathrm{jet}\,\nabla\psi\Big(\prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}}\Phi(\mathfrak{T})_{X^{f(\mathfrak{T})}}\Big) \tag{4.4}$$

The connection form $\Omega^{T^*\otimes E}$ of the product connection on $T^*M \otimes EM$ acts as a derivation

$$\Omega^{T^*\otimes E}(\mathfrak{T})_{X^{f(\mathfrak{T})}} \;:=\; \Omega(\mathfrak{T})_{X^{f(\mathfrak{T})}}\otimes \mathrm{id}_E \;+\; \mathrm{id}_{T^*}\otimes\Omega^E(\mathfrak{T})_{X^{f(\mathfrak{T})}}$$

on $T^*M\otimes EM$ for every black jet tree $\mathfrak{T}$. Consequently for every black tree $\mathfrak{T}$ with $f(\mathfrak{T}) = \infty$ occurring in equation (4.4) we need to choose whether the corresponding curvature expression is to act by minus the adjoint endomorphism $\Omega(\mathfrak{T})_{X^{f(\mathfrak{T})}}\alpha := -\alpha(\Omega(\mathfrak{T})_{X^{f(\mathfrak{T})}}\,\cdot\,)$ on T^*M or as $\Omega^E(\mathfrak{T})_{X^{f(\mathfrak{T})}}$ on EM. The possible choices are conveniently parametrized by a modified feedback map f^{new} with $f^{\mathrm{new}}(\mathfrak{T}) := f(\mathfrak{T})$ unless $\mathfrak{T}$ is a black tree with $f(\mathfrak{T}) = \infty$, whose corresponding curvature expression is to act on T^*M, in this case we set $f^{\mathrm{new}}(\mathfrak{T}) := k+1$. Observe that only the black trees with $f(\mathfrak{T}) \leq k$ matter in modifying the argument vector fields from $X_1,\ldots,X_k$ to $X^f_1,\ldots,X^f_k$, hence using the modified feedback map f^{new} instead of f won't change $X_r^f = X_r^{f^{\mathrm{new}}}$, $r = 1, \ldots, k$. Inserting the vector field X_{k+1} into formula (4.4) and changing the summation to modified feedback maps f^{new} we conclude

$$\begin{aligned}
&\nabla^{k+1}_{X_1,\ldots,X_{k+1}}\psi \\
&= \sum_{\substack{\text{jet forest}\\ |\mathfrak{F}|=k}} \sum_{\substack{\text{feedback}\\ f}} \Big[\prod^{\longrightarrow}_{\substack{\mathfrak{T}\subset\mathfrak{F}\text{ black}\\ f(\mathfrak{T})=\infty}} \Omega^{T^*\otimes E}(\mathfrak{T})_{X^{f(\mathfrak{T})}} \,\mathrm{jet}\,\nabla\psi\Big(\prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} \Phi(\mathfrak{T})_{X^{f(\mathfrak{T})}}\Big)\Big]_{X_{k+1}} \\
&= \sum_{\substack{\text{jet forest}\\ |\mathfrak{F}|=k}} \sum_{\substack{\text{modified}\\ f^{\mathrm{new}}}} \left(\prod^{\longrightarrow}_{\substack{\mathfrak{T}\subset\mathfrak{F}\text{ black}\\ f^{\mathrm{new}}(\mathfrak{T})=\infty}} \Omega^{E}(\mathfrak{T})_{X^{f^{\mathrm{new}}(\mathfrak{T})}}\right) \Big[\mathrm{jet}\,\nabla\psi\Big(\prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} \Phi(\mathfrak{T})_{X^{f^{\mathrm{new}}(\mathfrak{T})}}\Big)\Big]_{X_{k+1}^{f^{\mathrm{new}}}}
\end{aligned} \qquad (4.5)$$

by the coassociativity identity $\nabla^{k+1}_{X_1,\ldots,X_k,X_{k+1}}\psi = [\,\nabla^k_{X_1,\ldots,X_k}\nabla\psi\,]_{X_{k+1}}$ of iterated covariant derivatives. Evidently the adjoint antiautomorphism of $\mathcal{U}\mathrm{End}\,TM$ makes its appearance in modifying X_{k+1} into $X_{k+1}^{f^{\mathrm{new}}}$, because this is the proper way the representation of the Lie algebra End TM on T^*M extends to the universal enveloping algebra $\mathcal{U}\mathrm{End}\,TM$ or more succinctly because $(\mathfrak{X}\alpha)(X) = \alpha(\mathfrak{X}^*X)$ for all $\mathfrak{X} \in \mathcal{U}\mathrm{End}\,TM$. With formula (4.5) we have succeeded in expressing $\nabla^{k+1}\psi$ solely in terms of jet$\nabla\psi$. In turn the special Jet Formula expresses jet$\nabla\psi$ in terms of jet ψ, namely the polarization of (4.2) with respect to X reads

$$\begin{aligned}
\big[\,\mathrm{jet}^k_{X_1\cdot\ldots\cdot X_k}\nabla\psi\,\big]_Y \;=\; &\mathrm{jet}^{k+1}_{X_1\cdot\ldots\cdot X_k\cdot Y}\psi + \sum_{r=2}^{k}\sum_{\mu_1<\ldots<\mu_r} \mathrm{jet}^{k-r+1}_{X_1\cdot\ldots\cdot X_k\cdot\Phi(X_{\mu_1}\cdot\ldots\cdot X_{\mu_r})Y}\psi \\
&+ \sum_{r=1}^{k}\sum_{\mu_1<\ldots<\mu_r} \Omega^E(X_{\mu_1}\cdot\ldots\cdot X_{\mu_r})_Y\,\mathrm{jet}^{k-r}_{X_1\cdot\ldots\cdot X_k}\psi
\end{aligned} \qquad (4.6)$$

where of course $X_1 \cdot \ldots \cdot X_k$ in the sums on the right hand side refers to the product of all the remaining arguments with $X_{\mu_1}, \ldots, X_{\mu_r}$ omitted. Expanding every summand in (4.5) in this way using (4.6) compares nicely to the way a jet forest $\mathfrak{F}^{\mathrm{new}}$ of order $k+1$ extends a jet forest $\mathfrak{F}$ of order k. Namely the summation over all r and all $\mu_1 < \ldots < \mu_r$ turns into the summation over all ways to join a new black or white vertex of $\mathfrak{F}^{\mathrm{new}}$ with bud labelled $k+1$ to a subset of the white roots of $\mathfrak{F}$. Note that the Taylor series $\Omega^E(X)_Y = \frac{1}{2}R^E_{X,Y} + O(X^2)$ has no constant term while $\Phi(X)Y = Y - \frac{1}{6}R_{X,Y}X + O(X^3)$ agrees with the identity up to at least quadratic terms reflecting the condition imposed on the number of incoming edges for a black and white vertex respectively. □

Philosophically both the special and general Jet Formula are ultimately consequences of the general curvature identity (3.1). Thinking of this statement the other way around we may conclude that the general Jet Formula is uniquely characterized by two properties alone, first of all the resulting expression for $\nabla^k_{X_1,\ldots,X_k}\psi$ satisfies all $k-1$ applicable instances of the curvature identity (3.1) and secondly this expression reduces to the definition

$$\nabla^k_{X,\ldots,X}\psi \;=\; \mathrm{jet}^k_{X\cdot\ldots\cdot X}\psi$$

of jet$^k\psi$ in case all argument vector fields $X_1, \ldots, X_k$ agree. In principle we could thus prove the general Jet Formula by showing that the sum over all jet forests indeed satisfies these characterizing conditions. The reader is invited to check formula (4.3) directly using this argument, because it makes both applicable instances of the curvature identity (3.1) true as a consequence of the first and second Bianchi identity. In general this approach to proving the general Jet Formula seems rather difficult, although it is easily checked that with all argument vector fields agreeing there is but one non–vanishing summand in the stated formula for $\nabla^k_{X,\ldots,X}\psi$, the summand associated to the jet forest with k solitary white vertices:

Remark 4.3 (Partial Symmetrization)
Consider a non–leaf vertex V in a jet forest $\mathfrak{F}$ with the property that all incoming edges to V connect directly to leaves. Symmetrizing the summand in the general Jet Formula 4.2 associated to $\mathfrak{F}$ and a feedback map f avoiding the labels of the leaves connecting to V over all arguments associated to these leaves results in zero.

In light of this observation let us try to find formulas for the iterated covariant derivatives of the form $\nabla^{k+1}_{X,\ldots,X,Y,X,\ldots,X}\psi$. In this situation the partial symmetrization criterion 4.3 is strong enough to characterize the relevant jet forests with feedback completely. Let us call a jet tree $\mathfrak{T}$ different from a solitary white vertex a palm tree if all its non–leaf vertices occur on a single stem, a path from some leaf to the root. In other words a palm tree is a tree of maximal height among all trees with a fixed number of non–leaf vertices. In the absense of feedback partial symmetrization would imply that the only jet forests $\mathfrak{F}$ possibly contributing to $\nabla^{k+1}_{X,\ldots,X,Y,X,\ldots,X}\psi$ are the jet forests with solitary white vertices only except at most a single palm tree. Things are more complicated due to the presence of feedback, however it is still easy to argue by induction that the jet forests $\mathfrak{F}$ with feedback f possibly contributing to $\nabla^{k+1}_{X,\ldots,X,Y,X,\ldots,X}\psi$ consist entirely of solitary white vertices and palm trees. Moreover the bud labels of all non–leaf vertices in such a jet forest $\mathfrak{F}$ are strictly increasing not only in each palm tree in $\mathfrak{F}$ separately, but over the entire forest $\mathfrak{F}$ in the sense that the total order on the set of palm trees by bud labels is well–defined independent of the choice of representative non–leaf vertices. Eventually all palm trees $\mathfrak{T}$ in $\mathfrak{F}$ except possibly the maximal one are black and they are linked via feedback in the sense that $f(\mathfrak{T}) < \infty$ is the label of a leaf connecting to the minimal non–leaf vertex in the successor palm tree of $\mathfrak{T}$.

Taking this characterization of the relevant jet forests into account we conclude that the iterated covariant derivative $\nabla^{k+1}_{X,\ldots,X,Y}\psi$ is the sum over all jet forests $\mathfrak{F}$ with at most one non–leaf vertex with bud labelled $k+1$, because the vector field Y occurs to the very right decorating the leaf with maximal label. Consequently the general Jet Formula collapses to the special one in the form (4.2) as expected. Of course this can't be too surprising, because the general Jet Formula is essentially conditioned to make the special Jet Formula true!

An interesting, but less trivial example is the iterated covariant derivative $\nabla^{k+1}_{Y,X,\ldots,X}\psi$, which plays a prominent role in the so–called canonical or Spencer connection on jet bundles in the model of jet theory we are using at the moment. In order to sum the general Jet Formula in this case we need to count the relevant jet forests of linked palm trees and solitary white vertices. For starters we want to count the jet forests with only a single palm tree $\mathfrak{T}$. Let us assume that we meet $k_1, k_2, \ldots, k_r$ additional incoming edges following the stem from the leaf labelled 1 decorated with Y to the root. Exactly

$$\binom{k}{k_1+\ldots+k_r}\binom{k_1+\ldots+k_r-1}{k_r-1}\binom{k_1+\ldots+k_{r-1}-1}{k_{r-1}-1}\cdots\binom{k_1+k_2-1}{k_2-1}\binom{k_1-1}{k_1-1}$$

jet forests meet this assumption. The first binomial chooses the subset of all leaves connecting to non–leaf vertices besides the leaf labelled 1. The leaf of maximal label among these is the bud of the root and only the other $k_r - 1$ additional leaves of the root need to be chosen. Again the leaf of maximal label in the remaining $k_1 + \ldots + k_{r-1}$ leaves is the bud of the vertex second to the root, $k_{r-1} - 1$ additional leaves have to be determined, and so on. Similarly it is possible to count the number of jet trees with two or more palm trees and to sum the general Jet Formula for $\nabla^{k+1}_{Y,X,\ldots,X}\psi$ eventually. Needless to say the combinatorial details of this summation are intricate, the final result (4.8) however suggests a more streamlined approach. First of all let us define the formal power series Φ^* on TM with values in End TM as the unique solution to the differential equation

$$(\,N\Phi^*\,)\,\Phi^{-*} \;=\; \Phi^{-1}\,(\,N\Phi\,) \tag{4.7}$$

subject to the initial condition $\Phi^*(X) = \mathrm{id} + O(X^2)$. Written as formal power series in the homogeneous components of $\mathcal{R}(X)Y := \sum_{k\geq 0}\frac{1}{k!}(\nabla^k_{X,\ldots,X}R)_{X,Y}X$ the two power series Φ and Φ^* are closely related, $\Phi^* = \mathrm{rev}(\Phi)$ is simply given by reversing the order of the factors in Φ. In particular Φ^* really is the adjoint of Φ on a Riemannian manifold M as suggested by notation, because the homogeneous components of $\mathcal{R}$ are all symmetric endomorphisms by the classical symmetries of the Riemannian curvature tensor. A rigorous proof of this simple relation between Φ and Φ^* based on the defining differential equation (4.7) proceeds by induction on r in the congruences $\Phi^* = \mathrm{rev}(\Phi) + O(\mathcal{R}^{r+1})$ modulo polynomials in the homogeneous components of $\mathcal{R}$ of order at least $r+1$.

Consider now the set of palm forests consisting entirely of palm trees linked via feedback satisfying all the conditions above. In other words the palm trees in a palm forest $\mathfrak{F}$ are totally ordered by bud labels and all palm trees $\mathfrak{T}$ in $\mathfrak{F}$ except possibly the maximal one are black with feedback $f(\mathfrak{T})$ labelling a leaf connected directly to the minimal non–leaf vertex in the successor palm tree. Given such a palm forest $\mathfrak{F}$ with exactly $r+1$ leaves and a decoration of the leaves by vector fields $Y, X, \ldots, X$ the root of the maximal white or black palm tree in $\mathfrak{F}$ is decorated either by a vector field or an endomorphism depending polynomially of order r on X and linearly on Y. Summing this decoration over all palm forests with exactly $r+1$ leaves and with white or black maximal palm tree results in

$$-\,\tfrac{1}{r}\,[\,(\Phi - \mathrm{id})\Phi^*\,]_r(X\cdot\ldots\cdot X)Y \qquad\qquad -\,\tfrac{1}{r}\,[\,\Omega^E_{\Phi^*}\,]_r(X\cdot\ldots\cdot X)_Y$$

for all $r \geq 2$ or $r \geq 1$ respectively. Omitting the details of the argument, which is by straightforward induction on r, we use this intermediate result to derive the following formula

$$\begin{aligned}\nabla^{k+1}_{Y,X,\ldots,X}\psi \;=\; \mathrm{jet}^{k+1}_{X\cdot\ldots\cdot X\cdot Y}\psi \;&-\; \sum_{r=2}^{k}\frac{1}{r}\binom{k}{r}\,\mathrm{jet}^{k+1-r}_{X\cdot\ldots\cdot X\cdot[(\Phi-\mathrm{id})\Phi^*]_r(X\cdot\ldots\cdot X)Y}\psi\\ &+\; \sum_{r=1}^{k}\frac{1}{r}\binom{k}{r+1}\,\mathrm{jet}^{k-r}_{X\cdot\ldots\cdot X\cdot[\Omega_{\Phi^*}]_r(X\cdot\ldots\cdot X)_Y X}\psi\\ &-\; \sum_{r=1}^{k}\frac{1}{r}\binom{k}{r}\,[\,\Omega^E_{\Phi^*}\,]_r(X\cdot\ldots\cdot X)_Y\,\mathrm{jet}^{k-r}_{X\cdot\ldots\cdot X}\psi\end{aligned} \tag{4.8}$$

for the Spencer connection, which can be verified along the lines of the recursive argument leading to Theorem 3.2. Needless to say the author has done so for all orders up to $k = 25$.

At this point the reader may suspect that the symmetries underlying the general Jet Formula point to the existence of an underlying group theoretic framework. Indeed the most natural interpretation of the general Jet Formula is that it gives a closed formula for the exponential of the special Jet Formula considered as an element (sic!) of a suitable algebraic group. In the calculus of difference elements the special Jet Formula is in fact the difference element between jet ψ and jet$\nabla\psi$. Alternatively we could have started from the difference element between jet ψ and $\nabla(\text{jet}\,\psi)$ to get another version of the general Jet Formula upon exponentiation. Redefining Ω and Φ if necessary equation (4.8) appears just as fundamental and perhaps even more natural than the special Jet Formula. In any case it seems as suitable a starting point for an induction proving (a different version of) the general Jet Formula.

Although this approach is perfectly feasible, it has a very serious drawback. At each induction step we need to take the covariant derivative of the group element we have constructed so far. In this way the general Jet Formula we end up with will feature not only the power series Ω and Φ, but all their iterated covariant derivatives as well. However the whole point in the general Jet Formula is that it separates the material from the redundant information contained in iterated covariant derivatives. Piling up more and more redundant information during the induction we will eventually loose all control about symmetry properties of the different terms and camouflaged curvature terms aka Weitzenböck formulas.

Chapter 5

Regularity of the Supertrace of the Heat Kernel

Philosophically the heat kernel coefficients of generalized Laplacians studied in Section 2 are the images of a certain universal object governing jet calculus on a Riemannian manifold. Pursuing this philosophy we will replace the original heat kernel coefficients by "universal" coefficients in this section. The principal advantage of using these universal coefficients is that their Taylor series live in the space $\mathrm{Sym}\, T^* \otimes \mathcal{U}\mathfrak{hol}$ of formal power series with values in the universal enveloping algebra $\mathcal{U}\mathfrak{hol}$ of the holonomy algebra $\mathfrak{hol}$. Twisting the natural filtration on $\mathcal{U}\mathfrak{hol}$ appropriately we will construct a filtration $\mathbb{F}^\bullet(\mathrm{Sym}\, T^* \otimes \mathcal{U}\mathfrak{hol})$ on this space, whose very existence probes deep into the combinatorial structure of heat kernel coefficients. Having defined this filtration we will turn it into an algebraic analogue of Getzler's transformation in this section and use it to prove the regularity of the trace of the heat kernels for twisted Dirac operators. The subsequent calculation of the index density is postponed to the next section.

Recall that the universal enveloping algebra $\mathcal{U}\mathfrak{hol}$ of a Lie algebra $\mathfrak{hol}$ is the associative algebra with unit generated freely by $\mathfrak{hol}$ subject only to the condition that the commutator of two elements from $\mathfrak{hol}$ in the algebra $\mathcal{U}\mathfrak{hol}$ agrees with their Lie bracket in $\mathfrak{hol}$. Alternatively we can define $\mathcal{U}\mathfrak{hol}$ by its universal property, namely every representation E of the Lie algebra $\mathfrak{hol}$ extends uniquely to an algebra homomorphism $\mathcal{U}\mathfrak{hol} \longrightarrow \mathrm{End}\, E$. Moreover the universal enveloping algebra $\mathcal{U}\mathfrak{hol}$ comes equipped with a natural ascending filtration with $\mathcal{U}^{\leq r}\mathfrak{hol}$ being spanned by all formal products of at most r factors from $\mathfrak{hol}$ and the graded algebra associated to this filtration is isomorphic to the symmetric algebra $\mathrm{Sym}\,\mathfrak{hol}$. Together with polarization and induction this readily implies that $\mathcal{U}^{\leq r}\mathfrak{hol}$ is already spanned by elements of the form $\frac{1}{\tilde{r}!}\mathfrak{X}^{\tilde{r}}$ with $0 \leq \tilde{r} \leq r$ and $\mathfrak{X} \in \mathfrak{hol}$. The universal property allows us to identify the algebra $(\mathcal{U}\mathfrak{hol})^{\mathrm{op}}$ opposite to $\mathcal{U}\mathfrak{hol}$ with the universal enveloping algebra of the opposite Lie algebra $\mathfrak{hol}^{\mathrm{op}}$. Consequently the canonical Lie algebra homomorphism $\mathfrak{hol} \longrightarrow \mathfrak{hol}^{\mathrm{op}}$, $\mathfrak{X} \longmapsto -\mathfrak{X}$, extends to the involutive adjoint antiautomorphism of $\mathcal{U}\mathfrak{hol}$

$$* : \ \mathcal{U}\mathfrak{hol} \longrightarrow \mathcal{U}\mathfrak{hol}, \qquad \mathfrak{X}_1 \ldots \mathfrak{X}_r \longmapsto (-1)^r\, \mathfrak{X}_r \ldots \mathfrak{X}_1$$

with $(\mathfrak{X}\alpha)(\psi) = \alpha(\mathfrak{X}^*\psi)$ for every representation E and all $\psi \in E$, $\alpha \in E^*$ and $\mathfrak{X} \in \mathcal{U}\mathfrak{hol}$.

In the gauge fixed geometric context used previously in Sections 2 and 3 let us consider the connection Laplacian $\nabla^*\nabla$ acting on sections of a trivial vector bundle $E \times T$ carrying a non–trivial connection over a euclidian vector space T endowed with a Riemannian metric g such that the symbol metric g and its exponential map $\exp_0 : T_0T \longrightarrow T$ at the origin agree with the scalar product on T and the canonical identification $T_0T \cong T$ respectively. Moreover let us assume that parallel transport along radial geodesics $t \longmapsto tX$ is the identity $E \times \{0\} \longrightarrow E \times \{tX\}$. Fixing an orthonormal base $\{x_\mu\}$ for the scalar product $\langle\, ,\rangle$ on T we get a local orthonormal base $\{\Phi x_\mu\}$ for g and an explicit expansion of its cometric tensor

$$g^{-1} \;:=\; \sum_\lambda \Phi x_\lambda \otimes \Phi x_\lambda \;=\; \sum_{\mu\nu} \langle \Phi^* x_\mu, \Phi^* x_\nu \rangle x_\mu \otimes x_\nu \;=\; \sum_{\mu\nu} g^{\mu\nu}\, x_\mu \otimes x_\nu$$

where $g^{\mu\nu} := \langle \Phi\Phi^* x_\mu, x_\nu \rangle$. By definition of the connection form ω^E we clearly have

$$\nabla^2_{Y,Z} \;=\; \frac{\partial^2}{\partial Y \partial Z} + \omega^E_Y \frac{\partial}{\partial Z} + \omega^E_Z \frac{\partial}{\partial Y} - \frac{\partial}{\partial \Gamma_Y Z} + \omega^E_Y \omega^E_Z + (\frac{\partial}{\partial Y} \omega^E)_Z - \omega^E_{\Gamma_Y Z}$$

for two constant vector fields Y, Z, where Γ are the Christoffel symbols corresponding to the symbol metric g or equivalently the connection form of the Levi–Civita connection on the tangent bundle in the standard trivialization of $TT \cong T \times T$. Of course ω^E and Γ are completely unrelated in general, nevertheless similar formulas apply to both as they serve similar purposes. Tracing ∇^2 over the cometric tensor g^{-1} we arrive at the following expression for the connection Laplacian $\nabla^*\nabla$:

$$-\sum_{\mu\nu} g^{\mu\nu} \left[\Big(\frac{\partial^2}{\partial x_\mu \partial x_\nu} - \frac{\partial}{\partial \Gamma_{x_\mu} x_\nu} \Big) + \Big(2\omega^E_{x_\mu} \frac{\partial}{\partial x_\nu} + (\frac{\partial}{\partial x_\mu} \omega^E)_{x_\nu} - \omega^E_{\Gamma_{x_\mu} x_\nu} \Big) + \omega^E_{x_\mu} \omega^E_{x_\nu} \right]$$

Looking more closely at this formula the reader will observe that all coefficients of $\nabla^*\nabla$ are functions on T with values in the universal enveloping algebra $\mathcal{U}\mathfrak{hol}^E$ of the holonomy algebra $\mathfrak{hol}^E$ of the connection ∇. Accordingly we have ordered the terms into scalar differential operators with coefficients in $\mathcal{U}^{\leq 0}\mathfrak{hol}^E$ and differential operators with coefficients in $\mathcal{U}^{\leq 1}\mathfrak{hol}^E$ or $\mathcal{U}^{\leq 2}\mathfrak{hol}^E$ respectively. Evidently this property remains unchanged under conjugation by the square root $j^{\frac{1}{2}}$ of the Jacobian determinant, because equation (2.3) reads

$$j^{\frac{1}{2}}\, \nabla^*\nabla\, j^{-\frac{1}{2}} \;=\; \nabla^*\nabla + \frac{\partial}{\partial L} + \omega^E_L + \frac{1}{2} \operatorname{div}_g L - \frac{1}{4} g(L, L) \tag{5.1}$$

with $L := \operatorname{grad}_g \log j$ so that $\nabla^*\nabla$ is modified by scalar differential operators only except for the term ω^E_L with coefficients in $\mathcal{U}^{\leq 1}\mathfrak{hol}^E$. This observation has the striking consequence that the heat kernel coefficients $a_d \in C^\infty(T, \operatorname{End} E)$ for $\nabla^*\nabla$ acting on sections of $E \times T$ are actually the images of "universal" heat kernel coefficients $a_d \in C^\infty(T, \mathcal{U}\mathfrak{hol}^E)$ under the representation homomorphism $\mathcal{U}\mathfrak{hol}^E \longrightarrow \operatorname{End} E$. In fact the recursion formula (2.8)

$$(d + 1 + N)\, a_{d+1}(X) \;=\; -(j^{\frac{1}{2}}\, \nabla^*\nabla\, j^{-\frac{1}{2}})\, a_d(X)$$

makes perfect sense for functions with values in $\mathcal{U}\mathfrak{hol}^E$ while the initial value for the recursion is the image of $a_0(X) = 1$ under the representation. Note however the minor subtlety that the coefficients of $\nabla^*\nabla$ act by multiplication with their adjoint from the right on $\mathcal{U}\mathfrak{hol}^E$.

