Feynman path integrals as infinite dimensional oscillatory integrals

Sonia Mazzucchi
University of Trento (I) and I.N.d.A.M.

- Physical motivations: Feynman path integrals.
- Mathematical problem: rigorous definition of Feynman path integrals.
- Infinite dimensional oscillatory integrals: definition and classical results.
- New developments and applications.

Quantum description of a nonrelativistic $d-$dimensional particle:

State: wave function $\psi \in L^2(\mathbb{R}^d)$,
Time evolution: Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x) \\ \psi(0,x) = \psi_0(x), \qquad x \in \mathbb{R}^d \end{cases}$$

Feynman path integrals

P.A.M. Dirac, *The Lagrangian in quantum mechanics*, Phys. Zeitschr. d. Sowjetunion, 3, No 1, 64–72, 1933.
R.P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. 20, 367–387 (1948).

$$\psi(t,x) = \quad \text{“}const \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))D\gamma \quad \text{”}$$

$$S(\gamma) = S^0(\gamma) - \int_0^t V(\gamma(s))ds,$$

$$S^0(\gamma) = \int_0^t \frac{m}{2}\dot{\gamma}^2(s)ds,$$

The mathematical problem:
the integral is not well defined

$$\text{“}\quad C\int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))D\gamma \quad\text{”}$$

$$\text{“}\quad C=\int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S^0(\gamma)}D\gamma \quad\text{”}$$

R. Feynman: “one must feel as Cavalieri must have felt calculating the volume of a pyramid before the invention of the calculus”

Difficulties:

- Construction of an integration theory on an infinite dimensional space.
- Oscillatory behavior of the integrand.

M. Kac, On distributions of certain Wiener functionals. Trans. Amer. Math. Soc. 65, (1949). 1–13.

Heat equation $\frac{\partial}{\partial t}u = \frac{1}{2}\Delta u - Vu$

$$\frac{e^{iS_t(\gamma)}D\gamma}{\int e^{iS_t(\gamma)}D\gamma} \longrightarrow \frac{e^{-S_t(\gamma)}D\gamma}{\int e^{-S_t(\gamma)}D\gamma}$$

Feynman-Kac formula:

$$u(t,x) = \quad \text{“}const \quad \int e^{-S_t(\omega)}u_0(\omega(t))D\omega \quad \text{”}$$
$$= \mathbb{E}\Big[e^{-\int_0^t V(\omega(s)+x)ds)}u_0(\omega(t)+x)\Big]$$

R.H. Cameron, A family of integrals serving to connect the Wiener and Feynman integrals. J. Math. and Phys. 39, 126-140, 1960.

Feynman's complex measure

$$d\mu_F(\gamma) \equiv \quad \text{“}\frac{e^{\frac{i}{\hbar}S^0(\gamma)}D\gamma}{\int e^{\frac{i}{\hbar}S^0(\gamma)}D\gamma}\quad\text{”}$$

would have infinite total variation.

The Feynman functional

$$\text{“} \int f(\gamma) \frac{e^{\frac{i}{\hbar}S^0(\gamma)} D\gamma}{\int e^{\frac{i}{\hbar}S^0(\gamma)} D\gamma} = \int f(\gamma) d\mu_F(\gamma) \quad \text{”} := I_F(f)$$

Definition of the integral as a linear continuous functional on a suitable Banach algebra of “integrable functions”

Properties:

- simple transformation properties under ”translations and rotations in paths space”
- it should satisfy a Fubini-type theorem
- approximable by finite dimensional oscillatory integrals
- it should allow for the implementation of an infinite dimensional version of the stationary phase method for the study of the asymptotic behavior of the integral in the limit $\hbar \to 0$.

Possible definitions of "Feynman functional" :

- by means of analytic continuation of Wiener integrals [Cameron, Nelson, Doss, Kallianpur - Kannan - Karandikar,...]

- sequential approach: as limit of particular finite dimensional approximations [Nelson, Friedman, Truman, Fujiwara, Kumano-Go,...]

- White noise approach: as an infinite dimensional distribution [DeWitt, Kree, Hida, Kuo, Potthoff, Streit,...]

- via non standard analysis [Albeverio, Fenstad, Høegh-Krohn, Lindstrøm]

- via complex Poisson measures [Chebotarev, Maslov,...]

