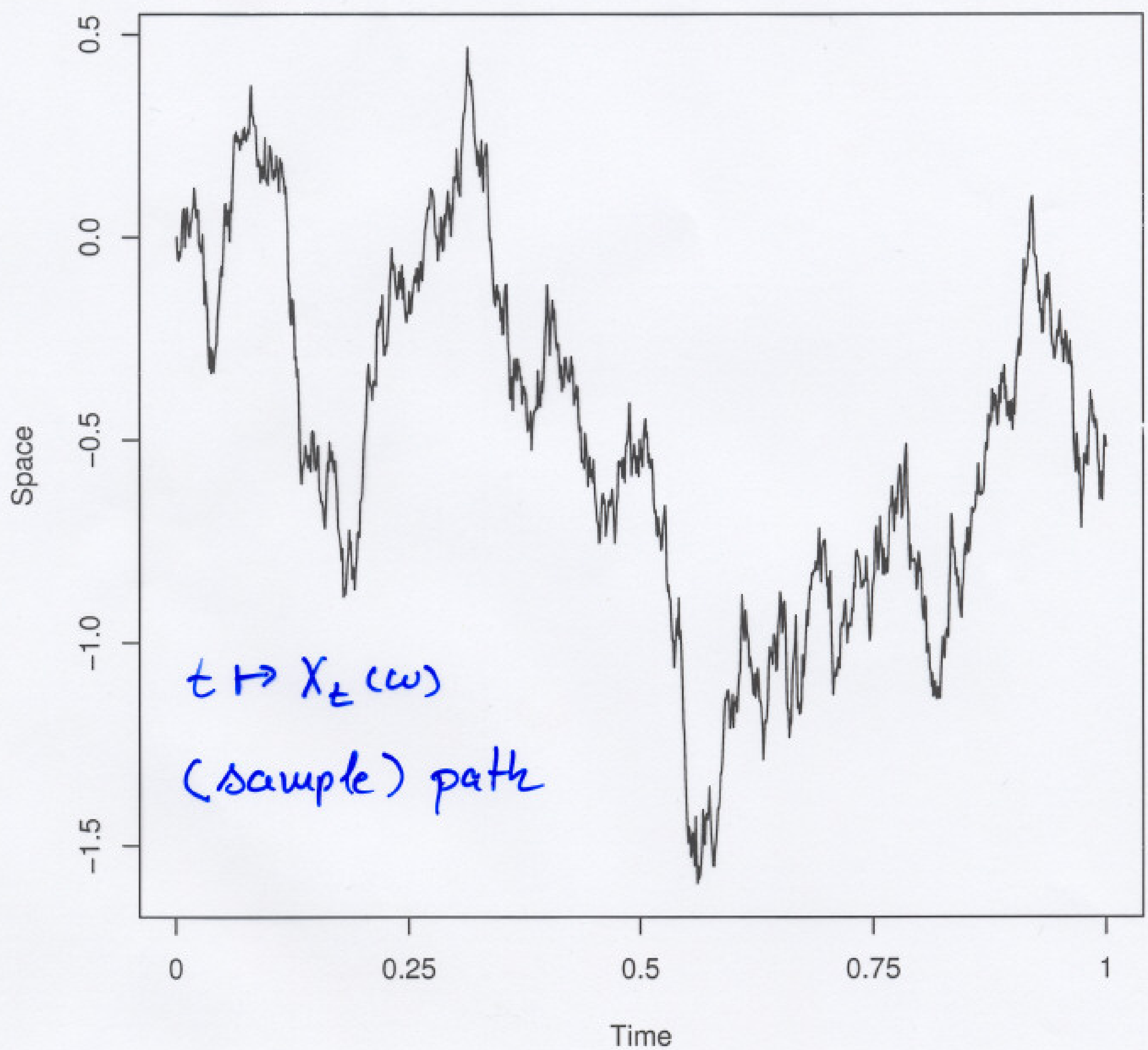
Sample path properties of stochastic processes

(R. Schilling, Marburg)