Remark 5.1 (Universal Heat Kernel Coefficients)
Consider a representation E of $\mathfrak{hol}^E$ and the associated trivial vector bundle $E \times T$ endowed with the connection ∇ with connection form ω^E. The heat kernel coefficients of the connection Laplacian $\nabla^*\nabla$ are the images of universal heat kernel coefficients $a_d \in C^\infty(T, \mathcal{U}\mathfrak{hol}^E)$ under the representation homomorphism $\mathcal{U}\mathfrak{hol}^E \longrightarrow \mathrm{End}\,E$.

The existence of universal heat kernel coefficients a_d, $d \geq 0$, can be seen as an algebraic analogue of the basic idea of the second proof of the Local Index Theorem in [BGV] using the scalar Laplacian acting on functions on a principal bundle over M. A more satisfactory explanation would be to realize the function $a(t, x, y)$ as a family of sections of the groupoid of diffeomorphisms of the principal bundle in question covering diffeomorphisms of M.

Eventually we want to solve the recursion formula (2.8) for the coefficients a_d, $d \geq 0$, in order to find their values $a_d(0)$ in the origin. For this purpose it is clearly sufficient to solve them for the Taylor series of $a_d \in \mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$ only. Restricting to Taylor series has the advantage that we know the Taylor series of the coefficients of $j^{\frac{1}{2}}\,\nabla^*\nabla\,j^{-\frac{1}{2}}$ in the origin from the formulas for the Taylor series for Φ^{-1} and ω^E derived in Section 3. Surprisingly we get an additional bonus from restricting to Taylor series in form of a quite remarkable filtration on $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$, which will become an algebraic substitute for Getzler's transform:

Definition 5.2 (Filtration for Heat Kernel Coefficients)
The space $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$ of formal power series with coefficients in the universal enveloping algebra $\mathcal{U}\mathfrak{hol}^E$ of the holonomy algebra $\mathfrak{hol}^E$ carries an ascending filtration

$$\ldots \subsetneq \mathbb{F}^{-1}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E) \subsetneq \mathbb{F}^{0}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E) \subsetneq \mathbb{F}^{1}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E) \subsetneq \ldots$$

defined by:

$$\mathbb{F}^r(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E) \;:=\; \bigoplus_{l \geq 0} \mathrm{Sym}^{\,l}T^* \otimes \mathcal{U}^{\leq \lfloor \frac{l+r}{2} \rfloor}\mathfrak{hol}^E$$

The filtration on the space $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$ induces a filtration on the space of Taylor series of differential operators acting on $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$. Namely a differential operator acting on $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$ is called filtered of degree $d \in \mathbb{Z}$ if it maps $\mathbb{F}^r(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ to $\mathbb{F}^{r+d}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ for all $r \in \mathbb{Z}$. The prototype example of such an operator is

$$(\,x^k \otimes \mathfrak{X}\,)\,\frac{\partial^m}{\partial X^m}$$

with $X \in T$, $x \in T^*$ and a coefficient $\mathfrak{X} \in \mathcal{U}^{\leq r}\mathfrak{hol}^E$ which is filtered of degree $m - k + 2r$. The crucial point in our first proof of the Local Index Theorem is that the connection Laplacian $\nabla^*\nabla$ and the squares D^2 of twisted Dirac operators are filtered of degree 2. In order to check this for $\nabla^*\nabla$ we use the Taylor expansion $\omega^E(X)_Y \;=\; \frac{1}{2}R^E_{X,Y} + \frac{1}{3}(\nabla_X R^E)_{X,Y} + O(X^3)$ of the connection form to calculate the Taylor series of the connection Laplacian $\nabla^*\nabla$ up to differential operators which are filtered of degree 0. Say for the summand

$$\sum_{\mu\nu} g^{\mu\nu}(\,X\,)\,\omega^E_{x_\mu}(\,X\,)\,\frac{\partial}{\partial x_\nu}$$

which is of first order and has coefficients in $\mathcal{U}^{\leq 1}\mathfrak{hol}^E$, this means to expand the coefficients $g^{\mu\nu}\omega_{x_\mu}$ up to order $O(X^3)$, because $m=1$, $r=1$ make $m-k+2r \leq 0$ for $k \geq 3$. Interestingly the connection form ω^E itself is $O(X)$ so that we can replace $g^{\mu\nu} = \delta^{\mu\nu} + O(X^2)$ by $\delta^{\mu\nu}$ without changing the result modulo operators filtered of degree 0. Similar considerations apply to all summands in the expansion of $\nabla^*\nabla$ and in the end we find

$$\begin{aligned}\nabla^*\nabla \;\equiv\; &-\sum_\mu \frac{\partial^2}{\partial x_\mu^2} - \sum_\mu \Big(R^E_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} + \frac{2}{3}(\nabla_\cdot R^E)_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} + \frac{1}{3}(\nabla_{x_\mu}R^E)_{\cdot,x_\mu}\Big) \\ &- \sum_\mu \Big(\frac{1}{4}R^E_{\cdot,x_\mu}R^E_{\cdot,x_\mu} + \frac{1}{6}R^E_{\cdot,x_\mu}(\nabla_\cdot R^E)_{\cdot,x_\mu} + \frac{1}{6}(\nabla_\cdot R^E)_{\cdot,x_\mu}R^E_{\cdot,x_\mu}\Big)\end{aligned} \tag{5.2}$$

modulo differential operators filtered of degree 0. Evidently this implies that the connection Laplacian is filtered of degree 2. It is certainly not a coincidence that this looks pretty much like the result of performing a Getzler transform on the square of a twisted Dirac operator.

Theorem 5.3 (Coefficients of the Connection Laplacian)
The heat kernel coefficients a_d, $d \geq 0$, of $\nabla^*\nabla$ are elements of $\mathbb{F}^{2d}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$, thus

$$a_d \;\in\; \bigoplus_{l\geq 0} \mathrm{Sym}^{\,l}T^* \otimes \mathcal{U}^{\leq d+\lfloor \frac{l}{2}\rfloor}\mathfrak{hol}^E$$

and the value $a_d(0)$ of the d–th coefficient along the diagonal lives in $\mathcal{U}^{\leq d}\mathfrak{hol}^E$. Moreover the symbol class $[a_d] \in \mathbb{F}^{2d}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)/\mathbb{F}^{2d-2}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ only depends on the Taylor series of $\nabla^*\nabla$ up to differential operators filtered of degree 0.

Proof: According to the Taylor expansion $j(X) = 1-\frac{1}{6}\mathrm{Ric}(X,X)+O(X^3)$ of the Jacobian determinant derived before the operator of multiplication by the scalar $j^{\frac{1}{2}}$ agrees with the identity up to operators filtered of degree -2. As we already know that $\nabla^*\nabla$ is filtered of degree 2 we conclude that conjugation by $j^{\frac{1}{2}}$ does not change $\nabla^*\nabla$ modulo operators filtered of degree 0. On the other hand Euler's number operator $N := \sum x_\mu \frac{\partial}{\partial x_\mu}$ acts by multiplication with l on $\mathrm{Sym}^{\,l}T^*$ so that the operator $d+1+N$ appearing in the recursion formula (2.8) of Minakshisundaram–Pleijel for $d \geq 0$

$$(\, d + 1 + N \,)\, a_{d+1} \;=\; -(\, j^{\frac{1}{2}} \circ \nabla^*\nabla \circ j^{-\frac{1}{2}} \,)\, a_d$$

is invertible on $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$ with inverse preserving the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$. With $j^{\frac{1}{2}}\nabla^*\nabla j^{-\frac{1}{2}}$ being filtered of degree 2 a straightforward induction starting with the initial value $a_0(X) = 1$ in $\mathbb{F}^0(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ proves the first assertion. The arguments for the second statement are similar and omitted. □

Expanding the summands of the operator $j^{\frac{1}{2}}\nabla^*\nabla j^{-\frac{1}{2}}$ more carefully than we have done above the reader may have noticed a couple of simplifications which altogether appear to be too lucky a coincidence and rather point to a intriguingly consistent efficiency. Recall that in Definition 3.4 we have defined the weight operator W as a derivation on the space of formal power series in R and R^E as well as their covariant derivatives by specifying the weights $W\nabla^k R^E = k\nabla^k R^E$ and $W\nabla^k R = (k+2)\nabla^k R$ compare table (3.3). Let us go back for a second and have a look at the expansion of $\nabla^2_{Y,Z}$ with constant vector fields Y and Z:

$$\nabla^2_{Y,Z} \;=\; \frac{\partial^2}{\partial Y \partial Z} + \omega^E_Y \frac{\partial}{\partial Z} + \omega^E_Z \frac{\partial}{\partial Y} - \frac{\partial}{\partial \Gamma_Y Z} + \omega^E_Y \omega^E_Z + (\frac{\partial}{\partial Y}\omega^E)_Z - \omega^E_{\Gamma_Y Z}$$

The summand $\omega^E_Y \frac{\partial}{\partial Z}$ is filtered of degree $3-N$ in the sense that the homogeneous component of ω^E of degree k as polynomial in X becomes a filtered operator of degree $3-k$. Similarly the summand $\omega^E_Y \omega^E_Z$ is filtered of degree $4-N$ while $\omega^E_{\Gamma_Y Z}$ is filtered of degree $2-N$ and so on. As soon as we express the degree N as a polynomial in X in terms of the weight W however all summands become filtered of degree $2-W$! Neither tracing over g nor conjugating by $j^{\frac{1}{2}}$ spoils this property, because $Wg^{-1} = Ng^{-1}$ and equation (5.1) becomes

$$j^{\frac{1}{2}}\, \nabla^*\nabla\, j^{-\frac{1}{2}} \;=\; \nabla^*\nabla + \frac{\partial}{\partial L} + \omega^E_L + \frac{1}{2}\operatorname{div} L + \frac{1}{4} g(\, L,\, L\,)$$

with $L := \operatorname{grad}_g \log j$ and $NL = (W-1)L$, the metric divergence $\operatorname{div}_g L = L(\log j) + \operatorname{div} L$ is replaced by the flat divergence div using the definition $\mathrm{vol}_g = j\mathrm{vol}$ of the Jacobian. Reconsidering the preceeding proof in light of this observation one readily concludes:

Remark 5.4 (Weight of the Heat Kernel Coefficients)
For every given $r \geq 0$ the class $[a_d]$ of the d–th heat kernel coefficient a_d, $d \geq 0$, in the quotient $\mathbb{F}^{2d}(\operatorname{Sym} T^* \otimes \mathcal{U}\mathfrak{hol}^E)/\mathbb{F}^{2d-r-1}(\operatorname{Sym} T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ is a polynomial of weight $\leq r$ in the Riemannian curvature tensor R and the curvature R^E of the connection on E together with their derivatives, where $\nabla^k R^E$ and $\nabla^k R$ are assigned the weights k and $k+2$ respectively.

Example 5.5 (Connection Laplacian for Flat Connections [Gi])
Consider the connection Laplacian $\nabla^*\nabla$ for a flat connection ∇. In this case the holonomy algebra $\mathfrak{hol} = \{0\}$ is trivial with $\mathcal{U}\mathfrak{hol} = \mathbb{R}$ and the definition of the filtration reduces to $\mathbb{F}^r(\operatorname{Sym} T^*) := \operatorname{Sym}^{\geq -r} T^*$. Consequently $a_d(0) = [a_d] \in \mathbb{F}^{2d}(\operatorname{Sym} T^*)/\mathbb{F}^{-1}(\operatorname{Sym} T^*)$ is a polynomial of weight $\leq 2d$ in the Riemannian curvature R and its covariant derivatives.

Let us turn to another class of generalized Laplacians which is perhaps even more interesting than connection Laplacians namely squares D^2 of twisted Dirac operators acting on sections of Clifford bundles. Barring intricacies arising from the representation theory of Clifford algebras the characteristic feature of a Clifford bundle $E \times T$ is that it is tensor product

$$E \times T \;\cong\; (\, \$ \otimes \mathrm{Hom}_{\mathrm{Cl}\,T}(\,\$\,, E)\,) \times T$$
$$E \times T \;\cong\; \big(\$^+ \otimes \mathrm{Hom}_{\mathrm{Cl}\,T}(\$^+, E) \oplus \$^- \otimes \mathrm{Hom}_{\mathrm{Cl}\,T}(\$^-, E)\big) \times T$$

endowed with a product connection $\omega^E = \omega \otimes \mathrm{id} + \mathrm{id} \otimes \omega^{\mathrm{twist}}$, where \$ and $\$^\pm$ are the spinor representations of the Clifford algebra $\mathrm{Cl}\,T$ of the euclidian vector space T of even and odd dimension respectively and $\mathrm{Hom}_{\mathrm{Cl}\,T}(\$, E)$ or $\mathrm{Hom}_{\mathrm{Cl}\,T}(\$^\pm, E)$ are called the twists. Whereas the spin connection ω couples to Riemannian geometry and can be identified with the Levi–Civita connection of the symbol metric g, we have no further information about the twisting connection ω^{twist} in general. Consequently the holonomy algebra $\mathfrak{hol}^E$ of the connection on

$E \times T$ is a subalgebra of the direct sum $\mathfrak{hol} \oplus \mathfrak{hol}^{\text{twist}}$ of the Riemannian holonomy algebra $\mathfrak{hol}$ and the holonomy algebra $\mathfrak{hol}^{\text{twist}}$ of the twisting connection. Its universal enveloping algebra $\mathcal{U}\mathfrak{hol}^E$ is thus a subalgebra of the product $\mathcal{U}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\text{twist}}$.

Strictly speaking the tensor product decomposition given above is valid only for modules over the Clifford algebra $\mathrm{Cl}\,T$ of the euclidian vector space T and its scalar product $\langle\,,\rangle$ whereas a Clifford bundle $E \times T$ in differential geometry comes equipped with a Clifford multiplication $\bullet_g : (T \otimes E) \times T \longrightarrow E \times T$ satisfying the Clifford relations for the metric g. The fiber $E \times \{X\}$ of such a Clifford bundle at a point $X \in T$ however can be turned into a module over $\mathrm{Cl}\,T$ by setting $Y\bullet := \Phi(X) Y \bullet_g$. Conversely given a module E over $\mathrm{Cl}\,T$ the vector bundle $E \times T$ becomes a Clifford bundle via $Y\bullet_g = \Phi^{-1} Y \bullet$ without modifying the tensor product decomposition. The general Weitzenböck formula for twisted Dirac operators

$$D^2 \;=\; \nabla^*\nabla \;+\; \frac{\kappa}{4} \;+\; \sum_{\mu<\nu} X_\mu \bullet_g X_\nu \bullet_g R^{\text{twist}}_{X_\mu, X_\nu} \tag{5.3}$$

written covariantly for a local orthonormal base $X_1, \ldots, X_n$ for the symbol metric g with its Clifford multiplication $\bullet_g$ and its scalar curvature κ reads

$$D^2 \;=\; \nabla^*\nabla \;+\; \frac{\kappa}{4}\,\mathrm{id} \otimes \mathrm{id} \;+\; 2\sum_{\mu<\nu} (x_\mu \wedge x_\nu) \star \otimes R^{\text{twist}}_{\Phi x_\mu, \Phi x_\nu}$$

in terms of tensor product decomposition and the local orthonormal base $\Phi x_1, \ldots, \Phi x_n$ where $(Y \wedge Z)\star = \frac{1}{2}(Y \bullet Z \bullet + \langle Y, Z\rangle)$ is the infinitesimal representation of the Lie algebra $\mathfrak{so}\,T$ on the spinor factor $\$$ or $\$^{\pm}$. Evidently the additional curvature term will not be an element of $\mathcal{U}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\text{twist}}$ unless the form part of the twisting curvature R^{twist} takes values in the Riemannian holonomy algebra $\mathfrak{hol}$. This restriction is not really a problem as we can always replace $\mathfrak{hol}$ by $\mathfrak{so}\,T \cong \Lambda^2 T$, but it distinguishes a preferred class of twists adapted to the Riemannian holonomy group. On a Kähler manifold the preferred twists are exactly the hermitean holomorphic bundles whereas self–dual bundles are preferred on hyperkähler manifolds. Presumably this condition has not yet been considered for the exceptional Riemannian holonomy groups $\mathbf{Spin}_7$ and $\mathbf{G}_2$.

With respect to the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ on $\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E$ the additional curvature term above is quadratic in $\mathfrak{hol}^E$ and thus filtered of degree 4 so that there is no way to apply Theorem 5.3 directly. Remedy however is easily found in taking the tensor product structure of $\mathcal{U}\mathfrak{hol}^E$ into account. In fact we can think of the heat kernel coefficients a_d, $d \geq 0$, as functions on T with values in $\mathcal{U}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\text{twist}}$. This additional structure allows us to twist the filtration by setting:

$$\mathbb{F}^r(\,\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\text{twist}}\,) \;:=\; \bigoplus_{l \geq 0} \mathrm{Sym}^{\,l} T^* \otimes \mathcal{U}^{\leq \lfloor \frac{l+r}{2} \rfloor}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\text{twist}}$$

In particular the troublesome curvature term is linear in $\mathfrak{hol}$ and thus filtered of degree 2 with respect to the twisted filtration. This filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\text{twist}})$ has another advantage in that the connection form ω^E of the connection on $E \times T$ is now the sum of the spin connection form ω and its twisting counterpart ω^{twist}. For the term $\sum_{\mu\nu} g^{\mu\nu} \omega^E_{x_\mu} \frac{\partial}{\partial x_\nu}$ considered before we thus get the sum of $\sum_{\mu\nu} g^{\mu\nu} \omega_{x_\mu} \frac{\partial}{\partial x_\mu}$ depending only the spin connection form ω and another term involving ω^{twist} which is filtered of degree 0. Omitting the tensor product sign for the natural inclusions of $\mathcal{U}\mathfrak{hol}$ and $\mathcal{U}\mathfrak{hol}^{\text{twist}}$ into $\mathcal{U}\mathfrak{hol}^E$ for convenience and arguing as before we find

$$\begin{aligned} D^2 \;\equiv\; -\sum_\mu \frac{\partial^2}{\partial x_\mu^2} \;-\; & \sum_\mu \Big(R_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} \;+\; \frac{2}{3}(\nabla.R)_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} \;+\; \frac{1}{3}(\nabla_{x_\mu}R)_{\cdot,x_\mu} \Big) \\ -\; & \sum_\mu \Big(\frac{1}{4}R_{\cdot,x_\mu}R_{\cdot,x_\mu} \;+\; \frac{1}{6}R_{\cdot,x_\mu}(\nabla.R)_{\cdot,x_\mu} \;+\; \frac{1}{6}(\nabla.R)_{\cdot,x_\mu}R_{\cdot,x_\mu} \Big) \\ +\; & \sum_{\mu<\nu} \Big(2\,(x_\mu\wedge x_\nu)\,R^{\text{twist}}_{x_\mu,x_\nu} \;+\; 2\,(x_\mu\wedge x_\nu)\,(\nabla.R^{\text{twist}})_{x_\mu,x_\nu} \Big) \end{aligned} \tag{5.4}$$

modulo differential operators filtered of degree 0. Note that the scalar curvature term $\frac{\kappa}{4}$ is filtered of degree 0 and thus does not occur explicitly on the right hand side. Evidently the square D^2 of a twisted Dirac operator is a differential operator filtered of degree 2. Mutatis mutandis the proof of Theorem 5.3 applies and we conclude:

Theorem 5.6 (Regularity of the Supertrace)
The heat kernel coefficients a_d, $d \geq 0$, of the square D^2 of a twisted Dirac operator with twisting holonomy $\mathfrak{hol}^{\text{twist}}$ belong to $\mathbb{F}^{2d}(\,\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\text{twist}}\,)$. In other words

$$a_d \;\in\; \bigoplus_{l\geq 0} \mathrm{Sym}^{\,l}T^* \,\otimes\, \mathcal{U}^{\leq d+\lfloor\frac{l}{2}\rfloor}\mathfrak{hol} \,\otimes\, \mathcal{U}\mathfrak{hol}^{\text{twist}}$$

and the value $a_d(0)$ of the d–th coefficient along the diagonal lives in $\mathcal{U}^{\leq d}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\text{twist}}$. Moreover its symbol class in $\mathbb{F}^{2d}(\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\text{twist}})/\mathbb{F}^{2d-2}(\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\text{twist}})$ only depends on the Taylor series of D^2 up to operators filtered of degree 0.

Contrary to the connection Laplacians considered before however there is no convincingly defined notion of weight allowing a generalization of Remark 5.4 to squares of twisted Dirac operators. Essentially the problem is that the Taylor series of the curvature $d\omega^E+\omega^E\wedge\omega^E$ of the connection on the bundle $E\times T$ in the origin is a sum of two terms satisfying

$$W\,d\omega^E \;=\; N\,d\omega^E \qquad\qquad W\,(\,\omega^E\wedge\omega^E\,) \;=\; (N-2)\,(\,\omega^E\wedge\omega^E\,)$$

and it is hard to believe that any adjustment can possibly reconsile this contradiction. Nevertheless a direct consequence of Theorem 5.6 is the regularity of the supertrace of the heat kernel for a twisted Dirac operator as time goes to zero. In fact the graded character $\mathrm{tr}_{\$^+-\$^-}$ of the spinor representation \$ in even dimension and the characters $\mathrm{tr}_{\$^\pm}$ of the two inequivalent spinor representations $\$^\pm$ in odd dimension have a very characteristic property:

Lemma 5.7 (Character of the Spinor Representation)
Consider the Lie algebra $\mathfrak{so}\,T\cong\Lambda^2T$ of the orthogonal group of an oriented euclidian vector space T of dimension n. In even dimensions n the unit volume $\mathrm{Vol}\in\Lambda^nT$ associated to the orientation provides a means to distinguish positive and negative half spinors. The graded character $\mathrm{tr}_{\$^+-\$^-}$ of the spinor representation $\$\;=\;\$^+\oplus\$^-$ vanishes on $\mathcal{U}^{<\frac{n}{2}}\mathfrak{so}\,T$ and satisfies

$$\mathrm{tr}_{\$^+-\$^-}(\,\mathfrak{X}_1\,\ldots\,\mathfrak{X}_{\frac{n}{2}}\,)\,\mathrm{Vol} \;=\; i^{-\frac{n}{2}}\,\mathfrak{X}_1\wedge\mathfrak{X}_2\wedge\ldots\wedge\mathfrak{X}_{\frac{n}{2}}$$

for all $\mathfrak{X}_1, \ldots, \mathfrak{X}_{\frac{n}{2}} \in \mathfrak{so}\,T$. *Similarly in odd dimensions* n *the unit volume* $\mathrm{Vol} \in \Lambda^n T$ *of the given orientation on* T *distinguishes between the two inequivalent spinor representations* $\$^\pm$. *Associated to their characters* $\mathrm{tr}_{\$^\pm}$ *are bilinear forms* $\mathrm{tr}_{\$^\pm}(\, X \bullet \mathfrak{X}\,)$ *for* $X \in T$, $\mathfrak{X} \in \mathcal{U}\mathfrak{so}\,T$, *which vanish on* $\mathcal{U}^{<\frac{n-1}{2}}\mathfrak{so}\,T$ *while for all* $X \in T$ *and* $\mathfrak{X}_1, \ldots, \mathfrak{X}_{\frac{n-1}{2}} \in \mathfrak{so}\,T$ *we find:*

$$\mathrm{tr}_{\$^\pm}(\, X \bullet \mathfrak{X}_1 \ldots \mathfrak{X}_{\frac{n-1}{2}}\,)\,\mathrm{Vol} \;=\; \pm\, i^{-\frac{n+1}{2}}\, X \wedge \mathfrak{X}_1 \wedge \mathfrak{X}_2 \wedge \ldots \wedge \mathfrak{X}_{\frac{n-1}{2}}$$

Proof: By definition of the spinor representation the element $e^{\mathfrak{X}}$, $\mathfrak{X} \in \mathfrak{so}\,T$, of a suitable completion of $\mathcal{U}\,\mathfrak{so}\,T$ acts by Clifford multiplication with $e^{\frac{1}{2}\mathfrak{X}} \in \mathrm{Cl}\,T$. In order to exponentiate $\mathfrak{X} \in \mathfrak{so}\,T \cong \Lambda^2 T$ in the Clifford algebra it is convenient to write it as

$$\mathfrak{X} \;=\; x_1\, p_1 \wedge q_1 \;+\; \ldots \;+\; x_k\, p_k \wedge q_k$$

with orthonormal vectors $p_1, q_1, \ldots, p_k, q_k \in T$ and constants $x_1, \ldots, x_k \in \mathbb{R}$. The different $p_r \wedge q_r = p_r\, q_r$ are Clifford commuting square roots of -1 and so by Euler's formula

$$e^{\frac{1}{2}\mathfrak{X}} \;=\; \prod_r \Big[\, \cos\frac{x_r}{2} \;+\; \sin\frac{x_r}{2}\, p_r\, q_r \,\Big] \;=\; \prod_r \cos\frac{x_r}{2} \;+\; \ldots \;+\; \prod_r \sin\frac{x_r}{2}\, p_1\, q_1 \,\ldots\, p_k\, q_k$$

where the omitted terms are even elements of $\mathrm{Cl}\,T$ of intermediate degree. In even dimensions n we may assume $k = \frac{n}{2}$ without loss of generality while $p_1, q_1, \ldots, p_{\frac{n}{2}}, q_{\frac{n}{2}}$ is an oriented orthonormal base. In these dimensions only the multiples of the unit volume in the Clifford algebra $\mathrm{Cl}\,T$ have non–vanishing graded trace on $\$$ and by convention $\mathrm{Vol} = p_1\, q_1 \ldots p_{\frac{n}{2}}\, q_{\frac{n}{2}}$ acts as $\pm i^{-\frac{n}{2}}$ on the half spinor representations $\$^\pm$. In particular we conclude:

$$\mathrm{tr}_{\$^+ - \$^-}(\, e^{\mathfrak{X}}\,) \;=\; \prod_r \sin\frac{x_r}{2}\; \mathrm{tr}_{\$^+ - \$^-}(\, \mathrm{Vol}\,) \;=\; 2^{\frac{n}{2}}\, i^{-\frac{n}{2}} \prod_r \sin\frac{x_r}{2}$$