- as an infinite dimensional oscillatory integral [Ito, Albeverio, Høegh-Krohn, Elworthy, Truman, Brzeźniak,...] $\Longrightarrow$ infinite dimensional version of the stationary phase method [Albeverio, Høegh-Krohn, Albeverio, Rezende, Brzeźniak]

Oscillatory integrals on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$$

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \qquad f : \mathbb{R}^N \to \mathbb{C}, \qquad \hbar \in \mathbb{R}^+$$

Examples:

- Fresnel integrals $\int_{\mathbb{R}} e^{\frac{i}{\hbar}x^2} f(x)dx$
- Airy integrals $\int_{\mathbb{R}} e^{\frac{i}{\hbar}x^3} f(x)dx$

Definition: [Hörmander]

$\phi \in S(\mathbb{R}^n)$, $\phi(0) = 1$

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\Phi(x)} \phi(\epsilon x) f(x)dx$$

$$\equiv \widetilde{\int_{\mathbb{R}^n}} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$$

Infinite dimensional oscillatory integrals

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx$$

$(\mathcal{H}, \langle\cdot,\cdot\rangle)$ real separable Hilbert space,
$f : \mathcal{H} \to \mathbb{C}$

Definition:

D.Elworthy and A.Truman. Feynman maps, Cameron-Martin formulae and anharmonic oscillators. *Ann. Inst. H. Poincaré Phys. Théor.*, 41(2):115–142, 1984.

$\{P_n\}_{n\in\mathbb{N}}$, $P_n \leq P_{n+1}$, $P_n \uparrow I_{\mathcal{H}}$

$$\lim_{n\to\infty} (2\pi i\hbar)^{n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x) dP_n x$$
$$\equiv \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx$$

Fresnel integrable functions

The description of the largest class of Fresnel integrable function is still an open problem, even in finite dimension!

The Banach algebra $\mathcal{F}(\mathcal{H})$

$$f \in \mathcal{F}(\mathcal{H}) \qquad f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y),$$

μ_f complex bounded variation measure on $\mathcal{H}$

$$f \cdot g(x) = f(x)g(x), \qquad \|f\| := \|\mu_f\|$$

Parseval-type equality:

$L : \mathcal{H} \to \mathcal{H}$ self-adjoint and trace class, $(I - L)$ invertible, $f \in \mathcal{F}(\mathcal{H})$
$g(x) = e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x)$ is Fresnel integrable and

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} g(x)dx = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx$$

$$= (\det(I - L))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x,(I-L))^{-1}x\rangle} \mu_f(dx)$$

Mathematical realization of Feynman path integrals: infinite dimensional oscillatory integral on $(\mathcal{H}_t, \langle\ ,\ \rangle)$

$$\mathcal{H}_t = \{\gamma : [0,t] \to \mathbb{R}^d, |\, \gamma(t) = 0, \int_0^t |\dot{\gamma}(s)|^2 ds < \infty\}$$

$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1(s) \cdot \dot{\gamma}_2(s) ds.$ "Cameron-Martin space"

$$\begin{cases} i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + \frac{1}{2}xA^2x\psi + V_1(x)\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

Theorem 1. *[Albeverio and Brzeźniak, Elworthy and Truman]*
Let $\psi_0, V_1 \in \mathcal{F}(\mathbb{R}^d)$. *Then the functional on* $\mathcal{H}_t$

$$\gamma \mapsto e^{-\frac{i}{2\hbar}\int_0^t (\gamma(s)+x)A^2(\gamma(s)+x)ds} e^{-\frac{i}{\hbar}V_1(\gamma(s)+x)ds)}\psi_0(\gamma(0)+x)$$

is Fresnel integrable and the solution of the Schrödinger equation (1) is represented by the infinite dimensional Fresnel integral:

$$\psi(t,x) = \text{"} \int e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma \text{"}$$

$$= \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,\gamma\rangle} e^{-\frac{i}{2\hbar}\int_0^t (\gamma(s)+x)A^2(\gamma(s)+x)ds} e^{-\frac{i}{\hbar}V_1(\gamma(s)+x)ds)}$$

$$\psi_0(\gamma(0)+x)d\gamma$$

Asymptotic expansion as $\hbar \to 0$

$$\int e^{\frac{i}{\hbar}\Phi(x)} f(x)dx \quad \sim_{\hbar\to 0} \quad ?????$$

$$\Downarrow$$

Stationary phase method:

[Stokes, Kelvin, van der Corput, ...]