DMV-Jahrestagung, Bonn 2006



$(\Omega, \mathfrak{A}, \mathbb{P})$ probability space

$\mathbb{P}$ — probability measure

$\mathfrak{A}$ — σ-algebra on Ω

Ω — "arbitrary" set, $\omega \in \Omega \rightsquigarrow$ space of functions (stochastic) process = family of random variables

$$X_t(\cdot) : (\Omega, \mathcal{O}\!\mathcal{L}) \xrightarrow{\text{measurable}} (\mathbb{R}^d, \text{ Borel sets})$$
$$\omega \mapsto X_t(\omega)$$

dynamic point of view

$$X_\bullet(\cdot) : (\Omega, \mathcal{O}\!\mathcal{L}) \times \text{Parameter} \xrightarrow{\text{m'ble}} (\mathbb{R}^d, \text{ Borel sets})$$
$$(\omega, t) \mapsto X_t(\omega)$$

$\boxed{\text{why random? } 1}$ • description of X_t:

finite-dimensional distributions

$t_1 < t_2 < \dots < t_N$, $B_1, B_2, \dots, B_N \subset \mathbb{R}^d$ Borel

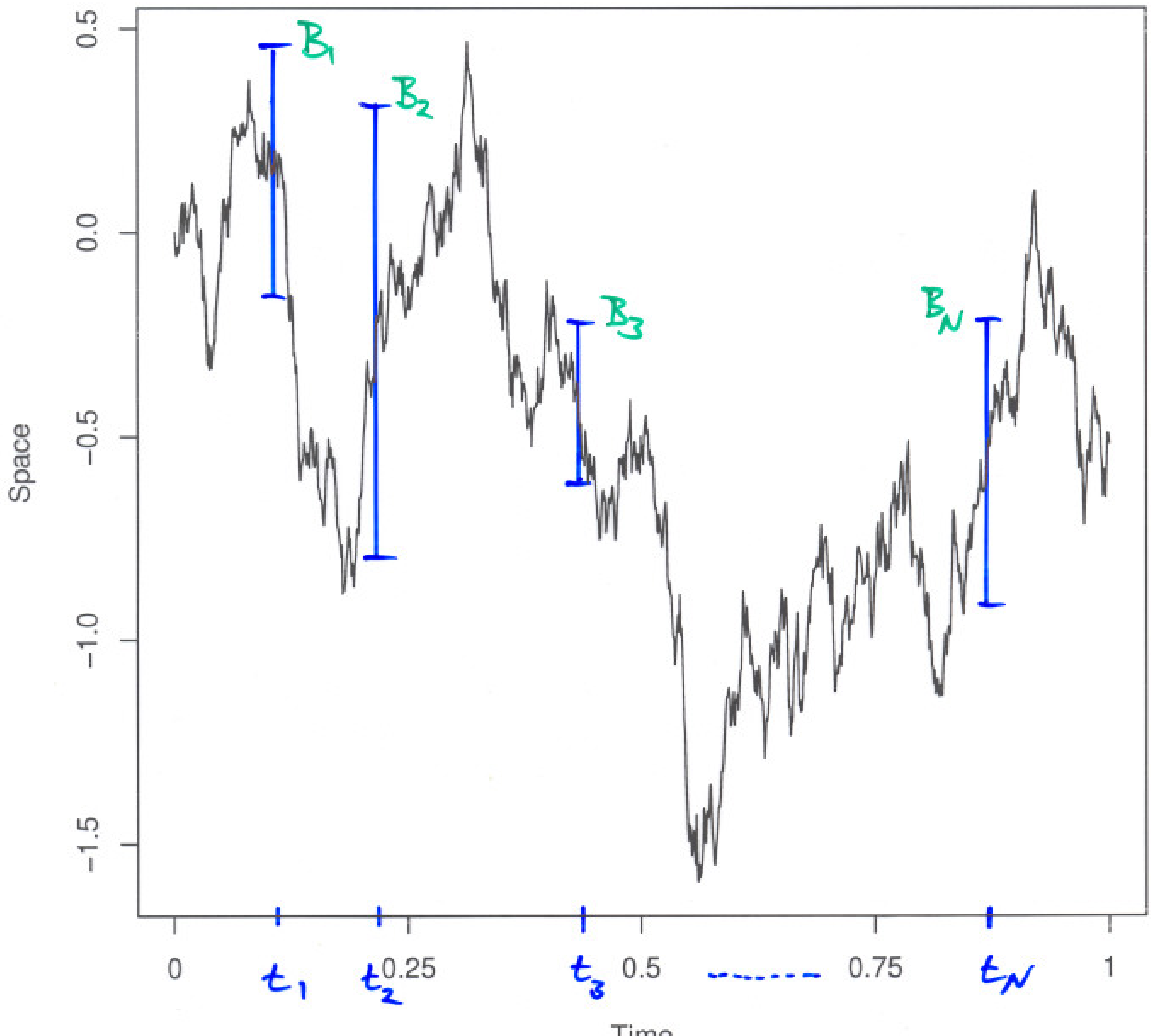
$$(*) \begin{cases} \mathbb{P}(X_{t_1} \in B_1, \dots, X_{t_N} \in B_N) = \underbrace{p_{t_1, \dots t_N}(B_1 \times \dots \times B_N)}_{\text{measure on } (\mathbb{R}^d)^N} \\ \qquad \qquad \| \\ \mathbb{P}((X_{t_1}, \dots, X_{t_N}) \in B_1 \times \dots \times B_N) \end{cases}$$