Separating the homogeneous terms of this equation results in the formulas

$$\mathrm{tr}_{\$^+ - \$^-}(\, \frac{1}{r!}\mathfrak{X}^r\,) \;=\; \begin{cases} 0 & \text{if } r < \frac{n}{2} \\ i^{-\frac{n}{2}}\, \mathrm{Pf}(\mathfrak{X}) & \text{if } r = \frac{n}{2} \end{cases} \tag{5.5}$$

where $\mathrm{Pf}(\mathfrak{X}) := \prod_r x_r$ is the Pfaffian of $\mathfrak{X}$. Recall that the filtration on the universal enveloping algebra is defined in such a way that $\mathcal{U}^{<k}\mathfrak{so}\,T$ is spanned by elements of the form $\frac{1}{r!}\mathfrak{X}^r$ with $0 \leq r < k$. Hence the character $\mathrm{tr}_{\$^+ - \$^-}$ vanishes on $\mathcal{U}^{<\frac{n}{2}}\mathfrak{so}\,T$. The second statement follows via polarization from the more customary definition of the Pfaffian:

$$\mathrm{tr}_{\$^+ - \$^-}(\, \frac{1}{(\frac{n}{2})!}\, \mathfrak{X}^{\frac{n}{2}}\,)\,\mathrm{Vol} \;=\; i^{-\frac{n}{2}}\, \mathrm{Pf}(\,\mathfrak{X}\,)\,\mathrm{Vol} \;:=\; i^{-\frac{n}{2}}\, \frac{1}{(\frac{n}{2})!}\, \mathfrak{X} \wedge \mathfrak{X} \wedge \ldots \wedge \mathfrak{X}$$

In odd dimensions n we may assume similarly $k = \frac{n-1}{2}$ and can extend $p_1, q_1, \ldots, p_k, q_k$ to an oriented orthonormal base $z, p_1, q_1, \ldots, p_{\frac{n-1}{2}}, q_{\frac{n-1}{2}}$. However the only odd elements in the Clifford algebra $\mathrm{Cl}\,T$ of non–vanishing trace on $\$^\pm$ are again the multiples of the unit volume Vol acting by convention as $\pm i^{-\frac{n+1}{2}}$ on $\$^\pm$. Observing that the element $X \bullet e^{\frac{1}{2}\mathfrak{X}}$ is odd and arguing as before we conclude that the bilinear form $\mathrm{tr}_{\$^\pm}(X \bullet \mathfrak{X})$ vanishes on $\mathcal{U}^{<\frac{n-1}{2}}\mathfrak{so}\,T$ and verify the stated formula via polarization. $\square$

A different argument to the same end somewhat closer in spirit to Getzler's approach uses the filtration of the Clifford algebra directly instead of the filtration of $\mathcal{U}\mathfrak{so}\,T$. All $\mathfrak{X} \in \mathfrak{so}\,T$ act by elements of filtration degree 2 in the Clifford algebra, but only elements of top degree have non–vanishing graded trace on the spinor representation \$ in even dimensions n while in odd dimensions n the only odd elements of non–vanishing trace on $\$^{\pm}$ are again of top degree. Using the universal enveloping algebra instead of the Clifford algebra however avoids the problem that the filtration becomes trivial in degrees greater n and offers the possibility to use other interesting representations or equivalently characters. In general the vanishing order of the character tr_E of a representation E in the filtration of $\mathcal{U}\mathfrak{so}\,T$ is precisely the minimal order $d \geq 0$ of a Casimir $\mathfrak{X} \in \mathrm{Zent}^{\leq d}\mathcal{U}\mathfrak{so}\,T$ with non–vanishing trace on E. In particular it is possible to construct characters with arbitrarily high vanishing order, of course the associated representations are rather complicated. Say the spinor representation above has both $\$^+$ and $\$^-$ irreducible of the same dimension and with the same Casimir eigenvalues for all generators of the center of $\mathcal{U}\mathfrak{so}\,T$ except for the Pfaffian of degree $\frac{n}{2}$. The future will show whether interesting polynomials in the curvature tensor R can be proved to integrate to zero over M by using characters with vanishing order even higher than $\frac{n}{2}$.

Chapter 6

Symbols of the Heat Kernel Coefficients

Proving the regularity of the trace of the heat kernel as time goes to zero is not sufficient to prove the Local Index Theorem of course. In this section we will complete the proof of the Local Index Theorem begun in the previous section by calculating the even part of the symbol class of the heat kernel coefficients a_d, $d \geq 0$, for the connection Laplacian $\nabla^*\nabla$ and squares D^2 of twisted Dirac operators. Of fundamental importance is a proper understanding of how the form and the endomorphism part of the Riemannian curvature tensor R exchange roles in a mirror version of the classical Chern–Weil construction of characteristic classes.

In Chern–Weil theory the form factor of the curvature tensor R^E produces a differential form under the wedge product while we use a character to reduce the endomorphism factor to a number. The mirror version on the other hand uses the form factor of the curvature tensor R^E to construct the heat kernel coefficients while the character of the representation defining the bundle EM reduces the endomorphism factor to a number which has to be multiplied by the Riemannian volume form to produce a differential form of top degree.

Evidently Chern–Weil theory and its mirror version will result in quite different differential forms when applied to general bundles EM. However for twisted spinor bundles the relevant part of the curvature tensor is essentially the Riemannian curvature tensor R, which is completely symmetric in the form and endomorphism factor, whereas the character of the spinor representation essentially reproduces the wedge product so that we eventually end up with the same differential form. Let us consider the $\widehat{A}$–class as a particularly interesting example and sketch the two different interpretations of the formula (6.1) involved.

Remark 6.1 (Special Logarithms and $\widehat{A}$–Class)
The relation $\log \frac{z}{F(z)} = -(z\frac{d}{dz})^{-1}(z\frac{F'(z)}{F(z)} - 1)$ *for the logarithmic derivative of a formal power series* $F(z) = z + O(z^2)$ *is very convenient to rewrite the definition of the* $\widehat{A}$*–class as:*

$$\widehat{A}(M) \;:=\; \det{}^{\frac{1}{2}}\Big(\frac{\frac{R}{4\pi i}}{\sinh\frac{R}{4\pi i}}\Big) \;=\; \exp\Big(\frac{1}{2}\,\mathrm{tr}\,\log\frac{tR}{\sinh tR}\Big)\Big|_{t=\frac{1}{4\pi i}}$$
$$= \exp\Big(-\frac{1}{2}\,(t\frac{d}{dt})^{-1}\,\mathrm{tr}\Big(\frac{tR}{\tanh tR} - 1\Big)\Big)\Big|_{t=\frac{1}{4\pi i}} \qquad (6.1)$$

The proper interpretation of the latter expression for the $\widehat{A}$–class in Chern–Weil theory is to think of the curvature tensor $R \in \Lambda^2T^* \otimes \mathrm{End}\,T$ as an element of the algebra $\Lambda T^* \otimes \mathrm{End}\,T$ and so formal power series in tR can be evaluated in $\Lambda T^* \otimes \mathrm{End}\,T[\![t]\!]$. Applying the trace tr to the endomorphism factor we reduce the resulting element of $\Lambda T^* \otimes \mathrm{End}\,T[\![t]\!]$ to a differential form which we need to exponentiate in the exterior algebra $\Lambda T^*[\![t]\!]$. In this calculation every t comes along with a 2–form so that the final evaluation at $t = \frac{1}{4\pi i}$ will effectively multiply the top degree piece with $(4\pi i)^{-\frac{1}{2}\dim T}$. The interested reader will notice that the formula above represents the $\widehat{A}$–class in terms of the Chern character of the tangent bundle.

In the mirror version of Chern–Weil theory on the other hand the expression above for the $\widehat{A}$–class is not interpreted as a differential form at all. Instead we think of the forms $\Lambda^2T^* \subset \mathrm{End}\ T$ as skew endomorphisms on T making the curvature tensor R an element of the algebra $\mathrm{End}\ T \otimes \mathcal{U}\mathfrak{hol}$ so that formal power series in tR have a well–defined interpretation. Once the trace tr over T of the resulting element of $\mathrm{End}\ T \otimes \mathcal{U}\mathfrak{hol}[\![t]\!]$ is taken it can be exponentiated in the algebra $\mathcal{U}\mathfrak{hol}[\![t]\!]$ returning the generating power series of the symbols

$$[\ a_d(0)\ ] \quad \in \quad \mathcal{U}^{\leq d}\mathfrak{hol}^E/\mathcal{U}^{\leq d-1}\mathfrak{hol}^E$$

of the values of the universal heat kernel coefficients a_d, $d \geq 0$, along the diagonal. The universal heat kernel coefficients become the real heat kernel coefficients in every representation, in particular the pointwise supertrace of the heat kernel coefficient a_d of the square of the Dirac operator on the spinor bundle is just the value of the character $\mathrm{tr}_{\$^+-\$^-}$ of the $\mathbb{Z}_2$–graded spinor representation on a_d. Pairing the coefficient of $t^{\frac{\dim M}{2}}$ of the generating formal power series (6.1) with this character and multiplying by the Riemannian volume form $(4\pi)^{-\frac{\dim M}{2}}\,\mathrm{vol}_g$ we get the index density of the Local Index Theorem.

Before we begin discussing the details of the arguments sketched above we want to justify the expansions of the connection Laplacian $\nabla^*\nabla$ and of the squares D^2 of twisted Dirac operators in equations (5.2) and (5.4) respectively modulo operators filtered of degree 0, although the expansion modulo operators filtered of degree 1 would actually be sufficient to prove the Local Index Theorem. The graded algebra associated to the filtration on the universal enveloping algebra $\mathcal{U}\mathfrak{hol}^E$ is the commutative algebra $\mathrm{Sym}\,\mathfrak{hol}^E$. With this in mind the standard short exact sequence associated to every filtration $\mathbb{F}^\bullet$

$$0 \longrightarrow \mathbb{F}^{2d-1}/_{\mathbb{F}^{2d-2}} \longrightarrow \mathbb{F}^{2d}/_{\mathbb{F}^{2d-2}} \longrightarrow \mathbb{F}^{2d}/_{\mathbb{F}^{2d-1}} \longrightarrow 0$$

becomes for the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ introduced in Definition 5.2:

$$0 \longrightarrow \bigoplus_{\substack{l\geq 0\\ l\ \mathrm{odd}}} \mathrm{Sym}^{\,l}T^* \otimes \mathrm{Sym}^{\,\frac{l-1}{2}+d}\mathfrak{hol}^E \longrightarrow \bigoplus_{l\geq 0} \mathrm{Sym}^{\,l}T^* \otimes \mathrm{Sym}^{\,\lfloor\frac{l}{2}\rfloor+d}\mathfrak{hol}^E \longrightarrow \qquad (6.2)$$
$$\longrightarrow \bigoplus_{\substack{l\geq 0\\ l\ \mathrm{even}}} \mathrm{Sym}^{\,l}T^* \otimes \mathrm{Sym}^{\,\frac{l}{2}+d}\mathfrak{hol}^E \longrightarrow 0$$

Restricting our ambitions to calculating only the symbol classes of the universal heat kernel coefficients $[a_d] \in \mathbb{F}^{2d}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)/\mathbb{F}^{2d-2}(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E) \cong \mathrm{Sym}\,T^* \otimes \mathrm{Sym}\,\mathfrak{hol}^E$ we get rid of virtually all problems stemming from the non–commutativity of endomorphisms. In this specific sense the symbol class $[a_d] \in \mathrm{Sym}\,T^* \otimes \mathrm{Sym}\,\mathfrak{hol}^E$ lives in the maximal quotient of $\mathbb{F}^{2d}(\mathrm{Sym}\, T^* \otimes \mathcal{U}\mathfrak{hol}^E)$ allowing us to replace the non–commutative multiplication of $\mathcal{U}\mathfrak{hol}^E$ with the commutative multiplication in Sym $\mathfrak{hol}^E$ in doing the recursion (2.8) simplifying calculations drastically. Looking once more at the sequence (6.2) we note that the even part of the symbol class $[a_d]$ as a formal power series on T depends on $\nabla^*\nabla$ only up to operators filtered of degree 1 and hence is a polynomial in the curvature R^E of the bundle EM.

At this point let us discuss a more subtle point in solving the recursion formula (2.8) of Minakshisundaram–Pleijel explicitly for the symbol classes $[a_d]$ of the connection Laplacian $\nabla^*\nabla$ and squares D^2 of twisted Dirac operators. The problem is that it is not completely self–evident how precisely the operator $j^{\frac{1}{2}}\Delta^* j^{-\frac{1}{2}}$ appearing in the recursion formula (2.8) acts on the heat kernel coefficient $a_d \in C^\infty(T, \mathrm{End}\ E) \cong C^\infty(T, E^*) \otimes E$ thought of as a section of the trivial bundle $E^* \times T$ keeping E fixed. Interestingly there are two logically independent arguments in this matter and their equivalence leads to a consistency condition on the Taylor expansion of Δ in the chosen trivialization, which is already non–trivial in the expansion (5.2) of $\nabla^*\nabla$.

In a first line of reasoning we use the fact that the differential operator $(\nabla^*\nabla)^*$ is the connection Laplacian on the dual bundle $E^* \times T$ with dual connection compare (2.4). The gauge fixing conditions originally imposed on $E \times T$ are clearly valid for $E^* \times T$ as well, the only way to make the canonical pairing parallel is to have parallel transport along radial geodesics $t \longmapsto tX$ being the identity $E^* \times \{0\} \longrightarrow E^* \times \{X\}$. Essentially this is good news allowing us to use expansion (5.2) with the connection form of E^*, which is ω^E acting by minus its adjoint on E^* or $\omega^E\eta := -\eta(\omega^E\,\cdot\,)$ for $\eta \in E^*$. Under the isomorphism $E^* \otimes E \cong \mathrm{End}\ E$ this action of $A \in \mathrm{End}\ E$ on E^* keeping E fixed corresponds to right multiplication by $-A$ so that the expansion (5.2) of $\nabla^*\nabla$ turns into the expansion

$$(\nabla^*\nabla)^* \;\equiv\; -\sum_\mu \frac{\partial^2}{\partial x_\mu^2} + \sum_\mu \Big(R^E_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} + \frac{2}{3}(\nabla.R^E)_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} + \frac{1}{3}(\nabla_{x_\mu}R^E)_{\cdot,x_\mu}\Big) \qquad (6.3)$$
$$-\sum_\mu \Big(\frac{1}{4}R^E_{\cdot,x_\mu}R^E_{\cdot,x_\mu} + \frac{1}{6}R^E_{\cdot,x_\mu}(\nabla.R^E)_{\cdot,x_\mu} + \frac{1}{6}(\nabla.R^E)_{\cdot,x_\mu}R^E_{\cdot,x_\mu}\Big)$$

up to operators filtered of degree 0, where all endomorphisms act by multiplication from the right on the heat kernel coefficients $a_d \in C^\infty(T, \mathrm{End}\ E)$. Alternatively we observe that for a differential operator Q with formal adjoint Q^* with respect to the Riemannian volume form $\mathrm{vol}_g = j\mathrm{vol}$ the operators $j^{\frac{1}{2}}\,Q\,j^{-\frac{1}{2}}$ and $j^{\frac{1}{2}}\,Q^*\,j^{-\frac{1}{2}}$ are formally adjoint with respect to the flat volume form vol, because

$$\int_T \langle (\,j^{\frac{1}{2}}\,Q^*\,j^{-\frac{1}{2}}\,)\,\eta, \psi\rangle\,\mathrm{vol} \;=\; \int_T \langle Q^*\,(\,j^{-\frac{1}{2}}\eta\,), (\,j^{-\frac{1}{2}}\psi\,)\rangle\, j\,\mathrm{vol}$$

for any two sections $\eta \in C^\infty_0(T, E^*)$ and $\psi \in C^\infty_0(T, E)$. Calculating the formal adjoint with respect to the flat volume form is easy, simply reverse the order of all operators and switch $\frac{\partial}{\partial x_\mu}$ to $-\frac{\partial}{\partial x_\mu}$. Of course the isomorphism $E^* \otimes E \cong \mathrm{End}\ E$ still requires us to have all endomorphisms in (5.2) acting from the right. Recalling the congruence $\nabla^*\nabla \equiv j^{\frac{1}{2}}\nabla^*\nabla j^{-\frac{1}{2}}$ up to operators filtered of degree 0 we immediately verify the expansion (6.3) because:

$$-\sum_\mu \Big(-\frac{\partial}{\partial x_\mu}\Big)\frac{2}{3}(\nabla.R^E)_{\cdot,x_\mu} \;=\; \sum_\mu \Big(\frac{2}{3}(\nabla.R^E)_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu} + \frac{2}{3}(\nabla_{x_\mu}R^E)_{\cdot,x_\mu}\Big)$$

Additional potentials as they appear e. g. in the expansion (5.4) of squares of twisted Dirac operators simply act by multiplication from the right as well.

According to remarks before the sequence (6.2) the symbol classes $[a_d]$ of the connection Laplacian $\nabla^*\nabla$ can be thought of as formal power series on T with values in $\mathrm{Sym}\,\mathfrak{hol}^E$ as $\mathbb{F}^{2d}(\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}^E)/\mathbb{F}^{2d-2}(\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}^E)\;\cong\;\bigoplus_{l\geq 0}\mathrm{Sym}^{\,l}T^*\otimes\mathrm{Sym}^{\,\lfloor\frac{l}{2}\rfloor+d}\mathfrak{hol}^E$. Eventually we are thus interested in polynomials on T with values in $\mathrm{Sym}\,\mathfrak{hol}^E$. For the moment however let us consider bilinear forms $B\;\in\;(T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}^E$ on T with values in $\mathrm{Sym}\,\mathfrak{hol}^E$ instead, although only the symmetric part of such a form defines a (quadratic) polynomial on T. The special element $R^E\;\in\;\Lambda^2T^*\otimes\mathfrak{hol}^E\;\subset\;(T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}^E$ suggests to use the notation $B_{X,Y}\;\in\;\mathrm{Sym}\,\mathfrak{hol}^E$ in general for these bilinear forms. Besides the trace $\mathrm{tr}\,B\;:=\;\sum_\mu B_{x_\mu,x_\mu}$ with an orthonormal base $\{x_\mu\}$ of T we have a multiplication defined by

$$(\,B\,\tilde{B}\,)_{X,Y}\;\;:=\;\;\sum_\mu B_{X,x_\mu}\,\tilde{B}_{x_\mu,Y}\tag{6.4}$$

for any two elements $B,\,\tilde{B}\;\in\;(T^*\otimes T^*)\otimes\mathrm{Sym}\;\mathfrak{hol}^E$. Contrary to the sketch in the introduction this multiplication is anti–isomorphic to the multiplication on $\mathrm{End}\,T$ under the metric isomorphism $T^*\otimes T^*\;\cong\;T^*\otimes T$ and in principle we will have to reverse the order of all products in the end, this nuisance however will have no effect at all on the final result.

In case we consider the squares D^2 of twisted Dirac operators we need to use the twisted filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}})$ to get well–defined symbol classes $[a_d]$, which then become formal power series on T with values in $\mathrm{Sym}\,\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$. Both the definition of the trace and the multiplication (6.4) on the bilinear forms $(T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$ continue to make sense in this situation. Note that the twisting curvature $R^{\mathrm{twist}}\;\in\;\Lambda^2T^*\otimes\mathfrak{hol}^{\mathrm{twist}}$ appears as a constant function on T with values in $\mathfrak{hol}\otimes\mathfrak{hol}^{\mathrm{twist}}$ in this context, because we assume that its form part takes values in the holonomy algebra $\mathfrak{hol}\subset\Lambda^2T^*$. The generating series for the even part of the symbols $[a_d]$ can be written in terms of the trace and the multiplication on $(T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}^E$ or $(T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$:

Theorem 6.2 (Symbol Classes of the Heat Kernel Coefficients)
The generating series for the even part of the symbol classes $[a_d]\;\in\;\mathrm{Sym}\,T^*\otimes\mathrm{Sym}\,\mathfrak{hol}^E$ *of the universal heat kernel coefficients of the connection Laplacian considered as a formal power series* $[a(t,\cdot)]^{\mathrm{ev}}\;:=\;\sum_{d\geq 0}t^d[a_d]^{\mathrm{ev}}$ *in* t *with coefficients in* $\mathrm{Sym}^{\,\mathrm{ev}}T^*\otimes\mathrm{Sym}\,\mathfrak{hol}^E$ *reads:*

$$[\,a(t,X)\,]^{\mathrm{ev}}\;\;=\;\;\exp\left(\;-\;\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\frac{tR^E}{\tanh\,tR^E}\;-\;1\Big)\;-\;\frac{1}{4t}\Big(\frac{tR^E}{\tanh\,tR^E}-1\Big)_{X,X}\right)$$

Using similarly the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^*\otimes\mathcal{U}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}})$ *we find the slightly modified result*

$$[\,a(t,X)\,]^{\mathrm{ev}}\;\;=\;\;\exp\left(\;-\;\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\frac{tR}{\tanh\,tR}\;-\;1\Big)\;-\;\frac{1}{4t}\Big(\frac{tR}{\tanh\,tR}-1\Big)_{X,X}\;-\;2tR^{\mathrm{twist}}\right)$$

for squares D^2 *of twisted Dirac operators on Clifford bundles* EM *over* M*.*

Evidently this formula for the even part of the symbol classes is a variant of Mehler's Formula for formal power series on T with values in $\mathrm{Sym}\,\mathfrak{hol}^E$ and is proved in the same way using an ansatz [BGV]. Of course we are lucky to know the final result in advance from other proofs of the Local Index Theorem. Nevertheless the reader should keep in mind that we do not assume that the vector bundle $E \times T$ is a twisted spinor bundle and this difference accounts for the factor $\frac{1}{2}$ apparently missing in the formulas above. Lack of time or possibly determination prevents us from discussing the odd part of the symbol classes $[a_d]$ as polynomials on T.