The asymptotic behavior of the integral $\int e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$ is determined by the critical set

$$\{x \in \mathbb{R}^n \mid \Phi'(x) = 0\}$$

Generalization to the infinite dimensional case

[Albeverio, Høegh-Krohn, Brzeźniak, Rezende, ...]

$$I(\hbar) := \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{\hbar}V(x)} f(x)dx, \qquad V, f \in \mathcal{F}(\mathcal{H})$$

- the phase function $\Phi(x) = \langle x, x\rangle/2 - V(x)$ has only nondegenerate stationary points
- $I(\hbar)$ is a C^∞ function
- its asymptotic expansion when $\hbar \to 0$ depends only on the derivatives of V and g at the critical points.
- Borel summability of the expansion [Rezende]

The semiclassical limit of quantum mechanics

$$\psi(t,x) = \text{"} \quad \int e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma \quad \text{"} \qquad \sim_{\hbar\to 0} \quad ?????$$

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + \frac{1}{2}xA^2x\psi + V_1(x)\psi \\ \psi(0,x) = e^{\frac{i}{\hbar}f(x)}\chi(x), \qquad \chi \in C_0^\infty(\mathbb{R}^d), f \in C^\infty(\mathbb{R}^d) \end{cases}$$

the infinite dimensional oscillatory integral representation for the solution of the Schrödinger equation

$$\widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,\gamma\rangle} e^{-\frac{i}{2\hbar}\int_0^t(\gamma(s)+x)A^2(\gamma(s)+x)ds} e^{-\frac{i}{\hbar}V_1(\gamma(s)+x)ds)} e^{\frac{i}{\hbar}f(\gamma(0)+x)}\chi(\gamma(0)+x)d\gamma$$

has an asymptotic expansion in powers of $\hbar$, depending only on classical features of the system.

$$\Downarrow$$

Independent derivation of Maslov's results on the WKB-type asymptotics of the solution of Schrödinger equation

Quantum mechanical trace formula

Application to the trace of the Schrödinger group

$$Tr[e^{-\frac{i}{\hbar}Ht}]$$

and its asymptotic behavior when $\hbar \downarrow 0$.

[S. Albeverio, Ph. Blanchard, R. Høegh-Krohn, Feynman path integrals and the trace formula for the Schrödinger operators, Comm. Math. Phys. 83 n.1, 49–76 (1982).]
[S. Albeverio, A.M. Boutet de Monvel-Berthier, Z. Brzeźniak, The trace formula for Schrödinger operators from infinite dimensional oscillatory integrals, Math. Nachr. 182, 21–65 (1996).]

Proof of Gutzwiller trace formula:

connection between the spectrum of the quantum mechanical energy operator

$$H = -\frac{\Delta}{2} + V$$

and the classical periodic orbits of the system.

extension of the classical Selberg formula for the heat kernel on manyfolds with constant negative curvature

Phase space Feynman path integrals

$$\psi(t,x) = \quad \text{“} \int_{\{(q,p)|q(t)=x\}} e^{\frac{i}{\hbar}S(q,p)}\phi(q(0))DqDp \quad \text{''}$$

$$S(q,p) = \int_0^t [p(s)\dot{q}(s) - H(q(s),p(s))]ds$$

Realization as an infinite dimensional oscillatory integral:

[S.Albeverio, G. Guatteri, S. Mazzucchi, Phase space Feynman path integrals, J. Math. Phys. 43, 2847–2857 (2002).]