i.e. path is observed at time t_j
in the set B_j

$$\mathbb{P}(\omega : X_{t_1}(\omega) \in B_1, \ldots\ldots, X_{t_N}(\omega) \in B_N)$$

$$\overset{!}{=} P_{t_1 \ldots t_N}(B_1 \times \ldots \times B_N)$$

$$= \text{probab. measure on } (\mathbb{R}^d)^N$$



why "random" 2 • realisation of X_t

(*) indicates that $\{P_{t_1 \dots t_N}(\cdot)\}_{\substack{t_1 \dots t_N \\ N \in \mathbb{N}}}$ come from ONE $\mathbb{P}$ as pullbacks under the maps

$$(X_{t_1} \dots X_{t_N}) : \Omega \longrightarrow (\mathbb{R}^d)^N$$

Theorem (Kolmogorov) Given $\{P_{t_1 \dots t_N}(\cdot)\}_{\substack{t_1 \dots t_N \\ N \in \mathbb{N}}}$ probab. measures which are consistent i.e.

$$P_{t_1 \dots t_j \dots t_N}(B_1 \times \dots \times \mathbb{R}^d \times \dots \times B_N)$$

$$= P_{t_1 \dots t_{j-1} t_{j+1} \dots t_N}(B_1 \times \dots \times B_{j-1} \times B_{j+1} \dots B_N)$$

then there exists exactly one $\mathbb{P}$ on $\Omega = (\mathbb{R}^d)^{[0,\infty)}$ with $\mathbb{P} \circ (X_{t_1}, \dots, X_{t_N})^{-1} = P_{t_1 \dots t_N}(\cdot)$.

Consequence (canonical process)

$\omega \in \Omega$ is a function $[t \mapsto \omega(t)] \in (\mathbb{R}^d)^{[0,\infty)}$

$$\left.\begin{aligned} X_t(\omega) &= \omega(t) \\ X_t &: \Omega \to \mathbb{R}^d \end{aligned}\right\} \text{ projection onto coordinate } t$$

stochastic process = probability measure on a function space

why "random" 3 • construction of X_t

AIM produce consistent $p_{t_1 \dots t_N}(\cdot)$'s

DIRECTLY exist known consistent families (e.g. Gaussian, stable...)