Proof: Let us focus on the perhaps more interesting squares D^2 of twisted Dirac operators. Mutatis mutandis all arguments presented work equally well for connection Laplacians $\nabla^*\nabla$ and in any case the discussion simplifies in the latter case due to the simpler definition of the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^* \otimes \mathcal{U}\mathfrak{hol}^E)$. Starting with an ansatz

$$[\,a(t,X)\,]^{\mathrm{ev}} \;=\; \exp\left(\;-\;\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\,F(tR)\;-\;\frac{1}{4t}\,F(tR)_{X,X}\;-\;2tR^{\mathrm{twist}}\,\right)$$

with an even formal power series $F(z)$ in one variable z satisfying $F(z) = O(z^2)$ so that $F(tR)$ is defined by multiplication (6.4) and in the image of the $t\frac{d}{dt}$ as an element of the algebra $(T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}[\![t]\!] \subset (T^*\otimes T^*)\otimes\mathrm{Sym}\,\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}}[\![t]\!]$. Note that the coefficient $[a_d(X)]^{\mathrm{ev}}$ of t^d in the formal power series $[a(t,X)]^{\mathrm{ev}}$ is an even polynomial on T of degree $\leq 2d$ with values in $\mathrm{Sym}\,\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$. We want to choose the parameter $F(z)$ of our ansatz so that the sequence of these polynomials satisfy the recursion formula (2.8) of Minakshisundaram–Pleijel. Clearly both $\frac{d}{dt}$ and Euler's number operator N act by derivations or:

$$\begin{aligned}&(\,\frac{d}{dt}+\frac{N}{t}\,)\,[a(t,X)]^{\mathrm{ev}} &(6.5)\\ &=\;[a(t,X)]^{\mathrm{ev}}\left(-\frac{1}{2t}\mathrm{tr}\,F(tR)+\frac{1}{4t^2}F(tR)_{X,X}-\frac{1}{4t}[R\,F'(tR)]_{X,X}-2R^{\mathrm{twist}}-\frac{1}{2t^2}F(tR)_{X,X}\right)\end{aligned}$$

Problems due to non–commutativity could only arise through the additional potential R^{twist}, which clearly commutes with itself and all other relevant elements of $\mathrm{Sym}\,\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$. For the time being we are only interested in the even part of the symbol classes $[a_d]$, hence we can replace the operator $-j^{\frac{1}{2}}D^2j^{\frac{1}{2}}$ in the recursion formula (2.8) by its expansion (5.4)

$$-j^{\frac{1}{2}}\,D^2\,j^{-\frac{1}{2}}\;\;\equiv\;\;\sum_\mu\frac{\partial^2}{\partial x_\mu^2}\;+\;\sum_\mu R_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu}\;+\;\frac{1}{4}\sum_\mu R_{\cdot,x_\mu}R_{\cdot,x_\mu}\;-\;2R^{\mathrm{twist}}$$

modulo operators filtered of degree 1 (sic!), all coefficients act by multiplication from the right on $[a(t,X)]^{\mathrm{ev}}$ and the term $\sum_\mu R_{\cdot,x_\mu}\frac{\partial}{\partial x_\mu}$ switches sign. For any formal power series $F(z)$ we evidently have $F(tR)_{X,Y} = F(-tR)_{Y,X}$ for all $X,Y \in T$ so that $F(tR)_{X,Y}$ is symmetric in X and Y for an even formal power series F. Keeping this in mind we calculate for starters

$$\frac{\partial}{\partial Y}\,[a(t,X)]^{\mathrm{ev}}\;\;=\;\;[a(t,X)]^{\mathrm{ev}}\left(-\frac{1}{2t}F(tR)_{X,Y}\right)$$

to get some feeling for calculating the derivatives of $[a(t,X)]^{\mathrm{ev}}$ with respect to X, in particular

$$\begin{aligned}&(\,-j^{\frac{1}{2}}D^2j^{-\frac{1}{2}}\,)\,[a(t,X)]^{\mathrm{ev}} &(6.6)\\ &=\;[a(t,X)]^{\mathrm{ev}}\left(-\frac{1}{2t}\mathrm{tr}\,F(tR)+\frac{1}{4t^2}F^2(tR)_{X,X}+\frac{1}{2t}F(tR)R_{X,X}-\frac{1}{4}[z^2(R)]_{X,X}-2R^{\mathrm{twist}}\right)\end{aligned}$$

where $\frac{1}{2t}F(tR)R_{X,X} = 0$ vanishes, because the formal power series $F(z)z$ is odd. Comparing equations (6.5) and (6.6) we see that we can make things work by choosing F to satisfy the differential equation $-(z\frac{d}{dz}F)(z) - F(z) = F^2(z) - z^2$. However subject to the constraint $F(z) = O(z^2)$ this differential equation has a unique solution $F(z) = \frac{z}{\tanh z} - 1$. □

Corollary 6.3 (Index Theorem for Twisted Dirac Operators)
The index of a twisted Dirac operator acting on sections of a (complex) Clifford bundle EM over a compact manifold M of even dimension $\dim M$ *is given as the integral*

$$\text{index}\, D \;=\; \int_M \widehat{A}(\,M\,)\,\text{ch}\,\text{Hom}_{\text{Cl}\,TM}(\,\$M,\,EM\,)$$

Proof: According to McKean–Singer's "Fantastic Cancellations" the Dirac operator D defines a spectral symmetry $\Gamma(E^+M) \longrightarrow \Gamma(E^-M)$, $\psi \longmapsto D\psi$, off the kernel between the spectrum of D^2 acting on E^+M and E^-M respectively. Consequently the global supertrace of the heat kernel $k_t^{D^2}$ of D^2, which converges to the index of the Dirac operator as time t goes to ∞, is actually independent of time:

$$\text{index}\, D \;=\; \Big(\lim_{t\longrightarrow\infty}\Big)\int_M \text{tr}_{E^+M-E^-M}\, k_t^{D^2}(\,y,\,y\,)\,\text{vol}(y)$$

This equation is the key idea in all heat equation proofs of the Index Theorem. In particular we can replace the heat kernel $k_t^{D^2}$ by its asymptotic expansion and calculate the index of the Dirac operator as the limit of the global supertrace

$$\text{index}\, D \;=\; \lim_{t\longrightarrow 0^+}\int_M \frac{1}{\sqrt{4\pi t}^{\dim M}}\,\text{tr}_{E^+M-E^-M}\left(\sum_{d\geq 0} t^d\, a_d(y)\right)\text{vol}(y) \qquad (6.7)$$

as time t goes to zero. Localizing at a given point $y \in M$ by setting $T := T_yM$ and $E := E_yM$ while employing the usual gauge fixing conditions we are thus lead to study the supertrace of the formal power series $a(t,X) = \sum_{d\geq 0} t^d a_d(X)$ at the origin $X = 0$. As always in even dimensions this supertrace $\text{tr}_{E^+-E^-}$ splits via the isomorphism of complex Clifford modules $E \cong \$ \otimes \text{Hom}_{\text{Cl}\,T}(\$, E)$ into the product $\text{tr}_{\$^+-\$^-} \otimes \text{tr}_{\text{Hom}_{\text{Cl}\,T}(\$,E)}$ of the characters of the spinor representation and the twist. By Lemma 5.7 however the character of the spinor representation essentially reproduces the algebra homomorphism

$$\mathfrak{Pf}:\ \text{Sym}\,\mathfrak{hol} \longrightarrow \Lambda^{\text{ev}}T^*, \qquad \mathfrak{X}_1\cdot\ldots\cdot\mathfrak{X}_r \longmapsto \mathfrak{X}_1\wedge\ldots\wedge\mathfrak{X}_r$$

extending the inclusion $\mathfrak{hol} \subset \Lambda^2T^*$ in degree $\frac{\dim M}{2}$ up to an additional factor $i^{-\frac{\dim M}{2}}$ or

$$\frac{1}{\sqrt{4\pi t}^{\dim M}}\text{tr}_{E^+-E^-}[a(t,0)]\,\text{vol} \;\equiv\; (4\pi t i)^{-\frac{\dim M}{2}}\,(\,\mathfrak{Pf}\otimes\text{tr}_{\text{Hom}_{\text{Cl}\,T}(\$,E)}\,)([a(t,0)]) \;+\; O(t) \quad (6.8)$$

modulo differential forms of degree strictly less than dim M. In fact the values $a_d(0)$, $d \geq 0$, of the heat kernel coefficients live in $\mathcal{U}^{\leq d}\mathfrak{hol}\otimes\mathcal{U}\mathfrak{hol}^{\text{twist}}$ by Theorem 5.6. Consequently the coefficients $a_d(0)$ with $d < \frac{\dim M}{2}$ do not contribute at all to the left hand side while their classes $[a_d(0)]$ only contribute forms of less than top degree on the right. On the other hand the contributions from the coefficients $a_d(0)$ with $d > \frac{\dim M}{2}$ are all $O(t)$. Strictly speaking equation (6.8) is thus a statement only about the critical coefficient $a_d(0)$ with $d = \frac{\dim M}{2}$ and amounts to a reformulation of Lemma 5.7. In particular only the integral over the critical coefficient has any bearance on the limit in equation (6.7).

With the Pfaffian $\mathfrak{Pf}$ being an algebra homomorphism we can replace the multiplication on $(T^* \otimes T^*) \otimes \mathrm{Sym}\,\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$ used to construct the generating series $[a(t, X)]$ of the symbol classes by a similar multiplication defined on $(T^* \otimes T^*) \otimes \Lambda^{\mathrm{ev}} T^* \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$. Thinking of R and R^{twist} similarly as elements of $(T^* \otimes T^*) \otimes \Lambda^2 T^* \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$ and $\Lambda^{\mathrm{ev}} T^* \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$ respectively and applying Theorem 5.6 we arrive at

$$\begin{aligned}
&(\,\mathfrak{Pf} \otimes \mathrm{tr}_{\mathrm{Hom}_{\mathrm{Cl}\,T}(\$,E)}\,)([a(t,0)]) \\
&\quad= \mathrm{tr}_{\mathrm{Hom}_{\mathrm{Cl}\,T}(\$,E)} \exp\Big(-\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\frac{tR}{\tanh tR} - 1\Big) - 2tR^{\mathrm{twist}}\Big) \\
&\quad= \exp\Big(-\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\frac{tR}{\tanh tR} - 1\Big)\Big) \wedge \mathrm{tr}_{\mathrm{Hom}_{\mathrm{Cl}\,T}(\$,E)} \exp\Big(-2tR^{\mathrm{twist}}\Big)
\end{aligned}$$

where exp now refers to the exponential in $\Lambda^{\mathrm{ev}} T^* \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}$. In this equation every two form comes along with t so that multiplying by $(4\pi t i)^{-\frac{\dim M}{2}}$ as in equation (6.8) amounts to evaluating at $t = \frac{1}{4\pi i}$ on forms of top degree $\dim M$. Under this evalution the second factor turns into the definition of the Chern character of the twist whereas the first factor seems to turn into the $\widehat{A}(M)$–class according to equation (6.1). Recall however that we have used the form part of $R \in \Lambda^2 T^* \otimes \mathfrak{hol}$ already to construct the multiplication. An allusion to the classical symmetries between form and endomorphism in R completes the proof. $\square$

Among the well–known special cases of the Local Index Theorem we want to mention in particular the Theorem of Gauß–Bonnet–Chern, which is the Local Index Theorem for the Clifford bundle $\Lambda^{\mathrm{ev}} T^* M - \Lambda^{\mathrm{odd}} T^* M$ possibly twisted with an auxiliary vector bundle. It is quite remarkable that equation (5.5) originally used to prove the regularity of the supertrace only can be interpreted directly as an identity of rational characteristic classes

$$\mathrm{ch}\,(\$^+ M - \$^- M) = \mathrm{tr}_{\$^+ M - \$^- M}(\,e^{-\frac{R}{2\pi i}}\,) = i^{-\frac{\dim M}{2}}\,\mathrm{Pf}(\,-\frac{R}{2\pi i}\,) = \mathrm{Pf}(\,\frac{R}{2\pi}\,)$$

on an oriented even dimensional manifold M. By construction the Pfaffian is a form of purely top degree so that its product $\widehat{A}(M) \wedge \mathrm{ch}\,(\$^+ M - \$^- M) = \mathrm{Pf}(\frac{R}{2\pi})$ with the $\widehat{A}(M)$–class only picks up the constant term 1 of the latter. On the other hand the classical isomorphism $\mathrm{Cl}\,TM \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}\,\$M \cong \Lambda T^* M$ valid in even dimensions clearly becomes

$$\begin{aligned}
\Lambda^{\mathrm{ev}} T^* M \otimes_{\mathbb{R}} \mathbb{C} - \Lambda^{\mathrm{odd}} T^* M \otimes_{\mathbb{R}} \mathbb{C} &= (\,\$^+ M - \$^- M\,)^* \otimes (\$^+ M - \$^- M\,) \\
&\cong (-1)^{\frac{\dim M}{2}}(\,\$^+ M - \$^- M\,) \otimes (\$^+ M - \$^- M\,)
\end{aligned}$$

because by the classification of Clifford algebras the even subalgebras $\mathrm{Cl}^0 TM$ containing the spin groups are real or quaternionic matrix algebras for $\dim M \equiv 0 \bmod 4$ or $\$^{\pm *} M \cong \$^{\pm} M$ while they are complex matrix algebras for $\dim M \equiv 2 \bmod 4$ resulting in $\$^{\pm *} M \cong \$^{\mp} M$.

Chapter 7

Heat Kernel Coefficients of Generalized Laplacians

It is well known that the squares of twisted Dirac operators are generalized Laplacians with potentials given by curvature terms linking the differential operator directly to the geometry of the Riemannian manifold M, more precisely the form of their potentials is dictated by the Lichnerowicz–Weitzenböck formula. Formalizing this characteristic property we will introduce a class of generalized Laplacians in this section whose potentials will be the most general curvature terms arising from Weitzenböck formulas. The salient feature of twisted Laplacians in the class of all generalized Laplacians is that it is very easy to calculate their powers using iterated covariant derivatives. In turn the general Jet Formula 4.2 allows us to express these iterated covariant derivatives in terms of the Taylor series of sections in normal coordinates with the bundle trivialized by parallel transport along radial geodesics. According to Theorem 2.4 this information is precisely the input we need to calculate the infinite order jet of the heat kernel coefficients along the diagonal explicitly. Pursuing this strategy we will discuss the resulting formula for the heat kernel coefficients as a sum over all jet forests and its subsequent simplifications and ramifications for arbitrary generalized Laplacians in detail in this section.

The class of twisted Laplace operators $\Delta_\diamond$ acting on sections of twisted vector bundles EM over Riemannian manifolds M turns out to contain most interesting generalized Laplacians met in differential geometry. Incidentally many of these twisted Laplacians are really incarnations of a particular twisted Laplacian, the standard Laplacian, whatever their original motivation or definition. In order to define twisted Laplacians and the standard Laplacian let us first fix a viable notation for the application map $\mathrm{app} : \mathrm{End}\, EM \otimes EM \longrightarrow EM$ associated to every vector bundle EM over M.

Definition 7.1 (Twisted Bundles and Twisted Laplacians)
A twisted vector bundle EM is a vector bundle over a Riemannian manifold M together with a connection ∇ and a parallel map $\diamond : \Lambda^2 T^*M \longrightarrow \mathrm{End}\, EM$. The twisted trace

$$\mathrm{tr}_\diamond : \ T^*M \otimes T^*M \longrightarrow \mathrm{End}\, EM, \quad \alpha \otimes \beta \otimes \psi \longmapsto g(\alpha, \beta)\,\mathrm{id} - (\alpha \wedge \beta)\diamond$$

defines a generalized Laplacian $\Delta_\diamond := \mathrm{app} \circ (-\mathrm{tr}_\diamond) \circ \nabla^2$ acting on sections $\Gamma(EM)$ of the twisted vector bundle EM called the twisted Laplacian on EM.

Just as we can twist the spinor bundle on a spin manifold we can twist every twisted vector bundle EM by a coefficient bundle LM endowed with a connection to get a new twisted vector bundle $EM \otimes LM$ simply by having $\Lambda^2 T^*M$ act trivially on the coefficients LM. Every tensor or Clifford bundle on the other hand or more generally every vector bundle EM associated to the (spin) holonomy reduction of M is naturally a twisted vector bundle, because the infinitesimal representation $\star$ of the holonomy algebra defines a twist:

$$\Lambda^2 T^*M \longrightarrow \mathrm{End}\, EM, \quad \alpha \wedge \beta \longmapsto \mathrm{pr}_{\mathfrak{hol}}(\alpha \wedge \beta)\,\star$$

For every parallel subbundle of the forms the corresponding twisted Laplacian $\Delta_\star$ agrees with the Hodge Laplacian while on a symmetric space the operator $\Delta_\star$ is the Casimir of the isometry group for every homogeneous vector bundle. In the same vein the Partharasarty formula for the square of the Dirac operator on symmetric spaces generalizes to

$$D^2 \;=\; \Delta_\star + \frac{\kappa}{8} \;=\; \Delta_{2\star}$$

on the spinor bundle $\$M$ of a spin manifold M. Given its ubiquitousness it seems appropriate to call the twisted Laplacian $\Delta_\star$ the standard Laplacian on the vector bundle EM. Its very existence has deep implications in differential geometry like the strong Lefschetz theorem and most of the known vanishing theorems for Betti numbers in special holonomy.

Definition 7.2 (Standard Laplace Operator)
Consider a vector bundle EM on a Riemannian manifold M associated to the holonomy bundle of M and endowed with the connection ∇ arising from the Levi–Civita connection. The standard Laplacian on EM is the twisted Laplacian

$$\Delta_\star \;=\; \nabla^*\nabla \;+\; \tfrac{1}{2}\sum_{\mu\nu} \mathrm{pr}_{\mathfrak{hol}}(dX_\mu \wedge dX_\nu) \,\star\, R^E_{X_\mu, X_\nu}$$

associated to the parallel map $\star : \;\Lambda^2 T^*M \longrightarrow \mathrm{End}\, EM,\ \alpha \wedge \beta \longmapsto \mathrm{pr}_{\mathfrak{hol}}(\alpha \wedge \beta)\star$ *induced from the infinitesimal representation of the holonomy algebra $\mathfrak{hol}$ on the representation E.*

A general vector bundle EM endowed with a connection can be naturally thought of as a twisted vector bundle with trivial twist $\diamond : \;\Lambda^2 T^*M \longrightarrow \mathrm{End}\, EM$. In this case the twisted Laplacian $\Delta_\diamond$ is simply the connection Laplacian $\nabla^*\nabla$ of course. Turning to the general case we consider the fibered product $\mathrm{Hol}^E M \times_M \mathrm{Hol}\, M$ of the holonomy bundles $\mathrm{Hol}^E M$ and $\mathrm{Hol}\, M$ of EM and M respectively, which is naturally a principal bundle over M modelled on $\mathrm{Hol}^E \times \mathrm{Hol}$ endowed with the direct sum connection $\omega^{\mathrm{total}} := \omega^E \oplus \omega$. The minimal closed principal subbundles tangent to ω^{total} are all conjugated under global gauge transforms and chosing one such bundle $\mathrm{Hol}^{\mathrm{total}} M \subset \mathrm{Hol}^E M \times_M \mathrm{Hol}\, M$ defines the total holonomy group $\mathrm{Hol}^{\mathrm{total}} \subset \mathrm{Hol}^E \times \mathrm{Hol}$. Note that the projection $\mathrm{Hol}^{\mathrm{total}} \longrightarrow \mathrm{Hol}^E$ is surjective, for otherwise Hol^E is not the holonomy group of the connection on EM contrary to assumption. With the same argument the projection $\mathrm{Hol}^{\mathrm{total}} \longrightarrow \mathrm{Hol}$ is surjective, too, so that $\mathrm{Hol}^{\mathrm{total}} \subset \mathrm{Hol}^E \times \mathrm{Hol}$ can be thought of as a group correspondence between Hol^E and Hol. Clearly the parallel twists $\diamond : \;\Lambda^2 T^*M \longrightarrow \mathrm{End}\, EM$ are parametrized by the invariants

$$\diamond \;\in\; [\;\Lambda^2 T \otimes \mathrm{End}\; E\;]^{\mathrm{Hol}^{\mathrm{total}}}$$

in the representation $\Lambda^2 T \otimes \text{End } E$ of $\text{Hol}^{\text{total}}$. Consequently the kernel of the correspondence projection $\text{Hol}^{\text{total}} \longrightarrow \text{Hol}^E$ fixes all bivectors in the image of $\diamond^* : \text{End } E \longrightarrow \Lambda^2 T$. In particular a non–trivial invariant twist can only exist if the kernel of the projection $\text{Hol}^{\text{total}} \longrightarrow \text{Hol}^E$ is rather small so that the other projection $\text{Hol}^{\text{total}} \longrightarrow \text{Hol}$ is close to a surjective group homomorphism $\text{Hol}^E \longrightarrow \text{Hol}$. It is difficult to make this statement any more precise in this generality, but we hope the reader has got some idea about the analysis needed to study the set of all possible twists on a given vector bundle EM.

Twisted Laplacians $\Delta_\diamond$ are distinguished among all generalized Laplacians by the property that all their powers $\Delta_\diamond^k$, $k \geq 0$, can be neatly expressed in terms of iterated covariant derivatives alone. In fact the application map $\text{app} : \text{End } EM \otimes EM \longrightarrow EM$ is parallel by the very definition of the connection on $\text{End } EM$ as is its natural extension to multiple endomorphisms $\overrightarrow{\text{app}} : \bigotimes \text{End } EM \otimes EM \longrightarrow EM$ applied in the usual order with the rightmost one first. A straightforward induction using in addition the parallel powers

$$(-\text{tr}_\diamond)^k : \quad \bigotimes^{2k} T^*M \stackrel{(-\text{tr}_\diamond)^{\otimes k}}{\longrightarrow} \bigotimes^k \text{End } EM$$

of the twisted trace proves the closed formula $\Delta_\diamond^k = \overrightarrow{\text{app}} \circ (-\text{tr}_\diamond)^k \circ \nabla^{2k}$ for all $k \geq 1$. At this point it may seem as if this triviality can't possibly have any bearance on the problem of calculating the heat kernel coefficients explicitly. However the critical input we need in order to calculate the heat kernel coefficients via Theorem 2.4 is the total symbol $\text{ev}[\, e^{-t\Delta_\diamond}\,]^\sharp$, which is defined explicitly in the gauge fixed context of Section 2.

According to Section 2 the total symbol of a differential operator $\mathcal{D}$ acting on sections of a trivial vector bundle $E \times T$ over a euclidian vector space T is the element $\text{ev}[\,\mathcal{D}\,]^\sharp$ of $\text{Sym}\, T^* \otimes \text{End } E$ characterized by $\text{ev}[\,\mathcal{D}\psi\,] = \langle \text{ev}[\,\mathcal{D}\,], \psi \rangle_{\text{Sym}\, T^*}$ for all sections ψ of $E \times T$, where ψ is identified with its infinite order Taylor series $\text{jet}\, \psi \in \text{Sym}\, T^* \otimes E$ on the right and $\langle \cdot, \cdot \rangle_{\text{Sym}\, T^*}$ denotes the standard scalar product on $\text{Sym}\, T^*$ defined via Gram's permanent. Polarizing the characterization $\langle e^\alpha, e^\beta \rangle_{\text{Sym}\, T^*} = e^{\langle \alpha, \beta \rangle}$ of this scalar product with respect to $\beta \in T^*M$ we can easily pin down the total symbol of iterated partial derivatives

$$\text{ev}[\, \frac{\partial^k}{\partial X_1 \ldots \partial X_k} e^\alpha \,] \quad = \quad \alpha(X_1) \cdot \ldots \cdot \alpha(X_k) \quad = \quad \langle\, e^\alpha, X_1^\sharp \cdot \ldots \cdot X_k^\sharp \,\rangle_{\text{Sym}\, T^*}$$

for all $X_1, \ldots, X_k \in T$ and all $\alpha \in T^*$. The conventions concerning the relation between polynomials and symmetric forms discussed in Section 4 make this equation read:

$$\text{jet}\, e^\alpha (\, X_1 \cdot \ldots \cdot X_k \,) \quad = \quad \langle\, e^\alpha, X_1^\sharp \cdot \ldots \cdot X_k^\sharp \,\rangle_{\text{Sym}\, T^*}$$

In other words the total symbol of the differential operator $\psi \longmapsto \text{jet}\, \psi(X_1 \cdot \ldots \cdot X_k)$ acting on sections ψ of a trivial vector bundle $E \times T$ is simply given by $X_1^\sharp \cdot \ldots \cdot X_k^\sharp \otimes \text{id}_E$. In order to translate the gauge fixed context of Section 2 into covariant language we have to fix a point $y \in M$ and define $T := T_yM$ and $E := E_yM$. Taking normal coordinates about y and trivializing the bundle EM using parallel transport along radial geodesics we can then make sense out of $\text{jet}\, \psi \in \text{Sym}\, T_y^*M \otimes E_yM$ for a section $\psi \in \Gamma(EM)$. However the central argument (3.2) in the proof of the special Jet Formula tells us that $\text{jet}\, \psi$ agrees with the symmetrization of the iterated covariant derivatives of ψ. Put differently the general Jet Formula 4.2 is essentially a formula for the total symbol $\text{ev}[\,\nabla^k\,]^\sharp$ of ∇^k, $k \geq 1$.

Recall that in Section 4 we have specified inductive rules to turn a given decoration of the leaves of a jet forest $\mathfrak{F}$ of order $k \geq 1$ with vector fields $X_1, \ldots, X_k$ into a decoration of all black or white vertices of $\mathfrak{F}$ with endomorphisms on EM or vector fields on M respectively. Namely given decorations $X_1, \ldots, X_r$ and Y on the white source vertices of the incoming edges and the bud of a vertex V the decoration of V will either be $\Omega^E_r(X_1 \cdot \ldots \cdot X_r)_Y$ or $\Phi_r(X_1 \cdot \ldots \cdot X_r)Y$ depending on the color of V. Clearly the decorations of the roots of black or white jet trees $\mathfrak{T} \subset \mathfrak{F}$ are multilinear expressions $\Omega^E(\mathfrak{T})_{X(\mathfrak{T})}$ or $\Phi(\mathfrak{T})_{X(\mathfrak{T})}$ respectively in the argument vector fields $X(\mathfrak{T})$ of the leaves of $\mathfrak{T}$. Stressing the aspect of multilinear forms we drop the reference to the arguments from now on and think of $\Omega^E(\mathfrak{T})$ and $\Phi(\mathfrak{T})$ as sections of $\bigotimes^{|\mathfrak{T}|}T^*M \otimes \mathfrak{hol}^E M$ and $\bigotimes^{|\mathfrak{T}|}T^*M \otimes TM$. Similarly the notation $\Omega(\mathfrak{T}) \in \Gamma(\bigotimes^{|\mathfrak{T}|}T^*M \otimes \mathfrak{hol}M)$ refers to the multilinear form we get by the same rules replacing the curvature of EM by the Riemannian curvature tensor of M. The labelling of jet forests allows us to sort all occurring multilinear expressions into the right slots in expressions like:

$$\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ black}} \Omega^E(\mathfrak{T}) \otimes \prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} \Phi(\mathfrak{T})^\sharp \;\in\; \Gamma\Big(\bigotimes^{|\mathfrak{F}|}T^*M \;\otimes \bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ black}} \mathfrak{hol}^E M \;\otimes\; \mathrm{Sym}^{\#\mathrm{White}\,\mathfrak{F}}T^*M \Big)$$

In the formulas to come it is tacitly understood that this correspondence between the factors in $\bigotimes^{|\mathfrak{F}|}T^*M$ and the leaves of trees $\mathfrak{T} \subset \mathfrak{F}$ depending on the labelling is established, in particular this notation is only defined for trees in a common jet forest! The ordering of the tensor product $\bigotimes \mathfrak{hol}^E M$ is uncritical, because we insist that the factors are indexed by trees $\mathfrak{T} \subset \mathfrak{F}$.

Nevertheless we still have to twist this notation to account for feedback. Given the fact that the summation over feedback maps f will disappear in a moment we keep this problem short and simply use the notation $\Omega^E(\mathfrak{T})_f$ and $\Phi(\mathfrak{T})_f$ for the multilinear maps created by the rules pertaining to feedback. Note that these expressions are multilinear not only in the arguments associated to $\mathfrak{T}$, but also in the arguments associated to all trees $\tilde{\mathfrak{T}}$ chained to $\mathfrak{T}$ by a given feedback map f. With this proviso we can read off the total symbol of the operator $(-\Delta_\diamond)^k = \overrightarrow{\mathrm{app}} \circ \mathrm{tr}^k_\diamond \circ \nabla^{2k}$ for all $k \geq 1$ from the general Jet Formula 4.2

$$\mathrm{ev}[\,(-\Delta_\diamond)^k\,]^\sharp \;=\; \sum_{\substack{\text{jet forest}\\ |\mathfrak{F}|=2k}} \sum_{\substack{\text{feedback}\\ f}} (\,\overrightarrow{m} \circ \mathrm{tr}^k_\diamond\,) \left(\bigotimes_{\substack{\mathfrak{T}\subset\mathfrak{F}\text{ black}\\ f(\mathfrak{T})=\infty}} \Omega^E(\mathfrak{T})_f \;\otimes \prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} \Phi(\mathfrak{T})^\sharp_f \right) \tag{7.1}$$

where the multiplication $\overrightarrow{m} : \bigotimes \mathrm{End}\, EM \longrightarrow \mathrm{End}\, EM$ is the appropriate replacement for $\overrightarrow{\mathrm{app}}$ in the absence of an argument to apply to. Note that in the spirit of the general Jet Formula all terms $\Omega^E(\mathfrak{T})$ have to be applied first before any endomorphism arising from the twisted trace $\mathrm{tr}^k_\diamond$. In other words $\overrightarrow{m}$ multiplies the endomorphisms arising from $\mathrm{tr}^k_\diamond$ all sorted to the left in their usual order and independently the terms $\Omega^E(\mathfrak{T}) \in \Gamma(\mathfrak{hol}^E M)$ all sorted to the right in ascending order of the bud labels of the trees $\mathfrak{T}$.