$\mathcal{H} = H_t \times L_t$, H_t Cameron-Martin space, $L_t = L^2([0,t])$
$\langle (q_1,p_1),(q_2,p_2)\rangle = \int_0^t \dot{q}_1(s)\dot{q}_2(s)ds + \int_0^t p_1(s)p_2(s)ds$

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = H\psi \\ \psi(0,x) = \phi(x) \end{cases} \qquad H(q,p) = \frac{p^2}{2m} + V_1(q) + V_2(p)$$

Assumptions: $V_1, \phi \in \mathcal{F}(\mathbb{R}^d)$, $\int_0^t V_2(p(s))ds \in \mathcal{F}(L_t)$.

$$\Downarrow$$

$$\psi(t,x) = \widetilde{\int_{H_t\times L_t}} e^{\frac{i}{\hbar}\int_0^t [p(s)\dot{q}(s) - \frac{p(s)^2}{2}]ds} e^{-\frac{i}{\hbar}\int_0^t V_1(q(s)+x)ds} e^{-\frac{i}{\hbar}\int_0^t V_2(p(s))ds}\phi(q(0))dqdp$$

Application to quantum theory of continuous measurements

if the quantum system is submitted to the measurement and interacts with a macroscopic measuring apparatus, Schrödinger equation is no longer valid.

$$\Downarrow$$

"collapse of the wave function"

Possible descriptions of continuous position measurement:

M.B. Mensky: "restricted path integrals"

$$\psi(t,x,[\omega]) = \text{``} \quad \int_{\{\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} e^{-k\int_0^t(\gamma(s)-\omega(s))^2 ds}\phi(\gamma(0))D\gamma \quad \text{''}$$

V.P. Belavkin: "stochastic Schrödinger equation"

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}x^2\psi(t,x)dt + \sqrt{\lambda}x\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad\qquad (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases}$$

[S. Albeverio, G. Guatteri, S. Mazzucchi, Probab. Theory Relat. Fields 125, 365–380 (2003).]
Generalization of the definition of the infinite dimensional oscillatory integral to **complex valued phase functions**.

Generalization of Parseval-type equality
$L = L_1 + iL_2$, $L_1, L_2 : \mathcal{H} \to \mathcal{H}$ self-adjoint, trace class, $[L_1, L_2] = 0$, $I + L_1$ invertible, $L_2 \geq 0$ is non negative. $y \in \mathcal{H}$, $f \in \mathcal{F}(\mathcal{H})$

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle} e^{\langle y,x\rangle} f(x)dx = \det(I+L)^{-1/2} \int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle k-iy,(I+L)^{-1}(k-iy)\rangle} \mu_f(dk)$$

$$\Downarrow$$

Application to the Feynman path integral representation for the solution of Belavkin equation

$$\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda \int_0^t |\gamma(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds} \cdot e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)} \psi_0(\gamma(0)+x) d\gamma \quad (2)$$

The Chern-Simons functional integral

[E. Witten, Commun. Math. Phys. 121, 353–389 (1989).]

M smooth 3-dimensional oriented manifold without boundary, G be a compact Lie group (the "gauge group"), g Lie algebra, A $g-$valued connection 1-form

Chern-Simons action

$$S_{CS}(A) \equiv \frac{k}{4\pi} \int_M \Big(\langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \Big),$$

Conjecture by Witten and Schwartz: the heuristic Feynman path measure of the form

$$d\mu_F(A) \equiv \frac{1}{Z} e^{iS_{CS}(A)} DA, \tag{3}$$

should allow for the computation of topological invariants of the manifold M.

Rigorous definition of the integral:

- G abelian: [Albeverio and Schaefer] via infinite dimensional Fresnel integrals; [Leukert and Schaefer] via White noise analysis.
- G not abelian, $M = \mathbb{R}^3$: [Albeverio, Sengupta, Hahn] via white noise analysis.
- G not abelian, M compact: [Hahn]
- Asymptotic in k: [Albeverio and Mitoma]

Problem:

Restricted class of Fresnel integral function: Fourier transform of bounded measures on $\mathcal{H}$

$$\Downarrow$$

The oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{\hbar}\Phi(x)} f(x) dx$$

can be defined and explicitly computed iff

Φ =quadratic + Fourier transform of measure

$$\Downarrow$$

Representation of solution of Schrödinger equation only for

$$V(x) = \frac{a^2}{2}x^2 + V_1(x)$$

V_1 Fourier transform of bounded variation measure on $\mathbb{R}^d$

Aim: Polynomially growing potentials

Problem: [K. Yajima, Commun. Math. Phys. 181, 605–629 (1996).]

