Estimates for the spectral asymptotic in the Large Coupling Limit

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Motivation/Main result

- Consider singularly perturbed families of positive definite forms $h_\kappa(\psi,\phi) = h_b(\psi,\phi) + \kappa^2 h_e(\psi,\phi)$ with huge coupling constant κ.
- We have $H_\kappa \longrightarrow H_\infty$, where H_∞ is defined by a restriction of h_κ in $\text{null}(h_e) \neq \{0\}$.
- We argue: For an analysis of $\lambda_\kappa \longrightarrow \lambda_\infty$ it is sufficient to study $(\psi, H_\kappa^{-1}\phi)$ for $\psi,\phi \in \text{range}(E_\infty\{\lambda_\infty\})$.

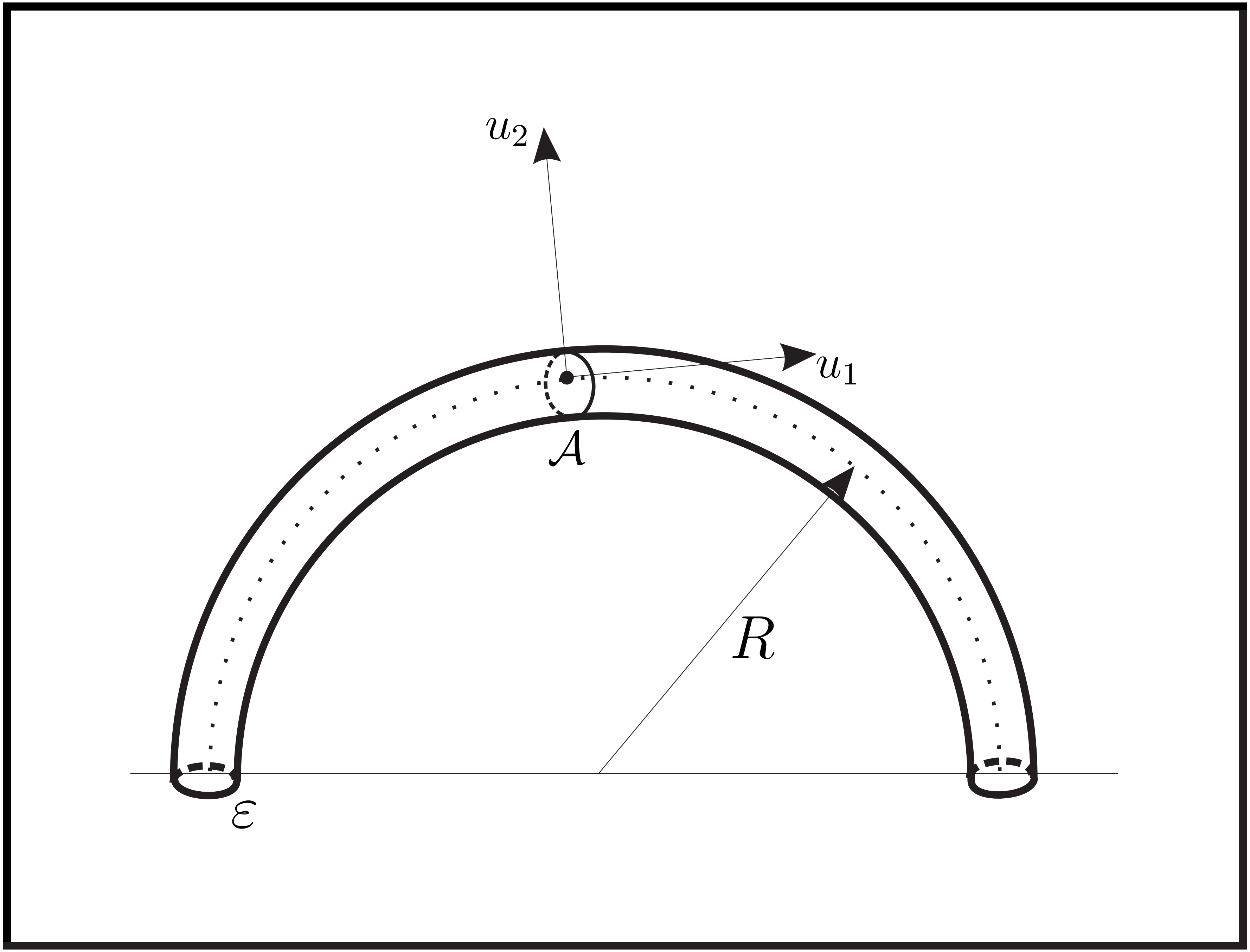
Multi-scale model problems

... in quantum as well as in classical mechanics

- $-\triangle + V + \kappa^2 \chi_{\mathcal{O}}$ (Brasche, Demuth, Kirsch, ...)
- $-\triangle + \kappa^2 \mathrm{div}^* \mathrm{div}$ (Beattie, Goerisch, Greenlee, ...)
- 1D-models $\left(u_2' + \frac{u_1}{R}\right)' \left(v_2' + \frac{v_1}{R}\right)' + 1/\epsilon^2 \left(u_1' - \frac{u_2}{R}\right) \left(v_1' - \frac{v_2}{R}\right)$