Comparing this with the formula say for the standard Laplacian $\Delta_\star$ in Definition 7.2 the reader may agree that this "normal" ordering is extremely strange, we would rather expect terms of the form $\mathrm{tr}_\diamond(\alpha \otimes \beta)$ alternating with terms arising from curvature. In fact it turns out that by judiciously redefining the order of multiplication in (7.1) we can effectively get rid of feedback altogether. Before discussing this argument we recall that feedback modifies the multilinear forms $\Omega^E(\mathfrak{T})$ and $\Phi(\mathfrak{T})$ by changing their arguments away from X to $(\mathfrak{X}_1 \ldots \mathfrak{X}_r)^* X$ with suitable $\mathfrak{X}_1, \ldots, \mathfrak{X}_r \in \mathfrak{hol}\, M$ depending on f. In turn the canonical adjoint antiautomorphism $*$ of $\mathcal{U}\mathrm{End}\, TM$ used here characterizes the dual representation of $\mathcal{U}\mathrm{End}\, TM$ on T^*M via $(\mathfrak{X}_1 \ldots X_r \alpha)(X) := \alpha((\mathfrak{X}_1 \ldots \mathfrak{X}_r)^* X)$. Instead of alternating the arguments we may thus let $\mathfrak{hol}\, M$ act directly on the slots of $\bigotimes^{|\mathfrak{F}|} T^*M$ dictated by f. Pressing this point home we define the application map

$$\mathrm{app}_f : \bigotimes^{|\mathfrak{F}|} T^*M \otimes \bigotimes_{\mathfrak{T} \subset \mathfrak{F} \text{ black}} \mathfrak{hol}^{\mathrm{total}} M \longrightarrow \bigotimes^{|\mathfrak{F}|} T^*M \otimes \bigotimes_{\substack{\mathfrak{T} \subset \mathfrak{F} \text{ black} \\ f(\mathfrak{T}) = \infty}} \mathfrak{hol}^E M$$

where $\mathfrak{hol}^{\mathrm{total}} M \subset \mathfrak{hol}^E M \oplus \mathfrak{hol}\, M$ is the total holonomy algebra bundle associated to the holonomy reduction $\mathrm{Hol}^{\mathrm{total}} M \subset \mathrm{Hol}^E M \times_M \mathrm{Hol}\, M$. In the $\mathfrak{hol}^{\mathrm{total}} M$–factors indexed by trees $\mathfrak{T}$ with $f(\mathfrak{T}) = \infty$ the map app_f is simply the projection $\mathfrak{hol}^{\mathrm{total}} M \longrightarrow \mathfrak{hol}^E M$, whereas the factors indexed by trees with $f(\mathfrak{T}) < \infty$ are projected to $\mathfrak{hol}\, M$ and then act by the dual representation on the slot $f(\mathfrak{T})$ of $\bigotimes^{|\mathfrak{F}|} T^*M$. With multiplication of endomorphisms failing to be commutative we need to specify that elements of $\mathfrak{hol}\, M$ acting on the same slot are applied in the order corresponding to multiplication in ascending order of bud labels, i. e. the rightmost tree with the maximal bud label is applied first. Defining the multilinear form $\Omega^{\mathrm{total}}(\mathfrak{T}) := \Omega^E(\mathfrak{T}) \oplus \Omega(\mathfrak{T})$ with values in $\mathfrak{hol}^{\mathrm{total}} M$ we can thus rewrite (7.1) using:

$$\bigotimes_{\substack{\mathfrak{T} \subset \mathfrak{F} \text{ black} \\ f(\mathfrak{T}) = \infty}} \Omega^E(\mathfrak{T})_f \otimes \prod_{\mathfrak{T} \subset \mathfrak{F} \text{ white}} \Phi(\mathfrak{T})^\sharp_f \;=\; \mathrm{app}_f \left(\bigotimes_{\mathfrak{T} \subset \mathfrak{F} \text{ black}} \Omega^{\mathrm{total}}(\mathfrak{T}) \otimes \prod_{\mathfrak{T} \subset \mathfrak{F} \text{ white}} \Phi(\mathfrak{T})^\sharp \right)$$

This is a very crucial point and the reader should not skip it too fast! With this reformulation of feedback we can make good use of $\mathrm{tr}_\diamond : T^*M \otimes T^*M \longrightarrow \mathrm{End}\, EM$ being parallel or

$$\mathfrak{X}^E \, \mathrm{tr}_\diamond(\, \alpha \otimes \beta \,) \;=\; \mathrm{tr}_\diamond(\, (\mathfrak{X}\alpha) \otimes \beta \,) \;+\; \mathrm{tr}_\diamond(\, \alpha \otimes (\mathfrak{X}\beta) \,) \;+\; \mathrm{tr}_\diamond(\, \alpha \otimes \beta \,)\, \mathfrak{X}^E \qquad (7.2)$$

for all $\mathfrak{X}^E \oplus \mathfrak{X} \in \mathfrak{hol}^{\mathrm{total}} M$ and all $\alpha, \beta \in T^*M$. Summing the expression

$$(\, \overrightarrow{m} \circ \mathrm{tr}^k_\diamond \circ \mathrm{app}_f \,) \left(\bigotimes_{\mathfrak{T} \subset \mathfrak{F} \text{ black}} \Omega^{\mathrm{total}}(\mathfrak{T}) \otimes \prod_{\mathfrak{T} \subset \mathfrak{F} \text{ white}} \Phi(\mathfrak{T})^\sharp \right)$$

for a fixed jet forest $\mathfrak{F}$ over all possible feedback maps f will thus provide plenty of commutators, which allow us to reorder the factors in the multiplication $\overrightarrow{m}$. Let us fix say arbitrary values for f for all black trees except for the tree $\mathfrak{T}$ with minimal bud label and sum over all possible values for $f(\mathfrak{T})$ in decreasing order starting with $f(\mathfrak{T}) = \infty$. The first summand will see $\Omega^E(\mathfrak{T})$ to the right of all $\mathrm{tr}_\diamond$–factors, summing the next two values will commute it past the rightmost $\mathrm{tr}_\diamond(\alpha \otimes \beta)$ by equation (7.2) and so on until $\Omega^E(\mathfrak{T})$ is sorted between

$$\ldots \, \mathrm{tr}_\diamond(\, \alpha_{2s-1} \otimes \alpha_{2s} \,)\, \Omega^E(\mathfrak{T})\, \mathrm{tr}_\diamond(\, \alpha_{2s+1} \otimes \alpha_{2s+2} \,) \, \ldots$$

where $2s-1$ or $2s$ is the bud label of $\mathfrak{T}$ so that we can't commute $\Omega^E(\mathfrak{T})$ past $\mathrm{tr}_\diamond(\alpha_{2s-1} \otimes \alpha_{2s})$. Note that this reordering can always be achieved independent of the values $f(\mathfrak{T})$ of the other trees, because the curvature term $\Omega(\mathfrak{T})$ of the tree with minimal bud label will always be applied last by app_f. Continuing in this manner with the other black trees in ascending order of bud labels we will eventually end up with product involving essentially the same factors as before, but reordered in a more plausible way. Simply by reordering a product however we can't get rid of feedback completely, because for a black tree $\mathfrak{T}$ with odd bud label $2s-1$, $s \geq 1$, the feedback value $f(\mathfrak{T}) = 2s$ is perfectly legal, but provides only an incomplete half commutator according to (7.2).

In order to cope with this nuisance let us modify the definition of jet forests by introducing transparent vertices. A transparent vertex V is a vertex with at least three incoming edges required to have an aligned twin bud, i. e. the two leaves with maximal labels in the subtree rooted at V are connected directly to V with labels $2s-1$ and $2s$ for some $s \geq 1$. In a modified jet forest the root of a tree $\mathfrak{T} \subset \mathfrak{F}$ may be black, white or transparent, nevertheless we will continue to distinguish only between black and white trees, trees with transparent roots are white. Transparent vertices are to simulate the situation, where the feedback map on a black tree $\mathfrak{T}$ with odd bud label $2s-1$ takes the value $f(\mathfrak{T}) = 2s$. Consequently the rule for decorating a transparent vertex with a vector field reads $-\Omega(X_1 \cdot \ldots \cdot X_r)_Y Z$, where Y and Z are the decorations of the twin buds with labels $2s-1$ and $2s$ respectively while $X_1, \ldots, X_r$ are the decorations of the source vertics of the other incoming edges. In the presence of transparent vertices the budding condition has to be modified, too, we can no longer insist that the bud of a vertex V, the leaf of maximal label in the subtree rooted at V, is connected directly to V. Although the twin buds of a transparent vertex are still required to connect directly to the vertex in question, for other non–leaf vertices intermediate transparent vertices are allowed on the branch to the bud. For the time being we refrain from formalizing the definition of modified jet forests, more details can be found in the Definition 7.3 of Laplace forests.

Summarizing our considerations so far we have seen that by multiplying the factors in equation (7.1) in a delicately chosen order and modifying the concept of jet forests slightly we can eliminate the sum over feedback maps. The new multiplication m^* doing the trick depends on the jet forest $\mathfrak{F}$ of order $|\mathfrak{F}| = 2k$ considered and is defined as the linear map

$$m^* : \quad \bigotimes^k \mathrm{End}\; EM \;\otimes \bigotimes_{\mathfrak{T} \subset \mathfrak{F} \text{ black}} \mathfrak{hol}^E M \;\longrightarrow\; \mathrm{End}\; EM$$

which shuffles the two different kinds of endomorphisms like a deck of cards before multiplication. More precisely in the final product all factors from $\bigotimes^k \mathrm{End}\, EM$ appear in their usual order from left to right and all factors from $\bigotimes \mathfrak{hol}^E M$ appear in ascending order of bud labels, however the factor of $\bigotimes \mathfrak{hol}^E M$ indexed by a black tree $\mathfrak{T}$ in $\mathfrak{F}$ with bud label $2s-1$ or $2s$ appears sandwiched between the factors from $\bigotimes^k \mathrm{End}\, EM$ indexed by s and $s+1$ respectively. Of course this does not mean that in the final product factors from $\bigotimes^k \mathrm{End}\, EM$ will alternate with factors from $\bigotimes \mathfrak{hol}^E M$, in a given jet forest $\mathfrak{F}$ we may have two black trees with bud labels $2s-1$ and $2s$ or none. Introducing the empty jet forest $\mathfrak{F}_{\mathrm{vac}} := \emptyset$ of order $|\mathfrak{F}_{\mathrm{vac}}| = 0$ and summing (7.1) over all $k \geq 0$ we eventually find the total symbol

$$\mathrm{ev}[\, e^{-t\Delta_\diamond} \,]^\sharp \;=\; \sum_{\substack{\text{(modified) jet forest} \\ |\mathfrak{F}| \text{ even}}} \frac{t^{\frac{|\mathfrak{F}|}{2}}}{(\frac{|\mathfrak{F}|}{2})!} \,(\, m^* \circ \mathrm{tr}_\diamond^{\frac{|\mathfrak{F}|}{2}} \,) \left(\bigotimes_{\mathfrak{T} \subset \mathfrak{F} \text{ black}} \Omega^E(\mathfrak{T}) \;\otimes \prod_{\mathfrak{T} \subset \mathfrak{F} \text{ white}} \Phi(\mathfrak{T})^\sharp \right) \tag{7.3}$$

of the operator $e^{-t\Delta_\diamond}$, where the summand corresponding to the empty jet forest $\mathfrak{F}_{\mathrm{vac}}$ can be interpreted without straining the notation too much as the identity id_E, which is the constant term in the formal power series $e^{-t\Delta_\diamond}$ of differential operators.

Equation (7.3) is the cornerstone of our calculation of the heat kernel coefficients of the twisted Laplacian $\Delta_\diamond$. The remaining steps in this calculation are essentially trivial and will unveil the secret wisdom of the formula given in Theorem 2.4. Consider an aligned pair of solitary white vertices in a modified jet forest $\mathfrak{F}$ labelled $2s-1$ and $2s$ for some $s \geq 1$. Tracing over these solitary white vertices alone we see that the summand corresponding to $\mathfrak{F}$ in the total symbol of the operator $e^{-t\Delta_\diamond}$ picks up the factor

$$\sum_{\mu\nu} \ldots \mathrm{tr}_\diamond(\, dx_\mu \otimes dx_\nu\,) \ldots \;\otimes\; \ldots x^\sharp_\mu \cdot x^\sharp_\nu \ldots \quad = \quad \ldots \mathrm{id}_E \ldots \;\otimes\; \ldots \,|\cdot|^2 \ldots$$

in equation (7.3). In particular in the flat case the total symbol of the operator $e^{-t\Delta_\diamond}$ is just $e^{t|\cdot|^2}$, because all summands in (7.3) vanish except for the summands associated to jet forests with only solitary white vertices. In the formula of Theorem 2.4 this total symbol is countered by the multiplication with $e^{-t|\cdot|^2}$ so that $j^{\frac{1}{2}}a(t) = 1$ as expected. Of course this argument only works in the flat case, but it suggests that the multiplication with $e^{-t|\cdot|^2}$ eliminates exactly those summands in (7.3), which are associated to jet forests $\mathfrak{F}$ with at least one aligned pair of solitary white vertices. In fact there are exactly $\binom{k}{r}$ ways to insert aligned pairs of solitary white vertices into a possibly empty jet forest $\mathfrak{F}^{\mathrm{red}}$ of order $2r$, $r \geq 0$, without such pairs in order to produce a jet forest $\mathfrak{F}$ of order $2k$ and conversely every jet forest $\mathfrak{F}$ of order $2k$ arises this way from a unique reduced jet forest $\mathfrak{F}^{\mathrm{red}}$ without aligned pairs of solitary white vertices. Consequently the total symbol of the operator $e^{-t\Delta_\diamond}$ in (7.3) splits of the factor $e^{t|\cdot|^2}$, which is subsequently killed by the multiplication with $e^{-t|\cdot|^2}$:

$$e^{-t|\cdot|^2}\,\mathrm{ev}[\, e^{-t\Delta_\diamond}\,]^\sharp \;=\; \sum_{\substack{\text{reduced (modified)}\\ \text{jet forest } |\mathfrak{F}| \text{ even}}} \frac{t^{\frac{|\mathfrak{F}|}{2}}}{(\frac{|\mathfrak{F}|}{2})!}\,(\, m^* \circ \mathrm{tr}_\diamond^{\frac{|\mathfrak{F}|}{2}}\,)\left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F} \text{ black}} \Omega^E(\mathfrak{T}) \;\otimes \prod_{\mathfrak{T}\subset\mathfrak{F} \text{ white}} \Phi(\mathfrak{T})^\sharp \right) \tag{7.4}$$

Motivated by this success vindicating Theorem 2.4 we want to eliminate solitary white vertices altogether. A straightforward way to do this is to omit all solitary white vertices in a reduced jet forest $\mathfrak{F}$ and shift the labels of the remaining leaves down accordingly to get a new jet forest $\mathfrak{F}^{\mathrm{new}}$ without solitary white vertices. The information lost this way can essentially be recovered from knowing the subset $L \subset \mathrm{Leaf}\,\mathfrak{F}^{\mathrm{new}}$ of leaves, which formed an aligned pair with some solitary white vertex in the original jet forest $\mathfrak{F}$. More precisely given a jet forest $\mathfrak{F}^{\mathrm{new}}$ without solitary white vertices and a subset $L \subset \mathrm{Leaf}\,\mathfrak{F}^{\mathrm{new}}$ of marked leaves we simply double each marked leaf and shift the labels of the leaves up accordingly to reproduce the orginal reduced jet forest $\mathfrak{F}$. Because we do not know whether the solitary white vertex originally took the label $2s-1$ or $2s$, it is not possible to reconstruct the original reduced jet forest $\mathfrak{F}$ exactly. Eventually however we are only interested in the sum (7.4) over all reduced jet forests and so we may sum over all $2^{\#L}$ possibilities to expand $\mathfrak{F}^{\mathrm{new}}$ to a reduced jet forest $\mathfrak{F}$ first. Actually this summation reduces to a partial symmetrization by:

$$\sum_\mu \ldots \mathrm{tr}_\diamond(\, dx_\mu \otimes \alpha \,+\, \alpha \otimes dx_\mu\,) \ldots \;\otimes\; \ldots\, x^\sharp_\mu \ldots \quad = \quad \ldots \mathrm{id}_E \ldots \;\otimes\; \ldots\, 2\,\alpha \ldots \tag{7.5}$$

In consequence it is possible to express the total symbol of the operator $e^{-t\Delta_\diamond}$ as a sum over modified jet forests without solitary white vertices or Laplace forests:

Definition 7.3 (Laplace Forest with Marked Leaves)
A Laplace forest of order $k \geq 0$ is a rooted forest $\mathfrak{F}$ without solitary white vertices together with a labelling of its leaves $\mathrm{Leaf}\,\mathfrak{F} \xrightarrow{\cong} \{1, \ldots, k\}$ and a coloring of its vertices by colors white, transparent and black such that the following three conditions are met:

- *All leaves are white while all black vertices are roots. Every black vertex has at least two, every white or transparent vertex at least three incoming edges unless it is a leaf.*
- *In the subtree rooted at a transparent vertex V the two leaves of maximal labels take consecutive labels and connect directly to V, every transparent vertex has a twin bud.*
- *For all black or white vertices V besides leaves the leaf of maximal label in the subtree rooted at V is connected to V via a branch with transparent intermediate vertices only.*

A Laplace forest with marked leaves is a Laplace forest $\mathfrak{F}$ together with a distinguished subset $L \subset \mathrm{Leaf}\,\mathfrak{F} \setminus \mathrm{Twin}\,\mathfrak{F}$ of its leaves with $\#L \equiv |\mathfrak{F}| \bmod 2$ avoiding the set $\mathrm{Twin}\,\mathfrak{F}$ of twin buds of transparent vertices of $\mathfrak{F}$ such that every maximal sequence $I \subset \mathrm{Leaf}\,\mathfrak{F} \setminus (L \cup \mathrm{Twin}\,\mathfrak{F})$ of leaves with consecutive labels avoiding both L and $\mathrm{Twin}\,\mathfrak{F}$ has even length $\#I$.

According to the color of their roots the trees $\mathfrak{T}$ in a Laplace forest $\mathfrak{F}$ will be called black or white trees, trees with transparent roots are white. Perhaps the most difficult aspect of this definition is the constraint imposed on the subset L of marked leaves. Recall that the marked leaves in the subset $L \subset \mathrm{Leaf}\,\mathfrak{F}$ make up aligned pairs of leaves with solitary white vertices in a reduced jet forest mapping to $\mathfrak{F}$. Doubling each leaf in L and shifting up the labels of the leaves accordingly the twin buds of transparent vertices must be aligned as well as the pairs of doubled leaves in L, equivalently the set of marked leaves $\mathrm{Leaf}\,\mathfrak{F} \setminus L$ is the disjoint union of pairs of leaves with consecutive labels among which are the twin buds of transparent vertices. In other words the constraint imposed on the subset L of marked leaves allows us to group all unmarked leaves in disjoint pairs of leaves with consecutive labels such that the twin buds of transparent vertices stay together in a pair.

Given a Laplace forest $\mathfrak{F}$ with marked leaves $L \subset \mathrm{Leaf}\,\mathfrak{F}$ any reduced jet forest obtained by doubling each marked leaf clearly has order $|\mathfrak{F}| + \#L$. The summation over the $2^{\#L}$ different possibilities to choose the solitary white vertices in the two copies of each doubled marked leaf results in a partial symmetrization according to (7.5), in particular the twisted trace factorizes over the symmetric multiplication

$$m_L : \quad \bigotimes^{|\mathfrak{F}|} T^*M \otimes \mathrm{Sym}^{\#\mathrm{White}\,\mathfrak{F}} T^*M \quad \longrightarrow \quad \bigotimes^{|\mathfrak{F}|-\#L} T^*M \otimes \mathrm{Sym}^{\#\mathrm{White}\,\mathfrak{F}+\#L} T^*M$$

of the slots associated to the labels of marked leaves in L and the twisted trace $\mathrm{tr}_\diamond^{\frac{|\mathfrak{F}|-\#L}{2}}$ over the remaining slots while picking up an additional factor $2^{\#L}$. In this way we have succeeded in rewriting equation (7.4) as a sum over all Laplace forests with marked leaves:

$$(2t)^{-N}\, e^{-t|\cdot|^2}\, \mathrm{ev}[\, e^{-t\Delta_\diamond}\,]^\sharp \tag{7.6}$$
$$= \sum_{\substack{\mathfrak{F} \text{ Laplace forest} \\ L \text{ marked leaves}}} \frac{t^{\frac{|\mathfrak{F}|-\#L}{2} - \#\mathrm{White}\,\mathfrak{F}}}{2^{\#\mathrm{White}\,\mathfrak{F}}\,(\frac{|\mathfrak{F}|+\#L}{2})!}\, m^* \circ \mathrm{tr}_\diamond^{\frac{|\mathfrak{F}|-\#L}{2}} \circ m_L \left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F} \text{ black}} \Omega^E(\mathfrak{T}) \otimes \prod_{\mathfrak{T}\subset\mathfrak{F} \text{ white}} \Phi(\mathfrak{T})^\sharp \right)$$

At first sight it appears that the right hand side is a Laurent series, there certainly are Laplace forests $\mathfrak{F}$ and subsets $L \subset \operatorname{Leaf}\mathfrak{F}$ of marked leaves which make $\frac{|\mathfrak{F}|-\#L}{2} - \#\operatorname{White}\mathfrak{F}$ negative. A priori we know from the derivation of Theorem 2.4 that the series is regular in t. Nevertheless it is worthwhile to think about an argument proving regularity directly, because in principle regularity could be the result of several summands cancelling each other. In our situation the critical summands vanish all by themselves, more precisely the summand associated to a Laplace forest $\mathfrak{F}$ with marked leaves $L \subset \operatorname{Leaf}\mathfrak{F}$ vanishes as soon as the set L contains all or all but one leaf of some white tree $\mathfrak{T}$ in $\mathfrak{F}$. Hence a summand in formula (7.6) vanishes unless every white tree $\mathfrak{T}$ contributes at least 2 to the difference $|\mathfrak{F}| - \#L$:

Lemma 7.4 (Gauß Type Vanishing)
Let $\mathfrak{T}$ be a white Laplace tree necessarily of order $|\mathfrak{T}| \geq 3$ without transparent vertices. For all subsets $L \subset \operatorname{Leaf}\mathfrak{T}$ containing all but one leaf the symmetrization of result and leaves in L via the multiplication $m_L : \bigotimes^{|\mathfrak{T}|}T^*M \otimes T^*M \longrightarrow T^*M \otimes \operatorname{Sym}^{|\mathfrak{T}|}T^*M$ kills the multilinear form $\Phi(\mathfrak{T}) \in \Gamma(\bigotimes^{|\mathfrak{T}|}T^*M \otimes TM)$ associated to $\mathfrak{T}$ in the sense $m_L(\Phi(\mathfrak{T})^\sharp) = 0$.

Proof: Thinking of $m_L(\Phi(\mathfrak{T})^\sharp)$ as a polynomial on TM it is sufficient to prove that its evaluation $m_L(\Phi(\mathfrak{T})^\sharp)(X)_Y = \langle \Phi(\mathfrak{T})_{X,\ldots,X,Y,X,\ldots,X}, X\rangle$ vanishes at all points and arguments $X, Y \in TM$. According to the rules for converting $\mathfrak{T}$ into a multilinear form applied to the root of $\mathfrak{T}$ only $\langle \Phi(\mathfrak{T})_{X,\ldots,X,Y,X,\ldots,X}, X\rangle$ equals $\langle \Phi_r(X \cdot \ldots \cdot X \cdot Z)X, X\rangle$ or $\langle \Phi_r(X \cdot \ldots \cdot X)Z, X\rangle$ depending on whether the bud is in L or not for some $r \geq 2$ and a vector field Z. In light of the consequence $\Phi_r(X \cdot \ldots \cdot X \cdot Z)X = -\frac{1}{r}\Phi_r(X \cdot \ldots \cdot X)Z$ of the constraint $\Phi_r(X \cdot \ldots \cdot X)X = 0$ there is no essential difference between the two possibilities and the vanishing $\langle \Phi_r(X \cdot \ldots \cdot X)Z, X\rangle = 0$ for $r \geq 2$ is just the Gauß Lemma 3.3. □

Theorem 7.5 (Heat Kernel Coefficients of Twisted Laplacians)
The generating series $a(t)$ for the heat kernel coefficients of a twisted Laplacian satisfies

$$
\begin{aligned}
&j^{\frac{1}{2}}\, a(t) \\
&= e^{t\Delta}\left[\sum_{\substack{\mathfrak{F}\ \text{Laplace forest}\\ L\ \text{marked leaves}}} \frac{t^{\frac{|\mathfrak{F}|-\#L}{2}-\#\operatorname{White}\mathfrak{F}}}{2^{\#\operatorname{White}\mathfrak{F}}\,(\frac{|\mathfrak{F}|+\#L}{2})!}\, m^* \circ \operatorname{tr}_\diamond^{\frac{|\mathfrak{F}|-\#L}{2}} \circ\, m_L\left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \text{black}} \Omega^E(\mathfrak{T}) \otimes \prod_{\mathfrak{T}\subset\mathfrak{F}\ \text{white}} \Phi(\mathfrak{T})^\sharp\right)\right]
\end{aligned}
$$

where Δ is the formal Laplace operator on $\operatorname{Sym}T^*M$ defined by the Riemannian metric.