for super-quadratic potentials the fundamental solution

$$E(t, 0, x, y)$$

of the time dependent Schrödinger equation is nowhere of class C^1 (as a function of (t, x, y))

Recent results [Albeverio, Mazzucchi 2003]:

- $dim(\mathcal{H}) < \infty$ Definition of $\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$, with $\Phi(x)$ even degree polynomial. Asymptotic expansion for $\hbar \downarrow 0$
- $dim(\mathcal{H}) = \infty$ Definition of $\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$ with $\Phi(x) = \frac{1}{2}\langle x, Bx\rangle + A(x,x,x,x)$
- Application to Schrödinger equation with polynomially growing potential.

Idea: generalization of Parseval type equality

$$f(x) = \int_{\mathbb{R}^N} e^{i\langle y,x\rangle} d\mu_f(y)$$

$$\int e^{\frac{i}{\hbar}\Phi(x)} f(x)dx = \int_{\mathbb{R}^N} \tilde{F}(k)\mu_f(dk)$$

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{iy\cdot x} e^{\frac{i}{\hbar}\Phi(x)} dx$$

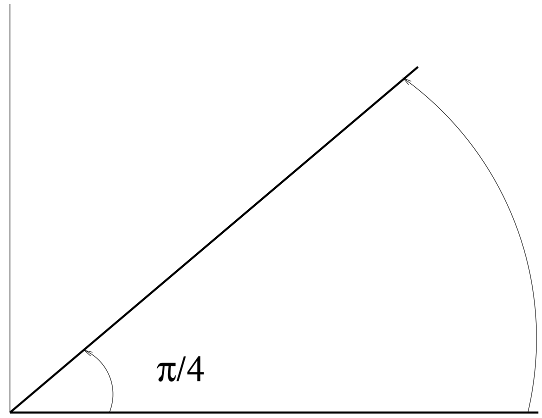

Figure 1

If $\Phi(x) = \langle x, x\rangle$ then

$$\tilde{F}(k) = (2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{2\hbar}\langle x,x\rangle} dx$$

can be explicitly computed and makes sense for any N, even for $dim(\mathcal{H}) = \infty$:

$$(2\pi i\hbar)^{-N/2} \int_{\mathbb{R}^N} e^{ik\cdot x} e^{\frac{i}{2\hbar}\langle x,x\rangle} dx = e^{-\frac{i\hbar}{2}\langle k,k\rangle}$$

If $\Phi(x) = x^2 + \alpha x^4$, $\alpha > 0$:

$$\tilde{F}(k) = \int_{\mathbb{R}} e^{ikx} e^{\frac{i}{\hbar}\alpha x^4} \frac{e^{\frac{i}{2\hbar}x^2}}{(2\pi i\hbar)^{1/2}} dx$$

$$= \int_{\mathbb{R}} e^{ikxe^{i\pi/4}} e^{-\frac{i}{\hbar}\alpha x^4} \frac{e^{-\frac{x^2}{2\hbar}}}{\sqrt{2\pi\hbar}} dx = \mathbb{E}[e^{ikxe^{i\pi/4}} e^{-\frac{i}{\hbar}\alpha x^4}]$$

$$f(x) = \int e^{ikx} d\mu_f(k)$$

$$(2\pi i\hbar)^{-1/2} \int e^{\frac{i}{2\hbar}x^2} e^{\frac{i}{\hbar}\alpha x^4} f(x)dx$$

$$= \int_{\mathbb{R}} \tilde{F}(k) d\mu_f(k) = \mathbb{E}[f(xe^{i\pi/4})e^{-\frac{i}{\hbar}\alpha x^4}] \quad (4)$$

General case $dim(\mathcal{H}) = N$

$B : \mathbb{R}^N \to \mathbb{R}^N$, $I - B > 0$,
$A : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $A \geq 0$, $\alpha \geq 0$,
$f \in \mathcal{F}(R^N)$

$$\widetilde{\int}_{\mathbb{R}^N} \frac{e^{\frac{i}{2\hbar}x\cdot(I-B)x}}{(2\pi i\hbar)^{N/2}} e^{\frac{i\alpha}{\hbar}A(x,x,x,x)} f(x)dx =$$

$$\int_{\mathbb{R}^N} \tilde{F}(k)\mu_f(dk) = \mathbb{E}[e^{-i\alpha\hbar A(x,x,x,x)} e^{\frac{1}{2}x\cdot Bx} f(e^{i\pi/4}\sqrt{\hbar}x)].$$