But also in other settings e.g. semi classical analysis

for magnetic potentials (Dauge, Helffer, ...)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$\mathcal{A}$
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Key technical result

Set $\eta_i^2(\lambda_\infty) = \max\limits_{\substack{d(\mathcal{S})=i \\ \mathcal{S}\subset\mathcal{E}_\infty}} \min\limits_{\psi\in\mathcal{S}\setminus\{0\}} \frac{(\psi,H_\kappa^{-1}\psi)-(\psi,H_\infty^{-1}\psi)}{(\psi,H_\kappa^{-1}\psi)}$,

for $\mathcal{E}_\infty := \text{range}\, E_\infty(\lambda_\infty)$, then

- for eigenvalues: $\sum\limits_{i=1}^{\#\lambda_\infty} \frac{|\lambda_\kappa^{(i)}-\lambda_\infty|}{\lambda_\infty} \simeq \sum\limits_{i=1}^{\#\lambda_\infty} \eta_i^2(\lambda)$
- and for eigenvectors:

  $\|E_\kappa - E_\infty(\{\lambda_\infty\})\|_{HS} \leq O(\sqrt{\sum_{i=1}^{\#\lambda_\infty} \eta_i^2(\lambda)})$

  and in particular $h_\kappa[v_\kappa - v_\infty] \simeq O(\eta_\#^2(\lambda_\infty))$

How to handle $\eta_i(\lambda_\infty)$

- variationally: $(\psi, H_\kappa^{-1}\phi) - (\psi, H_\infty^{-1}\phi) = \|H_\kappa H_\infty^{-1}\psi - \psi\|_{H_\kappa^{-1}}$
- geometrically: the quotient $\frac{(\psi,H_\kappa^{-1}\psi)-(\psi,H_\infty^{-1}\psi)}{(\psi,H_\kappa^{-1}\psi)}$ is $\sin^2\angle(H_\kappa^{-1}\psi, H_\infty^{-1}\psi)$ in energy space $(\mathcal{Q}, h_\kappa[\cdot])$.
- The theorem shows equivalence of error to the *a posteriori* quantity $\sin^2\angle_{h_\kappa}(H_\kappa^{-1}\mathcal{E}_\infty, H_\infty^{-1}\mathcal{E}_\infty)$.

Upper and lower estimates for $\eta_i(\lambda)$

- Adaptation of Brasche–Demuth ’05 yields

$$\frac{((I+H_e)^{-1}H_b^{-1}\psi, H_b^{-1}\psi)}{\kappa^2} \leq (\psi, H_\kappa^{-1}\psi) - (\psi, H_\infty^{-1}\psi)$$

- Special case: we assume

$$\|H_e^{1/2}H_b^{-1/2}f\| \geq \frac{1}{\beta}\|P_{\text{null}(H_e^{1/2}H_b^{-1/2})}f\|$$

and obtain $(\psi, H_\kappa^{-1}\psi) - (\psi, H_\infty^{-1}\psi) \leq \frac{\beta}{\kappa^2}K_\psi$

- K_ψ is a behavior of the referent system.

Regular perturbations

- $\beta < \infty$ is a regularity assumption on perturbation h_e

  ... not satisfied by the deep-well model problem.

- For the $\chi_{\mathcal{O}}$ potential use the laplace transform and

  the Feynman–Kac formula, as in

  Demuth–Brasche ’05, Demuth–Jeske–Kirsch ’93, ...

  ... or boundary layers as in Bruneau—Carbou ’02, ...