Invoking Theorem 2.4 with equation (7.6) we immediately verify the stated formula for the generating series $a(t) = \sum_{d\geq 0} t^d a_d$. Lest the reader objects that this formula is not really explicit we remark that the multilinear forms $\Omega^E(\mathfrak{T})$ and $\Phi(\mathfrak{T})$ are explicitly known polynomials in the curvature tensor R of M and the curvature R^E of the vector bundle EM together with all their covariant derivatives depending only on the isomorphism class of $\mathfrak{T}$ considered as a budding tree. Moreover the differential operator $e^{t\Delta}$ is just the flat Laplacian on polynomials on T_yM at every point $y \in M$, in other words it is in essence a symmetrized iterated trace. In particular a Laplace forest $\mathfrak{F}$ with marked leaves will only contribute to the value of some heat kernel coefficient along the diagonal if $|\mathfrak{F}|$ and $\#\operatorname{White}\mathfrak{F}$ have the same parity, and in this case it will contribute to the coefficient $a_d(0)$ with $d = \frac{|\mathfrak{F}| - \#\mathrm{White}\,\mathfrak{F}}{2}$ independent of the choice of marked leaves. For such forests the differential operator $e^{t\Delta}$ is a version of the unique $\mathbf{SO}\ T_yM$–invariant linear functional on $\mathrm{Sym}^{\#\mathrm{White}\,\mathfrak{F}+\#L}T_y^*M$.

In the second part of this section we want to generalize Theorem 7.5 to include even the most general Laplacians. In principle every such Laplacian can be written $\nabla^*\nabla + \mathcal{F}$ for a suitable potential $\mathcal{F} \in \Gamma(\mathrm{End}\ EM)$. Motivated by questions arising from the inverse scattering theory approach to solitons however we will consider Laplacians of the form $\Delta_{\diamond,\mathcal{F}} := \Delta_\diamond + \mathcal{F}$ on a twisted vector bundle EM with an auxiliary potential $\mathcal{F} \in \Gamma(\mathrm{End}\ EM)$. Of course it is no restriction at all to work on a twisted vector bundle, every vector bundle endowed with a connection is trivially twisted, but the arbitrariness in writing the potential as a sum of a curvature term and an auxiliary potential $\mathcal{F}$ must eventually be reflected in all our formulas for the heat kernel coefficients providing us with an implicit consistency check. Naturally the calculation of the heat kernel coefficients of twisted Laplacians $\Delta_{\diamond,\mathcal{F}}$ with potential proceeds more or less parallel to the calculations in the special case of twisted Laplacians $\Delta_\diamond$ without auxiliary potential $\mathcal{F} = 0$. In particular there is not too much of a difference between the formula given in Theorem 7.5 above and in the final Theorem 7.8. For this reason we will be rather sketchy in this second part and focus attention on the three critical steps in the calculation, which differ significantly from the special case.

Recall that in the special case of a twisted Laplacian $\Delta_\diamond$ we started the calculation with the formula $\Delta_\diamond^k = \overrightarrow{\mathrm{app}} \circ (-\mathrm{tr}_\diamond)^k \circ \nabla^{2k}$ for the powers of a twisted Laplace operator $\Delta_\diamond$. In a second step we applied the general Jet Formula 4.2 to calculate the total symbol $\mathrm{ev}[\,\Delta_\diamond^k\,]^\sharp$ of the powers $\Delta_\diamond^k$, $k \geq 0$, and then reordered the multiplication of the endomorphisms from $\overrightarrow{m}$ to m^* in a third step to eliminate the sum over feedback maps. Similarly the first and most difficult step of the calculations in the general case of a twisted Laplacian $\Delta_{\diamond,\mathcal{F}}$ with potential is to find the analogue of the formula for the powers $\Delta_{\diamond,\mathcal{F}}^k$, $k \geq 0$, suitable for manipulation. In a second step we apply the general Jet Formula to the iterated covariant derivatives not only of the section ψ, but also of the potential $\mathcal{F}$, in order to calculate the total symbol $\mathrm{ev}[\,\Delta_{\diamond,\mathcal{F}}^k\,]^\sharp$ and then reorder the multiplication from m^{sort} to m^* to eliminate the sum over feedback and hook maps. Concerning the first step a very simple–minded version of the jet and Laplace forests we have been using so far will serve as a bootstrap device:

Definition 7.6 (Red–White Forests)
A red–white forest of order k is a labelled rooted forest $\mathfrak{F}$ with labelling $\mathrm{Leaf}\,\mathfrak{F} \xrightarrow{\cong} \{1, \ldots, k\}$ together with a coloring of its vertices by colors red and white such that every leaf is white and every other vertex is a red root with at least two incoming edges and a twin bud.

Of course we will continue to distinguish trees $\mathfrak{T} \subset \mathfrak{F}$ in a red–white forest $\mathfrak{F}$ by the color of their roots in red and white trees and will write $\mathrm{Red}\,\mathfrak{F}$ and $\mathrm{White}\,\mathfrak{F}$ for the subsets of red and white trees respectively. Clearly a red tree $\mathfrak{T} \subset \mathfrak{F}$ consists of a red root connected directly to two or more leaves, such that the two leaves with maximal labels form a twin bud in that their labels are two consecutive numbers. Similarly a white tree $\mathfrak{T} \subset \mathfrak{F}$ is nothing but a solitary white vertex. A red–white forest $\mathfrak{F}$ of even order $|\mathfrak{F}|$ will be called aligned if the twin bud of every red root in $\mathfrak{F}$ is aligned in the sense that the two leaves with maximal labels take the labels $2s - 1$ and $2s$ for some $s \geq 1$.

Thinking of forests as abstract flow charts for calculations it is natural to ask about the rules pertaining to red–white forests. For formulating these rules it is convenient to use the notation $\overline{\mathrm{jet}}\,\mathcal{F}(X_1 \otimes \ldots \otimes X_r) := \nabla^r_{X_1,\ldots,X_r}\mathcal{F}$ for iterated covariant derivatives of unspecified order $r \geq 0$, for $r = 0$ both sides equal $\mathcal{F}$ by definition. Given a decoration of the leaves of a red–white forest $\mathfrak{F}$ by vector fields all non–leaf vertices are red roots and will be decorated by the iterated covariant derivatives $-\frac{1}{\dim M}\langle Y, Z\rangle(\overline{\mathrm{jet}}\,\mathcal{F})(X_1 \otimes \ldots \otimes X_r)$, $r \geq 0$, of the potential $\mathcal{F}$, where Y and Z are the decorations of the twin bud and $X_1, \ldots, X_r$ are the decorations of the remaining leaves in ascending order of their labels. In this way the root of every red tree $\mathfrak{T} \subset \mathfrak{F}$ is decorated by an endomorphisms $\overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T})_{X(\mathfrak{T})}$ on EM depending multilinearly on the argument vector fields $X(\mathfrak{T})$ decorating the leaves of $\mathfrak{T}$, while every solitary white vertex tree $\mathfrak{T}$ is decorated by its argument vector field $X(\mathfrak{T})$. Stressing the aspect of multilinear forms we will use the notation $\mathrm{id}(\mathfrak{T}) \in \Gamma(T^*M \otimes TM)$ for the identity associated to a white tree $\mathfrak{T}$ and $\overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T}) \in \Gamma(\bigotimes^{|\mathfrak{T}|}T^*M \otimes \mathrm{End}\,EM)$ for the multilinear form associated to a red tree $\mathfrak{T}$ in a red–white forest $\mathfrak{F}$ with the usual convention of ordering the multilinear forms according to the labelling of the leaves in the forest $\mathfrak{F}$ to define the multilinear form:

$$\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ red}} \overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T}) \otimes \bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} \mathrm{id}(\mathfrak{T}) \;\in\; \Gamma\Big(\bigotimes^{|\mathfrak{F}|}T^*M \otimes \bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ red}} \mathrm{End}\,EM \otimes \bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ white}} TM\Big) \tag{7.7}$$

The important point to notice here is that the multilinear form $\overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T})$ associated to a red tree is set up in such a way that tracing it over the twin bud

$$(-\mathrm{tr}_\diamond)(\,\overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T})\,) \;=\; \mathrm{id}_E \otimes \nabla^{|\mathfrak{T}|}\mathcal{F} \tag{7.8}$$

produces exactly the iterated covariant derivatives of the potential $\mathcal{F}$ we expect to see in the formula for $\Delta^k_{\diamond,\mathcal{F}}$. Although this way to treat the potential $\mathcal{F}$ is rather artificial it will turn out to be quite convenient to keep the notational overhead at bay.

Pairing the $\bigotimes TM$–factor of the multilinear form (7.7) with the iterated covariant derivatives of a section $\psi \in \Gamma(EM)$ associates to every red–white forest $\mathfrak{F}$ a differential operator on sections of EM with values in $\bigotimes^{|\mathfrak{F}|}T^*M \otimes \bigotimes^{\#\mathrm{Red}\,\mathfrak{F}}\mathrm{End}\,EM \otimes EM$. In order to end up with a differential operator on EM we need to restrict ourselves to aligned red–white forests so that we can reduce the $\bigotimes T^*M$–factor to a couple of endomorphisms on EM via powers of the twisted trace $\mathrm{tr}_\diamond$. Choosing a suitable order to apply the different endomorphisms arising from $\mathrm{tr}_\diamond$ and the potential to the EM–factor we get in fact a differential operator on EM. The appropriate order of application is most easily encoded in a linear map $\mathrm{app}^{\mathrm{sort}}$, which we choose in dependence on the aligned red–white forest $\mathfrak{F}$ such that the composition

$$\mathrm{app}^{\mathrm{sort}} \circ (-\mathrm{tr}_\diamond)^{\frac{|\mathfrak{F}|}{2}} : \;\bigotimes^{|\mathfrak{F}|}T^*M \otimes \bigotimes_{\mathfrak{T}\subset\mathfrak{F}\text{ red}} \mathrm{End}\,EM \otimes EM \;\longrightarrow\; EM$$

will see all endomorphisms arising via the twisted trace $\mathrm{tr}_\diamond$ applied in their usual order with the rightmost one first while the endomorphisms arising via red trees from the potential are applied directly before the twisted traces of their respective twin buds compare (7.8). Of course the twisted trace over the twin bud of a red tree $\mathfrak{T}$ is a multiple of the identity, which clearly commutes with the endomorphism $\nabla^{|\mathfrak{T}|}\mathcal{F}$, hence it is immaterial whether we choose to apply $\nabla^{|\mathfrak{T}|}\mathcal{F}$ directly before or directly after the identity or even drop the identity, however the order of application with respect to the other endomorphisms is critical. With this specific order of multiplication the powers $\Delta_{\diamond,\mathcal{F}}$ of a twisted Laplacian with potential $\mathcal{F}$ can be written as a linear combination of the differential operators on EM associated to aligned red–white forests $\mathfrak{F}$:

Lemma 7.7 (Powers of Laplacians with Potentials)
The powers $\Delta^k_{\diamond,\mathcal{F}}$, $k \geq 1$, of a twisted Laplacian $\Delta_{\diamond,\mathcal{F}}$ with potential $\mathcal{F}$ can be written as a sum over all aligned red–white forests of order $2k$:

$$\Delta^k_{\diamond,\mathcal{F}}\,\psi \;=\; \sum_{\substack{\textit{aligned red–white}\\ \textit{forest } |\mathfrak{F}|=2k}} \mathrm{app}^{\mathrm{sort}} \circ (-\mathrm{tr}_\diamond)^k \left[\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \textit{red}} \overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T}) \;\otimes\; \overline{\mathrm{jet}}\,\psi\Big(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \textit{white}} \mathrm{id}(\mathfrak{T})\,\Big) \right]$$

We will prove this Lemma by induction on k using the inductive structure on the set of aligned red–white forests, namely every aligned red–white forest $\mathfrak{F}^{\mathrm{new}}$ of order $2k+2$ arises from a unique aligned red–white forest $\mathfrak{F}$ of order $2k$ by shifting the labels of the leaves up by 2 to preserve alignment and adding two new leaves with labels 1 and 2 respectively. These two new leaves either form the twin bud of a new red root or can be separately chosen to stay solitary or connect to any of the already existing red roots of $\mathfrak{F}$.

Proof: Evidently there are exactly two aligned red–white forests of order 2, the forest with two solitary white vertices and the forest with a red root connected to two white leaves. For the former forest the only endomorphism to be applied is the twisted trace $(-\mathrm{tr}_\diamond)\nabla^2\psi$, which contributes $\Delta_\diamond\psi$ while the latter forest is responsible for providing the term $\mathcal{F}\psi$ involving the auxiliary potential $\mathcal{F}$ by (7.8). Consequently the statement is certainly correct for $k = 1$. Assume now by induction hypothesis that $\Delta^k_{\diamond,\mathcal{F}}\psi$ can be written as a sum over all aligned red–white forests of order $2k$ for some $k \geq 1$. The crucial point is that $\nabla^2\,\Delta^k_{\diamond,\mathcal{F}}\,\psi$ can be written as a sum over all aligned red–white forests $\mathfrak{F}^{\mathrm{new}}$ of order $2k+2$ with the property that the leaves labelled 1 and 2 are not the twin buds of a red tree in $\mathfrak{F}^{\mathrm{new}}$. Every such aligned red–white forest arises from a unique red–white forest $\mathfrak{F}$ of order $2k$ by shifting the labels of all leaves up by 2 and adding two new leaves labelled 1 and 2, which either stay solitary or connect independently to any of the already existing red roots of $\mathfrak{F}$. Fixing the forest $\mathfrak{F}$ and keeping the Leibniz rule for covariant derivatives in mind we observe

$$\nabla^2\left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \mathrm{red}} \overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T}) \;\otimes\; \overline{\mathrm{jet}}\,\psi\Big(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \mathrm{white}} \mathrm{id}(\mathfrak{T}) \Big) \right)$$
$$= \sum_{\mathfrak{F}^{\mathrm{new}}} \bigotimes_{\mathfrak{T}\subset\mathfrak{F}^{\mathrm{new}}\ \mathrm{red}} \overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T}) \;\otimes\; \overline{\mathrm{jet}}\,\psi\left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}^{\mathrm{new}}\ \mathrm{white}} \mathrm{id}(\mathfrak{T}) \right)$$

where the sum on the right is over all aligned red–white forests $\mathfrak{F}^{\mathrm{new}}$ of order $2k+2$ arising from the fixed red–white forest $\mathfrak{F}$ as above. In fact we may choose the leaves 1 and 2 in $\mathfrak{F}^{\mathrm{new}}$ independently to stay solitary or connect to any of the red roots of $\mathfrak{F}$. Staying solitary these leaves will increase the number of covariant derivatives taken of the section ψ while they will increase the number of covariant derivatives taken of the potential factor $\overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T})$

arising from a red tree $\mathfrak{T}$ in case they connect to the red root of $\mathfrak{T}$. On the other hand both the sorted application $\mathrm{app}^{\mathrm{sort}}$ and the twisted trace $\mathrm{tr}_\diamond$ are parallel so that we conclude

$$\nabla^2(\,\Delta^k_{\diamond,\mathcal{F}}\,\psi\,) \tag{7.9}$$
$$= \sum_{\substack{\text{aligned red–white}\\ |\mathfrak{F}^{\mathrm{new}}|\,=\,2k+2}} \mathrm{app}^{\mathrm{sort}} \circ (\,\mathrm{id}\otimes\mathrm{id}\otimes(-\mathrm{tr}_\diamond)^k\,) \left[\bigotimes_{\mathfrak{T}\subset\mathfrak{F}^{\mathrm{new}}\ \mathrm{red}} \overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T}) \;\otimes\; \overline{\mathrm{jet}}\,\psi(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}^{\mathrm{new}}\ \mathrm{white}} \mathrm{id}(\mathfrak{T})\;)\right]$$

where the summation is over all aligned red–white forests $\mathfrak{F}^{\mathrm{new}}$ of order $2k+2$ such that the leaves labelled 1 and 2 are not the twin buds of a red tree. If we replace $\mathrm{id}\otimes\mathrm{id}\otimes(-\mathrm{tr}_\diamond)^k$ by $(-\mathrm{tr}_\diamond)^{k+1}$ on the right hand side of (7.9) then the sorted application $\mathrm{app}^{\mathrm{sort}}$ will apply the additional twisted trace over the leaves labelled 1 and 2 last by definition so that the left hand side becomes $(\mathrm{app}\circ(-\mathrm{tr}_\diamond)\circ\nabla^2)\,\Delta^k_{\diamond,\mathcal{F}}\,\psi = \Delta_\diamond\,\Delta^k_{\diamond,\mathcal{F}}\,\psi$.

In this way we have succeeded in expressing $\Delta_\diamond\,\Delta^k_{\diamond,\mathcal{F}}\,\psi$ as a sum over all aligned red–white forests $\mathfrak{F}^{\mathrm{new}}$ of order $2k+2$ with the property that the leaves labelled 1 and 2 are not the twin buds of a red tree. Clearly in the remaining aligned red–white forests $\mathfrak{F}^{\mathrm{new}}$ of order $2k+2$ the leaves labelled 1 and 2 are the twin buds of a red tree and the sorted application $\mathrm{app}^{\mathrm{sort}}$ will apply the twisted trace $\mathrm{id}_E\otimes\mathcal{F}$ over these two leaves last. The net sum over these remaining forests $\mathfrak{F}^{\mathrm{new}}$ will thus contribute $\mathcal{F}\,\Delta^k_{\diamond,\mathcal{F}}\,\psi$ to the total sum $(\Delta_\diamond+\mathcal{F})\,\Delta^k_{\diamond,\mathcal{F}}\,\psi$ over all aligned red–white forests of order $2k+2$. □

With Lemma 7.7 proved we have overcome the most difficult problem in finding a closed formula for the heat kernel coefficients of twisted Laplacians with potentials. In a second step we need to apply the general Jet Formula 4.2 to the iterated covariant derivatives $\overline{\mathrm{jet}}\,\psi$ occurring in the formula for the powers $\Delta^k_{\diamond,\mathcal{F}}$, $k\geq 1$, of the generalized Laplacian $\Delta_{\diamond,\mathcal{F}}$ in order to find the total symbol $\mathrm{ev}[\,e^{-t\Delta_{\diamond,\mathcal{F}}}\,]^\sharp$ of the operator $e^{-t\Delta_{\diamond,\mathcal{F}}}$. Despite first appearance however things become significantly simpler if we decide to apply the general Jet Formula 4.2 not only to the iterated covariant derivatives of the section ψ but at the same time to the covariant derivatives of the potential $\mathcal{F}$, which appear in the potential terms $\overline{\mathrm{jet}}\,\mathcal{F}(\mathfrak{T})$.

Instead of making the resulting summation formula explicit it is more important to get a proper understanding of its general structure, which will allow us to make a crucial rearrangment of the different summands. Every aligned red–white forest $\mathfrak{F}$ of order $2k$ partitions the set $\{1,\ldots,2k\}$ into subsets labelled by the set of red trees in $\mathfrak{F}$ together with ∞, namely all solitary white vertex trees will have the labels of their leaves in the possibly empty set labelled by ∞, whereas the labels of the leaves of a red tree $\mathfrak{T}$ in $\mathfrak{F}$ form the subset labelled by $\mathfrak{T}$. It is convenient to strip the labels of the twin buds from the latter subsets in this partition to get a partition of the subset $\mathrm{Simple}_{\mathfrak{F}}\subset\{1,\ldots,2k\}$ of labels of leaves different from twin buds into possibly empty subsets labelled by $\mathrm{Red}\,\mathfrak{F}\cup\{\infty\}$. Expanding the covariant derivatives of both the section ψ and the potential $\mathcal{F}$ simultaneously via the general Jet Formula 4.2 results in a summation over all aligned red–white forests $\mathfrak{F}$ of order $2k$ and a jet forest with feedback for each (nonempty) subset in this partition of $\mathrm{Simple}_{\mathfrak{F}}$.

Let us now modify the notion of jet forests by allowing red in addition to black and white vertices subject to the condition that every red vertex is a root with at least two incoming edges and a twin bud. According to the color of their roots the trees in such a red–white jet forest $\mathfrak{F}$ will be called white, red or black trees respectively, moreover a red–white jet forest of even order $2k$ will be called aligned if all twin buds of red vertices are aligned. The definition of feedback maps for jet forests extends verbatim to aligned red–white jet forests $\mathfrak{F}$ of order $2k$, namely f is a map from the set of black trees in $\mathfrak{F}$ to $\{1, \ldots, 2k, \infty\}$ such that $f(\mathfrak{T})$ exceeds the bud label of $\mathfrak{T}$ for all black trees $\mathfrak{T}$ in $\mathfrak{F}$.

Besides feedback maps we will need the notion of hook maps for aligned red–white jet forests as well, namely maps h from the set of black trees in an aligned red–white jet forest $\mathfrak{F}$ to the set $\mathrm{Red}\,\mathfrak{F} \cup \{\infty\}$ of its red trees extended by ∞ such that for every black tree $\mathfrak{T}$ the bud label of $h(\mathfrak{T}) < \infty$ exceeds the bud label of $\mathfrak{T}$ unless $h(\mathfrak{T}) = \infty$ of course. Hook maps provide the information needed to reconstruct the partition of the set $\mathrm{Simple}_{\mathfrak{F}} \subset \{1, \ldots, 2k\}$ of labels of leaves different from twin buds of red vertices into subsets labelled by $\mathrm{Red}\,\mathfrak{F} \cup \{\infty\}$. While the labels of the leaves of a red or white tree $\mathfrak{T}$ are in the subset labelled $\mathfrak{T}$ or ∞ respectively the labels of the leaves of a black tree $\mathfrak{T}$ are in the subset labelled $h(\mathfrak{T})$. A feedback map f is said to be compatible with a hook map h if either $f(\mathfrak{T}) = \infty$ or $f(\mathfrak{T})$ is in the subset of the partition of $\mathrm{Simple}_{\mathfrak{F}}$ labelled $h(\mathfrak{T})$ for all black trees $\mathfrak{T}$. In particular a compatible feedback map f avoids the labels of the twin buds of red vertices.

The reason for introducing the notion of red–white jet forests with hook maps and compatible feedback is that the summands occurring in the simultaneous expansion of all iterated covariant derivatives of both the section ψ and the potential $\mathcal{F}$ in the formula for the powers $\Delta^k_{\diamond,\mathcal{F}}$, $k \geq 0$, of twisted Laplacians with potentials are in bijection to the set of aligned red–white jet forests $\mathfrak{F}$ of order $2k$ with compatible feedback f and hook map h. In fact the straightforward expansion of all the iterated covariant derivatives occurring in Lemma 7.7 results in a sum over all aligned red–white forests $\mathfrak{F}^{\mathrm{rw}}$ of order $2k$ together with jet forests with feedback for every (nonempty) subset in the associated partition of $\mathrm{Simple}_{\mathfrak{F}^{\mathrm{rw}}} \subset \{1, \ldots, 2k\}$. Given on the other hand an aligned red–white jet forest $\mathfrak{F}$ of order $2k$ with compatible feedback f and hook map h the hook map reconstructs the partition of $\mathrm{Simple}_{\mathfrak{F}}$ and thus the underlying red–white forest $\mathfrak{F}^{\mathrm{rw}}$. Forgetting the red vertices in $\mathfrak{F}$ and collecting the remaining jet trees together appropriately associates moreover a jet forest $\mathfrak{F}|_L$ with feedback $f|_L$ to every (nonempty) subset $L \subset \mathrm{Simple}_{\mathfrak{F}}$ of this partition. Clearly the feedback map f must be compatible with the hook map h to get meaningful feedbacks $f|_L$ for the jet forests $\mathfrak{F}|_L$.