CHAINING

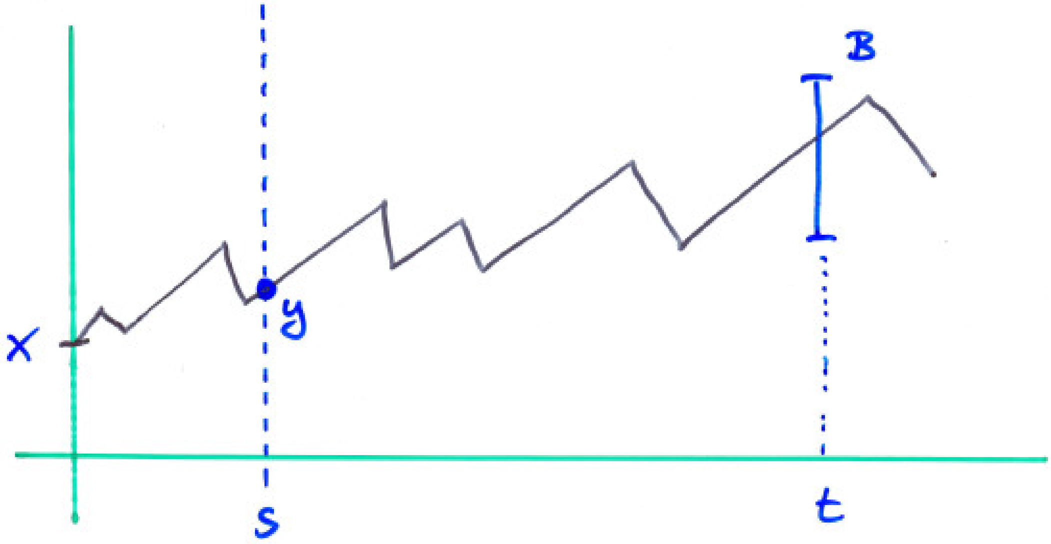
$$\underset{\substack{\uparrow \\ \text{future}}}{\mathbb{P}(F} \mid \overset{\substack{\text{past} \\ \downarrow}}{H)} = \frac{\mathbb{P}(F \cap H)}{\mathbb{P}(H)}$$

$$\mathbb{P}(X_{t_{N+1}} \in B_{N+1} \mid X_{t_j} \in B_j \quad j = 1, 2, \dots, N)$$

$$\overset{\text{ass}}{=} \mathbb{P}(X_{t_{N+1}} \in B_{N+1} \mid X_{t_N} \in B_N)$$

Markov property — no memory

Notation: $\mathbb{P}^x(\cdot) = \mathbb{P}(\cdot \mid X_0 \in \{x\})$



$$\mathbb{P}^x(X_t \in B) = \int_{\mathbb{R}^d} \underbrace{\mathbb{P}^y(X_{t-s} \in B)} \, \underbrace{\mathbb{P}^x(X_s \in dy)}_{P_s(x,\,dy)}$$

one-step probabilities

by iteration:

$$\mathbb{P}^x(X_{t_1} \in B_1, \dots, X_{t_N} \in B_N) = \int_{B_1} \dots \int_{B_N} \prod_{j=1}^{N} P_{t_j - t_{j-1}}(y_{j-1}, dy_j)$$

$(y_0 = x,\ t_0 = 0)$

From now on: MARKOV.

Q: how to get $\mathbb{P}^x(X_t \in B) = p_t(x,B)$

operator semigroups

- $T_t u(x) := \int_{\mathbb{R}^d} u(y)\, p_t(x,dy) = \mathbb{E}^x u(X_t)$

$$= \int_\Omega u(X_t)\, d\mathbb{P}^x$$

- $T_t 1_B(x) = p_t(x,B) = \mathbb{P}^x(X_t \in B)$

- $T_t \circ T_s u(x) = \int T_s u(x)\, p_t(x,dy)$

$$= \int u(z) \underbrace{\int p_s(y,dz)\, p_t(x,dy)}_{= p_{t+s}(x,dz)}$$

Chapman-Kolmogorov eqns

$$= T_{t+s} u(x)$$

- $T_0 = id$

- stochastic

$0 \le u \le 1 \implies 0 \le T_t u \le 1$

$\boxed{Q:}$ which u? which (Banach-) space?

$T_t T_s = T_{t+s}$, $T_0 = id$ $\rightsquigarrow$ functional eqn.

$T_t =$ "e^{tA}" where $A = \frac{d}{dt}\Big|_{t=0} T_t$

sense?