Model problem for a non-regular perturbation

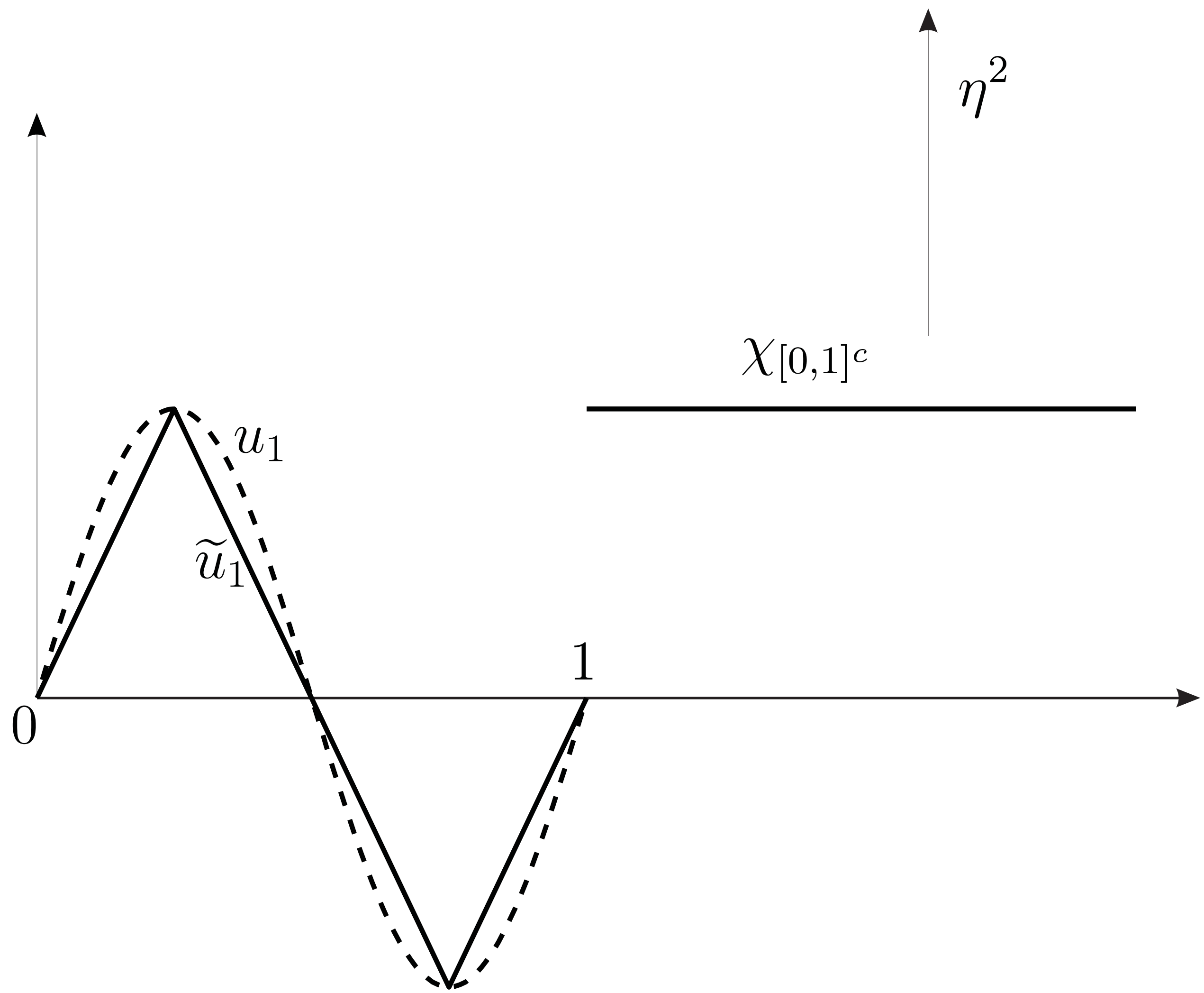
Consider $H_\kappa = -\partial_{xx} + \kappa^2 \chi_{[1,\infty)}$ in $\mathcal{H}^1_0[0,\infty\rangle$

- A direct expansion gives $\frac{\lambda_1^\infty - \lambda_1^\kappa}{\lambda_1^\infty} = \frac{2}{\kappa} + O\left(\frac{1}{\kappa^2}\right)$.
- For $\kappa \geq 5$ our $\eta_i(\lambda_\infty)$ approach gives

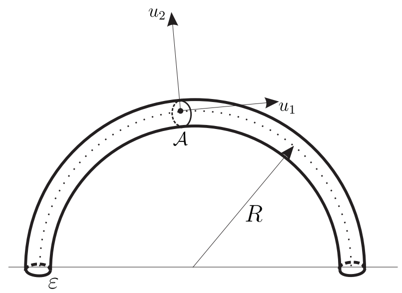
$$\frac{2}{3+\kappa} \leq \text{eigen}\textbf{value}\text{ error} \leq \frac{10}{3\kappa} + \frac{1}{\sqrt{\kappa}} O\left(\frac{1}{\kappa}\right)$$

$$\frac{2}{1+\kappa} \leq \text{eigen}\textbf{vector}\text{ error} \leq \frac{10}{\kappa} + \frac{1}{\sqrt{\kappa}} O\left(\frac{1}{\kappa}\right)$$

Non-regular case



A regular test case:



- Consider the 1D model

$$H_\kappa = \left(u_2' + \tfrac{u_1}{R}\right)' \left(v_2' + \tfrac{v_1}{R}\right)' + 1/\epsilon^2 \left(u_1' - \tfrac{u_2}{R}\right)\left(v_1' - \tfrac{v_2}{R}\right)$$

- One computes that $\beta \leq \frac{\sqrt{1+R^2}}{R}$

- And one obtains

$$\text{referent}(\lambda_\infty)\varepsilon^2 \leq \text{error} \leq \frac{\varepsilon^2\sqrt{1+R^2}}{R}\text{Referent}(\lambda_\infty)$$

Conclusion and outlook

- Quantitative version od Weidmann's convergence results ('84)
- Problem is reduces to a study of an equivalent "local" problem (study $(\psi, H_\kappa^{-1}\phi)$ for $\psi, \phi \in \mathcal{E}_\infty$)
- Regular case, qualified by $\beta < \infty$, covers a lot of important examples
- Our approach can be seen as a form theoretic version of the Temple–Kato inequality.