In this way we can expand all iterated covariant derivates occurring in Lemma 7.7 simultaneously via the general Jet Formula 4.2 into a sum over all aligned red–white jet forests $\mathfrak{F}$ with compatible feedback f and hook map h. In essence the resulting formula summed over all $k \geq 0$ should be seen as the analogue of equation (7.1) in the presence of auxiliary potentials. Note that once we have expanded the iterated covariant derivatives of the potential $\mathcal{F}$ the rule for decorating a red root with decorations Y, Z on the twin buds and $X_1, \ldots, X_r$ on the sources of the other incoming edges need to be changed from iterated covariant derivatives to their symmetrizations $-\frac{1}{\dim M}\langle Y, Z\rangle\,\mathrm{jet}\,\mathcal{F}(X_1 \cdot \ldots \cdot X_r)$. In particular every red tree $\tilde{\mathfrak{T}} \subset \mathfrak{F}$ in a red–white jet forest $\mathfrak{F}$ defines a multilinear form $\mathrm{jet}\,\mathcal{F}(\tilde{\mathfrak{T}})$ with values in $\mathrm{End}\,EM$. Moreover the black trees $\mathfrak{T} \subset \mathfrak{F}$ with feedback $f(\mathfrak{T}) = \infty$ and hook map $h(\mathfrak{T}) = \tilde{\mathfrak{T}}$ act on the $\mathrm{jet}\,\mathcal{F}(\tilde{\mathfrak{T}})$ by iterated commutators:

$$\Big(\overrightarrow{\prod_{\substack{\mathfrak{T} \subset \mathfrak{F} \text{ black} \\ f(\mathfrak{T})=\infty,\, h(\mathfrak{T})=\tilde{\mathfrak{T}}}}} \mathrm{ad}\,\Omega^E(\mathfrak{T}) \Big)\ \mathrm{jet}\,\mathcal{F}(\tilde{\mathfrak{T}}) \tag{7.10}$$

This equation plays a key role in the third step of our calculation of the heat kernel coefficients of twisted Laplacians $\Delta_{\diamond,\mathcal{F}}$ with potential, which in analogy to the special case of twisted Laplacian will eliminate the sum over hook maps and compatible feedback by judiciously choosing the order of multiplication of the different End EM–factors. Changing the point of view to begin with let us ask for the conditions a hook map h has to satisfy in order to be compatible with a given feedback f avoiding the labels of the twin buds of a red–white jet forest $\mathfrak{F}$. Clearly if $f(\mathfrak{T}) < \infty$ labels a leaf in a red or white tree in $\mathfrak{F}$, then $h(\mathfrak{T})$ must be this red tree or ∞ respectively. Similarly if $f(\mathfrak{T}) < \infty$ labels a leaf in a black tree $\tilde{\mathfrak{T}}$ in $\mathfrak{F}$ necessarily with higher bud label than $\mathfrak{T}$, then we have equality $h(\mathfrak{T}) = h(\tilde{\mathfrak{T}})$. In other words the value $h(\mathfrak{T})$ of the hook map h on a black tree $\mathfrak{T}$ with $f(\mathfrak{T}) < \infty$ is uniquely determined by f alone. In a sense this means that summing over compatible feedback f and hook map h is essentially redundant except for the additional summands arising from black trees $\mathfrak{T}$ with $f(\mathfrak{T}) = \infty$, but $h(\mathfrak{T}) \neq \infty$. In light of equation (7.10) these summands provide exactly the additional commutators we need in order to cancel the summation over both feedback and hook maps by rearranging the multiplication.

Fixing a red–white jet forest $\mathfrak{F}$ and the values for feedback f and compatible hook map h for all black trees in $\mathfrak{F}$ except for the black tree $\mathfrak{T}$ with minimal bud label we want to sum over the possible values for $f(\mathfrak{T})$ and $h(\mathfrak{T})$ making f and h compatible. Unless $f(\mathfrak{T}) = \infty$ the value of $h(\mathfrak{T})$ is uniquely determined by $f(\mathfrak{T})$ and the values of h on the other black trees in $\mathfrak{F}$ with larger bud label. Comparing this with the argument used in the special case of twisted Laplacians without auxiliary potential we can use the summation over f and thus h in decreasing order to commute $\Omega^E(\mathfrak{T})$ past the different endomorphisms arising from the twisted trace $\mathrm{tr}_\diamond$ as long as we do not hit upon an endomorphisms $\mathrm{jet}\,\mathcal{F}(\tilde{\mathfrak{T}})$ arising from the potential. However at this point the summand arising from $f(\mathfrak{T}) = \infty$ and $h(\mathfrak{T}) = \tilde{\mathfrak{T}}$ allows us to commute $\Omega^E(\mathfrak{T})$ past the potential and the directly preceeding twisted trace over the twin buds of $\tilde{\mathfrak{T}}$, which is merely a multiple of the identity, recall that the feedback map f avoids the labels of twin buds of red vertices in $\mathfrak{F}$. Summing in this way over all compatible values for $f(\mathfrak{T})$ and $h(\mathfrak{T})$ we can shuffle $\Omega^E(\mathfrak{T})$ between the two twisted traces

$$\ldots\ \mathrm{tr}_\diamond(\ \alpha_{2s-1} \otimes \alpha_{2s}\ )\ \Omega^E(\mathfrak{T})\ \mathrm{tr}_\diamond(\ \alpha_{2s+1} \otimes \alpha_{2s+2}\ )\ \ldots$$

where $2s-1$ or $2s$ is the bud label of $\mathfrak{T}$ as before. Continuing this way with the other black trees in $\mathfrak{T}$ in turn in increasing order of bud labels we eventually remain with the summation over compatible feedback f and hook map h eliminated in favor of shuffling the endomorphisms on EM arising from the twisted trace, the potential and curvature judiciously before multiplication. More precisely the twisted traces $\mathrm{tr}_\diamond(\alpha \otimes \beta)$ are multiplied in their usual order with interspersed endomorphisms $\mathrm{jet}\,F(\mathfrak{T})$ appearing directly to the right of the twisted trace over the twin buds of $\mathfrak{T}$ as in the sorted application $\mathrm{app}^{\mathrm{sort}}$ while the curvature terms are shuffled into the product like a deck of cards in such a way that $\Omega^E(\mathfrak{T})$ is shuffled between the twisted trace over the pairs of leaves labelled $2s-1, 2s$ and $2s+1, 2s+2$ respectively, where $2s-1$ or $2s$ is the bud label of $\mathfrak{T}$.

With the three critical steps in the calculation of the heat kernel coefficients settled the remaining arguments generalize mutatis mutandis from the special case of twisted Laplacians to the general case. Summarizing these arguments we define red–white Laplace forests $\mathfrak{F}$ with marked leaves L as Laplace forests with additional red vertices subject to the condition that every red vertex is a root with at least two incoming edges and a twin bud. In other words the two leaves of maximal labels in a red tree connect directly to the root and take consecutive labels. Moreover the subset $L \subset \mathrm{Leaf}\,\mathfrak{F} \setminus \mathrm{Twin}\,\mathfrak{F}$ of marked leaves avoids the twin buds of both red and transparent vertices such that the complement $\mathrm{Leaf}\,\mathfrak{F} \setminus (\mathrm{Twin}\,\mathfrak{F} \cup L)$ is the disjoint union of pairs of leaves with consecutive labels. A given decoration of the leaves of a red–white Laplace forests $\mathfrak{F}$ by vector fields $X_1, \ldots, X_k$ can be turned into a decoration of all vertices in $\mathfrak{F}$ using the rules that white or black vertices are decorated by

$$\Phi_r(X_1 \cdot \ldots \cdot X_r)Y \qquad\qquad \Omega^E_r(X_1 \cdot \ldots \cdot X_r)_Y$$

respectively, where Y and $X_1, \ldots, X_r$ are the decorations of the bud and the sources of the remaining incoming edges. Likewise transparent and red vertices are decorated by

$$-\Omega_r(X_1 \cdot \ldots \cdot X_r)_Y Z \qquad\qquad -\frac{1}{\dim M}\langle Y, Z\rangle\,\mathrm{jet}\,\mathcal{F}(X_1 \cdot \ldots \cdot X_r)$$

where Y, Z and $X_1, \ldots, X_r$ are the decorations of the smaller and larger of the twin buds and the source vertices of the remaining incoming edges. In this way every tree in a red–white Laplace forest $\mathfrak{F}$ defines a multilinear form in the vector fields decorating its leaves written say $\Phi(\mathfrak{T})$ or $\mathrm{jet}\,\mathcal{F}(\mathfrak{T})$ for a white or red tree. For white trees including trees with a transparent root this multilinear form will take values in TM, while for red and black trees it will take values in $\mathrm{End}\,EM$ and $\mathfrak{hol}^E M$ respectively. With this notation in mind our considerations above culminate in the following explicit formula for the heat kernel coefficients of a twisted Laplacian $\Delta_{\diamond,\mathcal{F}}$ with potential $\mathcal{F} \in \Gamma(\mathrm{End}\,EM)$:

Theorem 7.8 (Heat Kernel Coefficients of Laplacians with Potentials)
The explicit formula for the generating series $j^{\frac{1}{2}}a(t)$ of the heat kernel coefficients of a twisted Laplacian $\Delta_{\diamond,\mathcal{F}}$ with potential $\mathcal{F} \in \Gamma(\mathrm{End}\,EM)$ can be stated as an equality between $e^{-t\Delta}j^{\frac{1}{2}}a(t)$ and a sum over all red–white Laplace forests $\mathfrak{F}$ with marked leaves $L \subset \mathrm{Leaf}\,\mathfrak{F}$:

$$\sum_{\substack{\mathfrak{F}\ \text{red–white Laplace}\\ L\ \text{marked leaves}}} \frac{t^{\frac{|\mathfrak{F}|-\#L}{2}-\#\mathrm{White}\,\mathfrak{F}}}{2^{\#\mathrm{White}\,\mathfrak{F}}\,(\frac{|\mathfrak{F}|+\#L}{2})!}\, m^*\circ\mathrm{tr}_\diamond^{\frac{|\mathfrak{F}|-\#L}{2}}\circ m_L\left(\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \text{black}}\Omega^E(\mathfrak{T})\otimes\bigotimes_{\mathfrak{T}\subset\mathfrak{F}\ \text{red}}\mathrm{jet}\,\mathcal{F}(\mathfrak{T})\otimes\prod_{\mathfrak{T}\subset\mathfrak{F}\ \text{white}}\Phi(\mathfrak{T})^\sharp\right)$$

Chapter 8

A Combinatorial Proof of the Local Index Theorem

There are various reasons to think that the formulas given in Theorems 7.5 and 7.8 for the generating series $j^{\frac{1}{2}}a(t)$ of the heat kernel coefficients of generalized Laplace operators Δ have not yet found their proper formulation. As far as the Local Index Theorem in the form of Theorem 6.2 is a reliable approximation we would rather expect to find a formula in terms of an exponential of some sort. Moreover it may not be readily clear to the reader how explicit these formulas really are. For this reason we will discuss some aspects of the formulas given in Theorems 7.5 and 7.8 in greater detail in this final section. In particular we will see that the formulas are explicit enough to give a combinatorial proof of the Local Index Theorem simply by summing the relevant summands of Theorem 7.8.

Perhaps the most profound difference between the combinatorial proof of the Local Index Theorem given in this section to other proofs is that it makes no reference to Mehler's formula at all in one way or other. In a sense the purely combinatorial interpretation (8.2) of the Taylor series of the hyperbolic tangens given in an excursion into combinatorics in the first part of this section can be seen as compensation for not using Mehler's formula in that it explains the appearance of the hyperbolic tangens in equation (6.1) for the $\widehat{A}$–genus say. At the same time of course this interpreation of the Taylor series of tanh z turns into a beautiful combinatorial interpretation of the Bernoulli numbers B_{2k}, $k \geq 1$ in light of the classical Taylor series $\tanh z = \sum_{k>0} \frac{4^k(4^k-1)}{(2k)!} B_{2k}$. In the second part of this section we will sum the contribution of all Laplace forests consisting entirely of black trees with two vertices to the formula given in Theorem 7.5, the expected exponential will reemerge in this process of summation through the combinatorial identity (8.10).

Before we begin let us make a few philosophical remarks about the combinatorial meaning of the summation over Laplace forests. The contribution of each forest is a product of the multilinear forms $\Phi(\mathfrak{T})$ and $\Omega^E(\mathfrak{T})$ associated to the Laplace trees $\mathfrak{T}$, which essentially depend only on the isomorphism class of the trees in the category of budding trees with morphisms respecting the budding edges. The labelling of the leaves of the trees $\mathfrak{T}$ in a given Laplace forest $\mathfrak{F}$ on the other hand is relevant for the way these multilinear forms are assembled into the contribution associated to the forest $\mathfrak{F}$. It seems possible to sum this contribution of a given jet forest $\mathfrak{F}$ over all automorphisms of $\mathfrak{F}$ as a budding tree or equivalently over all possible labellings of its leaves preserving the budding condition in order to simplify the appearance of the formulas given for the heat kernel coefficients. In a sense this section can be seen as vindication of this idea in that the Local Index Theorem is a statement about a single isomorphism class of Laplace forests, namely the isomorphism class of Laplace forests consisting entirely of black trees with two vertices.

Recall that a cycle $\gamma \in S_{r+1}$ in the symmetric group acting on $\{1,\ldots,r+1\}$ is a permutation with a single orbit on $\{1,\ldots,r+1\}$ or equivalently a permutation conjugated to the right shift defined by $\mathrm{shift}(\mu) := \mu+1$ for $\mu \leq r$ and $\mathrm{shift}(r+1) := 1$. Conjugation thus defines a bijection $S_r \stackrel{\cong}{\longrightarrow} \Gamma_{r+1}$, $\sigma \longmapsto \sigma \circ \mathrm{shift} \circ \sigma^{-1}$, between the subset $\Gamma_{r+1} \subset S_{r+1}$ of all cycles and the subgroup $S_r \subset S_{r+1}$ of permutations fixing 1. Excluding the trivial case $r = 0$ and thinking of a cycle as a walk on $\{1,\ldots,r+1\}$ we may say that a cycle $\gamma \in \Gamma_{r+1}$ with $r \geq 1$ never rests $\gamma(\mu) \neq \mu$ and thus goes ahead $\gamma(\mu) > \mu$ or steps back $\gamma(\mu) < \mu$. What links this point of view to heat kernel coefficients is that the signs of particularly interesting summands in Theorem 7.5 depend on the parity of the number of backwards steps

$$\mathrm{Back}\,\gamma \;:=\; \#\{\,\mu \mid \gamma(\mu) < \mu\,\}$$

of associated cycles γ. The concept corresponding to backward steps under the bijection $S_r \stackrel{\cong}{\longrightarrow} \Gamma_{r+1}$ is known as a "run" in the classical merge sort algorithm. Namely a run in a given permutation $\sigma \in S_r$ is a maximal subset of consecutive numbers in $\{1,\ldots,r+1\}$ on which σ is (necessarily strictly) increasing. Evidently the number of such runs is given by:

$$\mathrm{Run}\,\sigma \;:=\; 1 \;+\; \#\{\,\mu \mid \mu \neq r+1 \text{ and } \sigma(\mu) > \sigma(\mu+1)\,\}$$

Under the bijection $S_r \stackrel{\cong}{\longrightarrow} \Gamma_{r+1}$ the number of runs of a permutation σ becomes the number of backward steps of the cycle $\gamma := \sigma \circ \mathrm{shift} \circ \sigma^{-1}$. Presumably it is simplest to verify this statement graphically, a formal proof notes that the cycle γ necessarily steps backwards at $\sigma(r+1)$, because $\gamma(\sigma(r+1)) = \sigma(1) < \sigma(r+1)$ with σ fixing 1, and checks that σ induces a bijection between $\{\mu|\,\mu \neq r+1 \text{ and } \sigma(\mu) > \sigma(\mu+1)\}$ and $\{\mu|\,\mu \neq \sigma(r+1) \text{ and } \mu > \gamma(\mu)\}$.

Definition 8.1 (Counting Polynomials $\theta_{r+1}(w)$)
In order to count the cycles $\gamma \in \Gamma_{r+1}$ with a given number of backward steps or equivalently the permutations $\sigma \in S_r$ with a given number of runs we introduce the counting polynomials

$$\theta_{r+1}(w) \;:=\; \sum_{\gamma \in \Gamma_{r+1}} w^{\mathrm{Back}\,\gamma} \;=\; \sum_{\sigma \in S_r} w^{\mathrm{Run}\,\sigma}$$

for $r \geq 1$ and their generating formal power series $\theta(z,w) := \sum_{r\geq 1} \frac{\theta_{r+1}(w)}{r!} z^r$. The first few counting polynomials read $\theta_2(w) = w$, $\theta_3(w) = w + w^2$ and $\theta_4(w) = w + 4w^2 + w^3$.

The apparent symmetry $w^{r+1}\theta_{r+1}(\frac{1}{w}) = \theta_{r+1}(w)$ of the coefficients of the counting polynomials $\theta_{r+1}(w)$, $r \geq 1$, is evidently due to the involution $\gamma \longmapsto \gamma^{-1}$ on Γ_{r+1} satisfying $\mathrm{Back}\,\gamma^{-1} = r+1-\mathrm{Back}\,\gamma$. Using the alternative interpretation in terms of runs of permutations however makes it slightly simpler to write down the following recursion relation:

$$\theta_2(w) \;:=\; w \qquad\qquad \theta_{r+1}(w) \;:=\; \left[\, w(1-w)\frac{d}{dw} \;+\; rw \,\right]\theta_r(w) \qquad r > 1$$

In fact given a permutation $\sigma \in S_{r-1}$ acting on $\{1, \ldots, r\}$ fixing 1 with k runs there are exactly k different positions to insert $r+1$ without breaking any of the runs of σ in two, namely the k positions at the very end of these runs. In this way every $\sigma \in S_{r-1}$ with k runs gives rise to exactly k permutations $\sigma^{\text{new}} \in S_r$ with k runs and $r-k$ permutations $\sigma^{\text{new}} \in S_r$ with $k+1$ runs. On the other hand the recursion operator $w(1-w)\frac{d}{dw} + rw$ maps w^k to $kw^k + (r-k)w^{k+1}$. Writing the recursion relation in terms of the generating formal power series $\theta(z, w)$ of the polynomials $\theta_{r+1}(w)$, $r \geq 1$, we find the differential equation:

$$\left[(1 - zw)\frac{\partial}{\partial z} - w(1-w)\frac{\partial}{\partial w} \right] \log\left(1 + \theta(z, w) \right) = w \tag{8.1}$$

Actually $\theta(z, w)$ is the unique solution to the differential equation (8.1) with an expansion of the form $\sum_{r \geq 1} \frac{\theta_{r+1}(w)}{r!} z^r$ with formal power series $\theta_{r+1}(w)$, $r \geq 1$, in the origin $z = 0 = w$. Integrating the flow lines of the vector field $\frac{(1-zw)}{1-w}\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}$ away from its singularities we see that in the domain $w \neq 0, 1$ the general solution to (8.1) can be written in the form $1 + \theta(z, w) = (1-w)\,\vartheta(\, z(1-w) + \log w\,)$ with an arbitrary function ϑ. Clairvoyantly anticipating (8.2) we can use this presentation to find the solution we are interested in:

Lemma 8.2

$$\theta(z, w) = -1 + \left(1 - w\frac{e^{z(1-w)} - 1}{1-w} \right)^{-1}$$

Indeed expanding the expression stated for $\theta(z, w)$ into a formal power series we see that it has an expansion $\sum_{r \geq 1} \frac{\theta_{r+1}(w)}{r!} z^r$ of the required form with polynomials $\theta_{r+1}(w)$, $r \geq 1$. More precisely it is possible to invert the series $1 - \sum_{r>0} \frac{z^r}{r!} w(1-w)^{r-1}$ explicitly to get

$$\theta_{r+1}(w) = \sum_{p_1 + 2p_2 + \ldots + rp_r = r} \frac{r!}{1!^{p_1}\, 2!^{p_2} \, \ldots \, r!^{p_r}} \binom{p_1 + \ldots + p_r}{p_1 \;\; p_2 \;\; \ldots \;\; p_r} w^{p_1 + \ldots + p_r} (1-w)^{r - p_1 - \ldots - p_r}$$

where the summation takes the typical form of a summation over all partitions of r involving in particular the multinomial coefficients $\binom{p_1 + \ldots + p_r}{p_1 \; p_2 \; \ldots \; p_r}$. As we know that $\theta(z, w)$ is the unique solution to the differential equation (8.1) with such an expansion it remains to calculate

$$\left[w(1-w)\frac{\partial}{\partial w} \right] \left(1 - w\frac{e^{z(1-w)} - 1}{1-w} \right) = -(1-zw)\, w\, e^{z(1-w)} + w \left(1 - w\frac{e^{z(1-w)} - 1}{1-w} \right)$$

which is equivalent the differential equation (8.1) in the form:

$$\left[(1 - zw)\frac{\partial}{\partial z} - w(1-w)\frac{\partial}{\partial w} \right] \log \left(1 - w\frac{e^{z(1-w)} - 1}{1-w} \right) = -w$$

At the end of this brief excursion into combinatorics we are now in the position to derive the result we will need in our discussion of the Local Index Theorem. Namely the hyperbolic tangens enters the calculation of the index density as the generating series of the special values $\theta_{r+1}(-1)$, $r \geq 1$, of the counting polynomials $\theta_{r+1}(w)$, which in turn can be interpreted as the difference between the number of cycles in Γ_{r+1} stepping back an even and an odd number of times respectively. Evaluating the generating formal power series $\theta(z, w)$ for the counting polynomials $\theta_{r+1}(w)$, $r \geq 1$, at $w = -1$ results in the identification:

$$\theta(z,-1) \;=\; \sum_{r\geq 1} \frac{\theta_{r+1}(-1)}{r!} z^r \;=\; -1 \,+\, \frac{2}{1\,+\,e^{2z}} \;=\; -\tanh z \tag{8.2}$$

Note that the hyperbolic tangens is odd and indeed the special values $\theta_{r+1}(-1) = 0$ vanish for even r by the symmetry $w^{r+1}\theta_{r+1}(\frac{1}{w}) = \theta_{r+1}(w)$. Comparing the coefficients in equation (8.2) with the classical Taylor series of the hyperbolic tangens featuring the Bernoulli numbers $\tanh z = \sum_{k>0} \frac{4^k(4^k-1)}{(2k)!} B_{2k} z^{2k-1}$ we get a formula for the special values $\theta_{r+1}(-1)$ for odd r

$$B_{2k} \;=\; -\frac{2k}{4^k(4^k-1)}\,\theta_{2k}(-1) \;=\; -\frac{2k}{4^k(4^k-1)} \sum_{\sigma\in S_{2k-1}} (-1)^{\mathrm{Run}\,\sigma} \qquad k \geq 1$$

which interpretes the Bernoulli numbers B_{2k} combinatorially as the difference between the number of permutations in S_{2k-1} with an even and an odd number of runs respectively.

Coming back to the heat kernel coefficients of the trivially twisted connection Laplacian $\Delta = \nabla^*\nabla$ on sections of a vector bundle EM with connection over a manifold M we note that in the trivially twisted case $\mathrm{tr}_\diamond(\alpha\otimes\beta) = \langle\alpha,\beta\rangle\mathrm{id}_E$ always results in multiples of the identity. Eliminating these multiples of the identity reduces the delicately chosen multiplication m^* to the multiplication $\overrightarrow{m}$ of the curvature terms $\Omega^E(\mathfrak{T})$ in increasing order of bud labels while the trace becomes the iterated trace $\mathrm{tr}^r : \bigotimes^{2r} T^*M \longrightarrow \mathbb{R}M$ over suitable pairs of forms. With these simplifications Theorem 7.5 reads

$$j^{\frac{1}{2}}\,a(t) \;=\; e^{t\Delta}\left[\sum_{\substack{\mathfrak{F}\text{ Laplace forest}\\ L\text{ marked leaves}}} \frac{t^{\frac{|\mathfrak{F}|-\#L}{2}-\#\mathrm{White}\,\mathfrak{F}}}{2^{\#\mathrm{White}\,\mathfrak{F}}\,(\frac{|\mathfrak{F}|+\#L}{2})!}\,\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}} \circ m_L\left(\overrightarrow{\prod_{\mathfrak{T}\subset\mathfrak{F}\text{ black}}}\Omega^E(\mathfrak{T}) \otimes \prod_{\mathfrak{T}\subset\mathfrak{F}\text{ white}}\Phi(\mathfrak{T})^\sharp\right)\right] \tag{8.3}$$

for trivially twisted Laplacians. Looking more closely at this explicit formula for the heat kernel coefficients we see that a jet forest $\mathfrak{F}$ contributes to the value $a_d(0)$ of the heat kernel coefficient a_d along the diagonal only for $d = \frac{|\mathfrak{F}|-\#\mathrm{White}\,\mathfrak{F}}{2}$ provided $\#\mathrm{White}\,\mathfrak{F}$ and $|\mathfrak{F}|$ have the same parity. In particular jet forests with more than d trees are irrelevant to $a_d(0)$, because every tree $\mathfrak{T}$ will increase $\frac{|\mathfrak{F}|-\#\mathrm{White}\,\mathfrak{F}}{2}$ at least by 1. Moreover black trees contribute a factor linear in $\mathfrak{hol}^E M$ to the summands in (8.3) while white trees only contribute scalars. Consequently the value of the d–th coefficient a_d along the diagonal is a section $a_d(0) \in \Gamma(\mathcal{U}^{\leq d}\mathfrak{hol}^E M)$ for all $d \geq 0$ and its symbol class $[a_d(0)] \in \Gamma(\mathrm{Sym}^{\,d}\mathfrak{hol}^E M)$ only depends on jet forests $\mathfrak{F}$, which have exactly d black trees with two leaves and no white trees.