$\mathbb{E}$ denoting the expectation with respect to the centered Gaussian measure on $\mathbb{R}^n$ with covariance $\hbar I$

Infinite dimensional case

$(\mathcal{H}, \langle \ , \ \rangle, | \ |)$ real separable Hilbert space
$B : \mathcal{H} \to \mathcal{H}$, self-adjoint trace class operator $I - B > 0$,
$A : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, $A \geq 0$, $\alpha \geq 0$,
$f \in \mathcal{F}(\mathcal{H})$

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle + \frac{i}{\hbar}A(x,x,x,x)} f(x) dx$$

$$= \lim_{n\to\infty} \widetilde{\int}_{P_n\mathcal{H}} \frac{e^{\frac{i}{2\hbar}P_n x\cdot(I-B)P_n x}}{(2\pi i\hbar)^{N/2}} e^{\frac{i\alpha}{\hbar}A(P_n x,P_n x,P_n x,P_n x)} f(x) dP_n x$$

$$= \lim_{n\to\infty} \mathbb{E}[e^{-i\alpha\hbar A(P_n x,P_n x,P_n x,P_n x)} e^{\frac{1}{2}P_n x\cdot BP_n x} f(e^{i\pi/4}\sqrt{\hbar}P_n x)]$$

$$= \mathbb{E}_{\mu,\mathcal{B}}[e^{\frac{1}{2}\langle\omega,B\omega\rangle} e^{-i\hbar A(\omega,\omega,\omega,\omega)} f(e^{i\pi/4}\sqrt{\hbar}\omega)]$$

Parseval type equality still holds

with all the functions of $x \in \mathcal{H}$ are lifted to the abstract Wiener space $(i, \mathcal{H}, \mathcal{B})$ built on $\mathcal{H}$:

Schrödinger equation for the anharmonic oscillator

$$i\hbar\frac{d}{dt}\psi(t,x) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\psi(t,x) + \frac{a^2}{2}x^2\psi(t,x) + \lambda x^4\psi(t,x)$$

Assumptions: $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$

Feynman path integral

$$" \int_{\mathbb{R}^d \times H_t} \bar{\phi}(x) e^{\frac{i}{2\hbar}\int_0^t \dot{\gamma}(s)^2 ds} e^{-\frac{i\lambda}{\hbar}\int_0^t (\gamma(s)+x)^4 ds} e^{-\frac{i}{2\hbar}\int_0^t a^2(\gamma(s)+x)^2 ds} \psi_0(\gamma(t)+x)\, dx D\gamma = \langle \phi, e^{-\frac{i}{\hbar}Ht}\psi_0\rangle "$$

Analytic continuation (in λ) of an infinite dimensional oscillatory integral on $\mathbb{R} \times H_t$, which, by Parseval-type equality, is equal to the (Gaussian) Wiener integral:

$$(i)^{d/2} \int_{\mathbb{R}^d \times C_t} e^{i\frac{\lambda}{\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)^4 ds} e^{\frac{1}{2\hbar}\int_0^t a^2(\sqrt{\hbar}\omega(s)+x)^2 ds} \bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + e^{i\pi/4}x) W(d\omega) dx.$$

Possible developments

- Study of the semiclassical asymptotic behavior of the new oscillatory integrals
  - oscillatory integrals with complex phase function applied to the FPI representation for the solution of the stochastic Schrödinger equation
  - oscillatory integrals with polynomial phase function applied to the FPI representation for the solution of the Schrödinger equation with quartic potential

  Infinite dimensional version of the saddle point method???

- extension of the results valid for the quartic potential to polynomial potentials of higher degree

  Problem: the "rotation trick"

$$\frac{e^{\frac{i}{2\hbar}x^2}}{\sqrt{2\pi i\hbar}}dx \qquad \mapsto \qquad \frac{e^{-\frac{1}{2\hbar}x^2}}{\sqrt{2\pi\hbar}}dx$$

doesn't apply

- Application to quantum fields.