A more sophisticated argument to the same end bridges the gap between the explicit formula (8.3) for the heat kernel coefficients of connection Laplacians $\nabla^*\nabla$ and the considerations in Section 5. Note first that the bundle $\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M$ of Taylor series of universal heat kernel coefficients is actually an algebra bundle. At the time we introduced the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M)$ in Definition 5.2 we were more interested in the induced filtration on the space of operators on $\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M$ than in the algebra structure, nevertheless the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M)$ is an algebra filtration by the trivial inequality $\lfloor\frac{d}{2}\rfloor + \lfloor\frac{\tilde{d}}{2}\rfloor \leq \lfloor\frac{d+\tilde{d}}{2}\rfloor$. Consequently the extended filtration

$$\mathbb{F}^r(\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]) \quad := \quad \sum_{d\geq 0} \sqrt{t}^{\,d} \bigotimes T^*M \otimes \mathbb{F}^{d+r}(\,\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M\,)$$

turns $\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]$ into a filtered algebra bundle as well. Quite surprisingly every single piece of the explicit formula (8.3) for the heat kernel coefficients of $\nabla^*\nabla$ can be interpreted in this filtration. To begin with the multilinear forms $\Phi(\mathfrak{T})$ and $\Omega^E(\mathfrak{T})$ associated to a tree $\mathfrak{T}$ in a Laplace forest $\mathfrak{F}$ scaled by appropriate powers of t

$$\begin{array}{rcl} t^{\frac{|\mathfrak{T}|}{2}}\,\Omega^E(\mathfrak{T}) & \in & \Gamma(\,\mathbb{F}^{2-|\mathfrak{T}|}(\,\bigotimes^{|\mathfrak{T}|} T^*M \,\otimes\, \mathrm{Sym}\,T^*M \,\otimes\, \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]\,)\,) \\ t^{\frac{|\mathfrak{T}|}{2}-1}\,\Phi(\mathfrak{T})^\sharp & \in & \Gamma(\,\mathbb{F}^{1-|\mathfrak{T}|}(\,\bigotimes^{|\mathfrak{T}|} T^*M \,\otimes\, \mathrm{Sym}\,T^*M \,\otimes\, \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]\,)\,) \end{array}$$

are sections of the algebra bundle $\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]$ and so is their product

$$t^{\frac{|\mathfrak{F}|}{2} - \#\mathrm{White}\,\mathfrak{F}} \overrightarrow{\prod_{\mathfrak{T}\subset\mathfrak{F}\ \mathrm{black}}} \Omega^E(\mathfrak{T}) \;\otimes\; \prod_{\mathfrak{T}\subset\mathfrak{F}\ \mathrm{white}} \Phi(\mathfrak{T})^\sharp \tag{8.4}$$

which is a section of $\mathbb{F}^{2\#\mathrm{Black}\,\mathfrak{F} + \#\mathrm{White}\,\mathfrak{F} - |\mathfrak{F}|}(\,\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]\,)$. As every black tree has at least two, every white tree at least three leaves the symbol class of (8.4) in

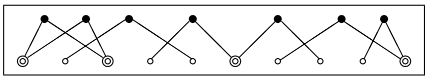
$$\mathbb{F}^0(\,\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]\,) \;/\; \mathbb{F}^{-1}(\,\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!]\,)$$

vanishes unless the Laplace forest $\mathfrak{F}$ consists entirely of black trees with exactly two leaves. On the other hand the operator $t^{-\frac{\#L}{2}} m_L$ and the iterated trace $\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}}$ are both filtered of degree 0 as is the operator $e^{t\Delta}$. This is clear for the iterated trace, because the contracted tensor factor $\bigotimes T^*M$ does not really participate in the definition of the filtration. For the multiplication m_L of marked leaves the increase in the degree in the polynomial factor is compensated by multiplication with $t^{-\frac{\#L}{2}}$. Eventually the operator $t\Delta$ is filtered of degree 0 and hence so is $e^{t\Delta}$, because the multiplication by t, which is filtered of degree 2, is countered by decreasing the degree of the polynomial factor by 2. The latter property is ultimately the justification for the Definition 5.2 of the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M)$ of course. Consequently the generating series $a(t) = \sum_{d\geq 0} t^d a_d$ for the heat kernel coefficients is a section of the filtration subbundle $\mathbb{F}^0(\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!])$ and its symbol class

$$[\,a(t,\,\cdot\,)\,] \;=\; e^{t\Delta}\left[\sum_{\substack{\mathfrak{F}\ \text{special Laplace}\\ L\ \text{marked leaves}}} \frac{t^{\frac{|\mathfrak{F}|-\#L}{2}}}{(\frac{|\mathfrak{F}|+\#L}{2})!}\,(\,\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}} \circ m_L\,)\left(\prod_{\mathfrak{T}\subset\mathfrak{F}} (\,\tfrac{1}{2}R^E\,)_{\cdot,\cdot}(\mathfrak{T}) \;\otimes\; 1\right)\right] \tag{8.5}$$

in $\Gamma(\mathrm{Sym}\,T^*M \otimes \mathrm{Sym}\,\mathfrak{hol}^E M[\![t]\!])$ only depends on special Laplace forests, i. e. Laplace forests consisting entirely of black trees with two leaves. In particular the symbol class is a polynomial in the curvature tensor R^E alone not involving any covariant derivatives.

In order to get a combinatorial interpretation of the iterated trace $\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}}$ appearing in formula (8.5) we recall that the set L of marked leaves for a special Laplace forest $\mathfrak{F}$ needs to be chosen in such a way that the unmarked leaves $\mathrm{Leaf}\,\mathfrak{F}\setminus L$ come in disjoint pairs of leaves with consecutive labels. Contracting these distinguished pairs of unmarked leaves into new vertices we get a bipartite graph with two different kinds of vertices, black roots and white leaves, the latter coming in two subvarieties either marked or contracted. Instead of the original labelling we will only retain the induced total ordering on the set of leaves. A not too simple example of such a contracted graph arising from a special Laplace forest of order 14 looks like



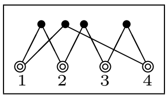
(8.6)

where the total ordering of the leaves, a remnant of the original labelling, is indicated by sorting the leaves in ascending order from left to right. Pictures like (8.6) by the way imply that the number of sets of marked leaves in a Laplace forest of order k without transparent vertices is given by the Fibonacci numbers $\sum_{r=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-r}{r}$, expanding r arbitrary "contracted" in $k-r$ leaves into a pair will see the remaining leaves form a legal subset of $k-2r$ marked leaves in k. In a general Laplace forest the number of sets of marked leaves is a product of Fibonacci numbers depending of the position of the twin buds.

Studying picture (8.6) we observe that the contracted graphs decompose into two kinds of connected subgraphs, closed circles with black roots alternating with contracted leaves and paths starting and ending in marked leaves with a chain of black roots and contracted leaves in between. For convenience we will refer to these subgraphs as closed and broken reels respectively, their length is the number of white leaves. The sample graph in picture (8.6) has thus a closed reel of length 2 and three broken reels of length 2 and twice 3 respectively.

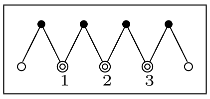
The definition of contracted graphs is set up in such a way that the iterated trace $\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}}$ sums the pairs of forms in $\bigotimes T^*M$ corresponding to a pair of contracted leaves over a local orthonormal base $\{X_\mu\}$ of the euclidian vector bundle TM. This description of the action of the iterated trace is very similar to the definition of the trace $\mathrm{tr}\,B := \sum_\mu B_{x_\mu,x_\mu}$ and the multiplication $(B\tilde{B})_{X,Y} := \sum_\mu B_{X,x_\mu}\tilde{B}_{x_\mu,Y}$ on the space of bilinear forms on a euclidian vector space T with values in $\mathrm{Sym}\,\mathfrak{hol}^E$ we introduced in the course of calculating the symbol classes of the heat kernel coefficients in Section 6. Comparing both definitions we easily see that the contribution of a closed or broken reel in the contracted graph of a special Laplace forest $\mathfrak{F}$ with marked leaves L to the summand associated to $\mathfrak{F}$ and L in formula (8.5) can be written in terms of multiplication and trace on the space of bilinear forms, say:

$$\mathrm{tr}(\,\frac{1}{2}R^E\,)^4 \quad =$$



$$= \quad \frac{1}{2^4}\sum_{\mu_1\,\mu_2\,\mu_3\,\mu_4} R^E_{X_{\mu_1},X_{\mu_2}}\,R^E_{X_{\mu_1},X_{\mu_4}}\,R^E_{X_{\mu_2},X_{\mu_3}}\,R^E_{X_{\mu_3},X_{\mu_4}} \tag{8.7}$$

$$(\,\frac{1}{2}R^E\,)^4_{X,X} \quad =$$



$$= \quad \frac{1}{2^4}\sum_{\mu_1\,\mu_2\,\mu_3} R^E_{X,X_{\mu_1}}\,R^E_{X_{\mu_1},X_{\mu_2}}\,R^E_{X_{\mu_2},X_{\mu_3}}\,R^E_{X_{\mu_3},X} \tag{8.8}$$

Of course the numbers on the contracted leaves refer to the associated summation index and not to any labelling of the leaves. In general the summand associated to a special Laplace forest $\mathfrak{F}$ with marked leaves L in equation (8.5) is the product of the factors $\pm\mathrm{tr}(\frac{1}{2}R)^r$ or $\pm(\frac{1}{2}R)^{r-1}_{X,X}$ contributed by the closed and broken reels respectively of length r in the contracted graph of $\mathfrak{F}$, for the moment there is no need to worry about the proper sign.

Evidently the contracted graph allows us to recover the special Laplace forest $\mathfrak{F}$ with its set of marked leaves L only up to an ambiguity arising from the two possible ways to connect the two leaves in a contracted pair of unmarked leaves to black roots. A superficial glance may make one believe that each closed or broken reel of length r contributes the factor 2^r and 2^{r-2} respectively to the number of preimage forests. However the closed reel of length 2 is anomalous in that the four ways to expand the two contracted leaves only results in two different special Laplace forests. Avoiding this problem for the moment we are content to simply assert that the number of preimage forests with marked leaves of a given contracted graph is the product of the numbers of preimages of its reels while the number of preimages of reels only depend on their length and whether they are closed or broken.

This factorization of the number of preimage forests of a contracted graph over the number of preimage forests of its reels as well as the corresponding factorization of the term $(\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}} \circ m_L)(\prod(\frac{1}{2}R^E)_{\cdot,\cdot}(\mathfrak{T}) \otimes 1\,)$ in (8.5) into factors $\pm\mathrm{tr}(\frac{1}{2}R^E)^r$ and $(\frac{1}{2}R^E)^{r-1}_{\cdot,\cdot}$ associated to reels is the conditio sine qua non for using the key combinatorial identity (8.10) making summation over all special Laplace forests with marked leaves feasible. In passing we note that even the factor $t^{\frac{|\mathfrak{F}|-\#L}{2}}$ in (8.5) factorizes into contributions t^r and t^{r-2} associated to closed and broken reels of length r respectively, because $\frac{|\mathfrak{F}|-\#L}{2}$ and $\frac{|\mathfrak{F}|+\#L}{2}$ are the numbers of contracted leaves and leaves respectively of the contracted graph arising from a special Laplace forest $\mathfrak{F}$ with marked leaves L.

The upshot of the preceeding considerations is that the summation over special Laplace forests $\mathfrak{F}$ with marked leaves L can be turned into a summation over all contracted graphs weighted by the number of preimage forests with marked leaves in such a way that the summand associated to each contracted graph factorizes completely into contributions from the reels except for the factorial factor $(\frac{|\mathfrak{F}|+\#L}{2})!$, which only depends on the total number $\frac{|\mathfrak{F}|+\#L}{2}$ of leaves of the contracted graph. Up to sign the contribution of each reel in turn depends only on its length and on whether it is closed or broken. Schematically the resulting summation over all contracted graphs makes formula (8.5) for the heat kernel coefficiens read

$$[\,a(\,t,\,\cdot\,)\,] \;=\; e^{t\Delta}\left[\sum_{k\geq 0}\frac{1}{k!}\sum_{\substack{G\text{ contracted graph}\\ \text{with }k\text{ leaves}}}\;\prod_{\text{reel in }G}(\quad\ldots\quad)\right] \tag{8.9}$$

A fundamental philosophical tenet of working with forests says that the sum over all forests is the same as taking the exponential of the sum over all trees. In order to make this principle sufficiently precise for our needs let us consider a function W from the finite subsets of $\mathbb{N}$ to a commutative algebra with 1 such that $W(\emptyset) = 1$ and $W(I) = W(\tilde{I})$ for all subsets I and $\tilde{I}$ of $\mathbb{N}$ of the same cardinality. In this case the following identity holds true

$$\sum_{k\geq 0}\frac{1}{k!}\sum_{\substack{P\text{ partion}\\ \text{of }\{1,\ldots,k\}}}\prod_{I\in P}W(I) \;=\; \exp\left(\sum_{r\geq 1}\frac{1}{r!}W(\{1,\ldots,r\})\right) \tag{8.10}$$

where the inner sum on the left is over (unordered) partitions of $\{1,\ldots,k\}$, i. e. partitions thought of as subsets $\emptyset \notin P \subset \mathfrak{P}(\{1,\ldots,k\})$ of the power set of $\{1,\ldots,k\}$. Identity (8.10) is essentially a reformulation of the statement that there are exactly $\frac{k!}{1!^{p_1}p_1!\ldots k!^{p_k}p_k!}$ different partitions P containing p_1 sets of size 1, p_2 sets of size 2 and so on.

Instead of a partition of $\mathbb{N}$ every contracted graph with k leaves defines a partition on the totally ordered set of its leaves given by the subsets of leaves of reels. Identifying the set of leaves with $\{1, \ldots, k\}$ via the total ordering we see that the summation over all contracted graphs in (8.9) agrees with the left hand side of identity (8.10) if we define the function W on subsets $I \subset \{1, \ldots, k\}$ with values in the commutative algebra $\Gamma(\mathrm{Sym}\, T^*M \otimes \mathrm{Sym}\, \mathfrak{hol}^E M)$ as the net contribution of all possible reels on I to the product over reels in (8.9). In fact the simplest way to generate all contracted graphs with k leaves is to choose a partition on the set $\{1, \ldots, k\}$ and a closed or broken reel for each set in this partition independently. The product of the $W(I)$ over all subsets I in a partition of $\{1, \ldots, k\}$ thus reproduces precisely the sum over all contracted graph with this prescribed partition.

Clearly for calculating the value $W(\{1, \ldots, r\})$ of the function W we need to sum the contributions of the reels on $\{1, \ldots, r\}$ to the product in (8.9), which are made up from the curvature terms $\pm t^r\, \mathrm{tr}(\frac{1}{2}R^E)^r$ or $\pm t^{r-2}\, (\frac{1}{2}R^E)^{r-1}_{\cdot,\cdot}$ for closed and broken reels respectively and the weighting factor counting the number of different forests we get by expanding contracted leaves into pairs. Interestingly $\mathrm{tr}(\frac{1}{2}R^E)^r = 0$ and $(\frac{1}{2}R^E)^{r-1}_{\cdot,\cdot} = 0$ vanish for odd or even r respectively so that $W(\{1, \ldots, r\})$ will depend either only on closed or only on broken reels. Let us have a closer look at the case of closed reels or equivalently r even first. As a graph a closed reel on $\{1, \ldots, r\}$ is a circle of with r black roots alternating with the r white leaves $\{1, \ldots, r\}$. Choosing an orientation associates a cycle $\gamma \in \Gamma_r$ to this circle and establishes a bijection between closed reels and pairs $\{\gamma, \gamma^{-1}\}$ of cycles in Γ_r. Looking at example (8.7) we note that moreover the proper sign of the curvature term $\pm\, t^r\, \mathrm{tr}(\frac{1}{2}R^E)^r$ associated to a closed reel is given by $(-1)^{\mathrm{Back}\,\gamma}$ independent of the choice of orientation as r is even. In this context the anomaly of the closed reel of length 2 reflects the fact that the associated cycle is the unique cycle with $\gamma = \gamma^{-1}$. Pretending that even in case $r = 2$ there are 2^r different preimage forests for a closed reel of length r we sum over all cycles and divide the result by 2 to account for the two different choices of orientation to find

$$W(\{1, \ldots, r\}) \;=\; 2^r\, t^r\, \frac{\theta_r(-1)}{2}\, \mathrm{tr}(\,\tfrac{1}{2}R^E\,)^r \;=\; \frac{\theta_r(-1)}{2}\, \mathrm{tr}(\,tR^E\,)^r \tag{8.11}$$

for all even $r \geq 2$. Turning to r odd or equivalently broken reels we observe that the underlying graph of a broken reel can be closed to a circle with alternating black roots and white leaves by introducing a ficticious white leaf connected to the unmarked leaves of the broken reel over two ficticious black roots. Associated to the two orientations of this circle are two cycles in Γ_{r+1} so that the broken reels are in bijection to pairs $\{\gamma, \gamma^{-1}\}$ of cycles in Γ_{r+1} this time. Example (8.8) tells us that the proper sign of the curvature term $\pm\, t^{r-2}(\frac{1}{2}R^E)^{r-1}$ is given by $-\,(-1)^{\mathrm{Back}\,\gamma}$ independent of cycle γ or γ^{-1} we choose as $r+1$ is even. Counting the number of preimage forests with marked leaves we conclude

$$W(\{1, \ldots, r\}) \;=\; -\,2^{r-2}\, t^{r-2}\, \frac{\theta_{r+1}(-1)}{2}\, (\,\tfrac{1}{2}R^E\,)^{r-1}_{\cdot,\cdot} \;=\; -\,\frac{\theta_{r+1}(-1)}{4t}\, (\,tR^E\,)^{r-1}_{\cdot,\cdot} \tag{8.12}$$

for all odd $r \geq 3$ and $W(\{1\}) = 0$, because there are no broken reels of length 1. For the particular function W from the subsets of $\mathbb{N}$ to $\Gamma(\mathrm{Sym}\, T^*M \otimes \mathrm{Sym}\, \mathfrak{hol}^E M)$ determined by equations (8.11) and (8.12) the right hand side of the key identity (8.10) reads

$$\begin{aligned}\exp\Big(\sum_{r\geq 1}\frac{1}{r!}W(\{1,\ldots,r\})\Big) &= \exp\Bigg(\frac{1}{2}\sum_{\substack{r\geq 2\\ r\text{ even}}}\frac{\theta_r(-1)}{r!}\mathrm{tr}(tR^E)^r - \frac{1}{4t}\sum_{\substack{r\geq 3\\ r\text{ odd}}}\frac{\theta_{r+1}(-1)}{r!}(tR^E)^{r-1}_{\cdot,\cdot}\Bigg)\\ &= \exp\Big(-\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}(\,tR^E\ \tanh\, tR^E\,)+\frac{1}{4t}(\frac{\tanh\, tR^E}{tR^E}-1)_{\cdot,\cdot}\Big)\end{aligned}$$

in terms of the interpretation (8.2) of the Taylor series of the hyperbolic tangens. The combinatorial identity (8.10) thus turns the schematical formula (8.9) for the symbol class $[a(t,\cdot)]$ of the heat kernel coefficients of connection Laplacians $\nabla^*\nabla$ into the explicit formula:

$$[\,a(t,X)\,] \;=\; e^{t\Delta}\exp\Big(\,-\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}(\,tR^E\ \tanh\, tR^E\,)\;+\;\frac{1}{4t}(\,\frac{\tanh\, tR^E}{tR^E}\;-\;1\,)_{X,X}\Big) \tag{8.13}$$

In a final step we have to evaluate the exponential of the flat Laplacian Δ in (8.13). Contemplating this problem for some time the reader will certainly agree that the result of applying $e^{t\Delta}$ to the exponential of a quadratic form in X is again the exponential of a quadratic form. Perhaps it is a good idea to try some toy example first, namely the exponential of the quadratic form ax^2+b on the real line with $a,\,b\,\in\,\mathbb{R}$. In this case we find

$$e^{-t\frac{\partial^2}{\partial x^2}}\ \exp\Big(\,ax^2\,+\,b\,\Big)\;=\;\exp\Big(\,\frac{a}{1\,+\,4ta}x^2\,+\,b\,-\,\frac{1}{2}\,\log(\,1\,+\,4ta\,)\,\Big) \tag{8.14}$$

for t close enough to 0, because the right hand side satisfies the differential equation

$$\frac{\partial}{\partial t}\exp\Big(\,\frac{a}{1+4ta}x^2+b-\frac{1}{2}\log(1+4ta)\,\Big)\;=\;-\frac{\partial^2}{\partial x^2}\exp\Big(\,\frac{a}{1+4ta}x^2+b-\frac{1}{2}\log(1+4ta)\Big)$$

and reduces to $\exp(\,ax^2+b\,)$ at time $t\,=\,0$. Equation (8.14) remains valid if a and b are no longer assumed constant, but can be functions of t as well. In the argument given above we need only rename t into τ for a moment and evaluate $e^{-\tau\frac{\partial^2}{\partial x^2}}\exp(a_tx^2+b_t)$ at $t\,=\,\tau$ in the end. With this toy model settled it is not too difficult to guess the appropriate generalization

$$e^{t\Delta}\ \exp\Big(\,(\,A_t\,)_{X,X}\,+\,\mathrm{tr}\,B_t\,\Big)\;=\;\exp\Big(\,(\,\frac{A_t}{1\,+\,4t\,A_t}\,)_{X,X}\,+\,\mathrm{tr}\,B_t\,-\,\frac{1}{2}\,\mathrm{tr}\,\log(\,1\,+\,4t\,A_t)\Big)$$

to symmetric bilinear forms on a euclidian vector space T_yM with values in a commutative algebra like $\mathrm{Sym}\,\mathfrak{hol}^E_yM$. Using this identity together with Remark 6.1 in the form $\log\frac{\tanh z}{z}\;=\;(z\frac{d}{dz})^{-1}\big(\frac{z}{\tanh z}-1-z\tanh z\big)$ we can rewrite equation (8.13) in the form:

$$\begin{aligned}[\,a(t,X)\,] &= e^{t\Delta}\ \exp\Big(\,-\,\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\,tR^E\ \tanh\, tR^E\Big)\;+\;\frac{1}{4t}\Big(\frac{\tanh\, tR^E}{tR^E}-1\Big)_{X,X}\Big)\\ &= \exp\Big(\,-\,\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\frac{tR^E}{\tanh\, tR^E}\,-\,1\Big)\;+\;\frac{1}{4t}\Big(1-\frac{tR^E}{\tanh\, tR^E}\Big)_{X,X}\Big)\end{aligned}$$

With this equation we have reproved Theorem 6.2 for trivially twisted connection Laplacians leading via Corollary 6.3 to the Local Index Theorem for the untwisted Dirac operator $D_{\$M}$ on a Riemannian spin manifold M. A straightforward procedure to extend this combinatorial proof of the Local Index Theorem to arbitrary twisted Dirac operators considers the square D^2 of a twisted Dirac operator as trivially twisted Laplace operator with potential, whose precise form is dictated by the general Weitzenböck formula (5.3) for twisted Dirac operators:

$$\mathcal{F} \;:=\; \frac{\kappa}{4} \;+\; 2\sum_{\mu<\nu}(X_\mu \wedge X_\nu)\star \otimes R^{\mathrm{twist}}_{X_\mu,X_\nu}$$

Evidently this potential is quadratic in $\mathcal{U}\mathfrak{hol}^E M$ and so the generating series $a(t)$ of the heat kernel coefficients of D^2 will fail to be of zeroth order in general in the filtration $\mathbb{F}^\bullet(\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M[\![\sqrt{t}]\!])$ used above. In Section 5 we found remedy to this problem in twisting the filtration on $\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}^E M$ to a filtration adapted to the inclusion $\mathcal{U}\mathfrak{hol}^E M \subset \mathcal{U}\mathfrak{hol}M \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}} M$. Once this twisted filtration is extend in the obvious way

$$\begin{aligned}&\mathbb{F}^r(\bigotimes T^*M \otimes \mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}M \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}M[\![\sqrt{t}]\!])\\ &\quad := \sum_{d\geq 0}\sqrt{t}^{\,d}\bigotimes T^*M \;\otimes\; \mathbb{F}^{d+r}(\,\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol} \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}M\,)\end{aligned}$$

all the pieces of the explicit formula given in Theorem 7.8 can be interpreted again in terms of the twisted filtration so that $a(t) \in \Gamma(\mathbb{F}^0(\mathrm{Sym}\,T^*M \otimes \mathcal{U}\mathfrak{hol}M \otimes \mathcal{U}\mathfrak{hol}^{\mathrm{twist}}M[\![\sqrt{t}]\!]))$. In this situation the symbol class $[\,a(t)\,]$ depends only on special red–white Laplace forests, red–white Laplace forests consisting entirely of black or red trees with two leaves. Splitting the curvature $R^E = R + R^{\mathrm{twist}}$ we see that the right hand side of (8.5) generalizes to

$$e^{t\Delta}\left[\sum_{\substack{\mathfrak{F}\text{ special red–white}\\ L\text{ marked leaves}}}\frac{t^{\frac{|\mathfrak{F}|-\#L}{2}}}{(\frac{|\mathfrak{F}|+\#L}{2})!}\,(\,\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}}\circ m_L\,)\left(\prod_{\substack{\mathfrak{T}\subset\mathfrak{F}\\ \text{black}}}(\,\tfrac{1}{2}R\,)_{\cdot,\cdot}(\mathfrak{T}) \,\otimes\, \prod_{\substack{\mathfrak{T}\subset\mathfrak{F}\\ \text{red}}}(\,-2R^{\mathrm{twist}}\,)_{\cdot,\cdot}(\mathfrak{T}) \,\otimes\, 1\right)\right] \tag{8.15}$$

where $(\,-2R^{\mathrm{twist}}\,)_{\cdot,\cdot}$ is short hand for the bilinear form $-\frac{2}{\dim M}\langle\cdot,\cdot\rangle\sum(X_\mu\wedge X_\nu)\star\otimes R^{\mathrm{twist}}_{X_\mu,X_\nu}$ with values in End EM associated to the potential $\mathcal{F}$.

Recall that the sets L of marked leaves in a red–white Laplace forest have to be chosen in such a way that the iterated trace $\mathrm{tr}^{\frac{|\mathfrak{F}|-\#L}{2}}$ contracts the twin buds of all red trees. Taking this observation as a clue an argument almost identical to the argument eliminating aligned pairs of solitary vertices in Section 7 implies that in the process of taking the iterated trace the contributions of the red trees to (8.15) completely decouple and split of the exponential of the potential $-2R^{\mathrm{twist}}$. The combinatorial structure of the remaining sum over special Laplace forests without red trees has been settled already and using the previous result we calculate the symbol class

$$\begin{aligned}&[\,a(t,X)\,]\\ &\quad=\;\exp\left(-\frac{1}{2}(t\frac{d}{dt})^{-1}\mathrm{tr}\Big(\frac{tR^E}{\tanh tR^E}-1\Big)+\frac{1}{4t}\Big(1-\frac{tR^E}{\tanh tR^E}\Big)_{X,X}\right)\exp\Big(-2\,tR^{\mathrm{twist}}\Big)\end{aligned}$$

of the series $a(t)$ of heat kernel coefficients for the square of a twisted Dirac operator D.

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