<u>want</u>

- $T_t 1_B(x) = p_t(x, B)$
- Chapman-Kolmogorov

$\forall B \, \forall t \, \forall x$

<u>cannot</u> use L^p naïvely (→ Dirichlet forms...)

<u>good candidate</u>

$$C_\infty(\mathbb{R}^d) = \{ u : \mathbb{R}^d \longrightarrow \mathbb{R}, \text{ cts}, \lim_{|x|\to\infty} u(x) = 0 \}$$

$\|\cdot\|_\infty$ = sup-norm

<u>Theorem</u> (Hille-Yosida-Ray) A (usually) unbounded linear operator A on C_∞ generates a strongly continuous stochastic semigroup $T_t : C_\infty \to C_\infty$ ("Feller semigroup") iff

① $\mathcal{D}(A) \subset C_\infty$ dense

② A is a generalized Laplace operator

$$u(x_0) = \sup_x u(x) \geq 0 \implies Au(x_0) \leq 0$$

("Positive maximum principle" PMP)

③ $\mathrm{range}(\lambda - A) = C_\infty \quad \forall \lambda > 0$

SO FAR:

$$\left.\begin{array}{lll} \text{(Feller) process} & & X_t \\ \text{(Feller) semigroup} & & T_t \\ \text{(Feller) generator} & & A \end{array}\right] \begin{array}{c} 1:1 \\ \text{correspondence} \end{array}$$

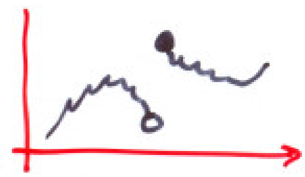
Example: Lévy processes

<u>Def^n</u> $X_t : \Omega \to \mathbb{R}^d$ <u>Lévy process</u> if

(L1) $\mathbb{P}^x (X_t - X_s \in B) = \mathbb{P}^0 (X_{t-s} \in B)$

(L2) $X_t - X_s$ independent of $X_\sigma,\ \sigma \le s\ (\le t)$

(L3) $t \mapsto X_t(\omega)$ càdlàg



<u>Consequences:</u>

$$p_s(x, B) = p_s(B - x), \quad T_s u = p_s * u$$

convolution operator

$$\hat{p}_s(\xi) = \mathbb{E}^x e^{i\xi(X_t - x)} = e^{-t\psi(\xi)}$$

Lévy-Khinchine formula

$$\psi(\xi) = i\ell\xi + \sum q_{jk}\xi_j\xi_k + \int_{y\neq 0} \left(1 - e^{iy\xi} + \frac{iy\xi}{1+|y|^2}\right) N(dy)$$

$\ell \in \mathbb{R}^d$, $(q_{jk}) \in \mathbb{R}^{d\times d}$ pos. definite, $N(dy)$ jump measure

$(\ell, q_{jk}, N(\cdot))$ characteristics of ψ resp. X_t

Pseudo-differential operators

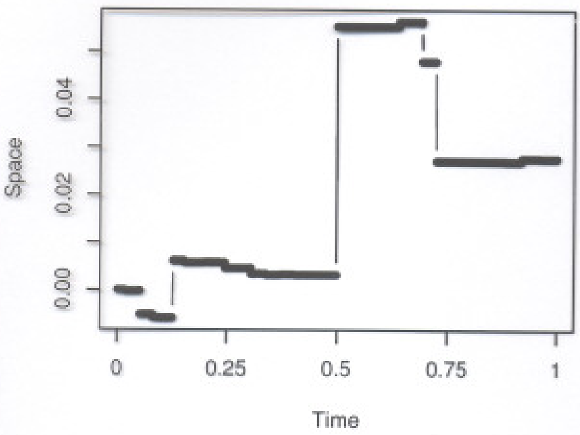
$$\widehat{\frac{d}{dt}\Big|_{t=0} T_t u} = \frac{d}{dt}\Big|_{t=0} \widehat{T_t u} = \frac{d}{dt}\Big|_{t=0} e^{-t\psi}\hat{u} = Au$$

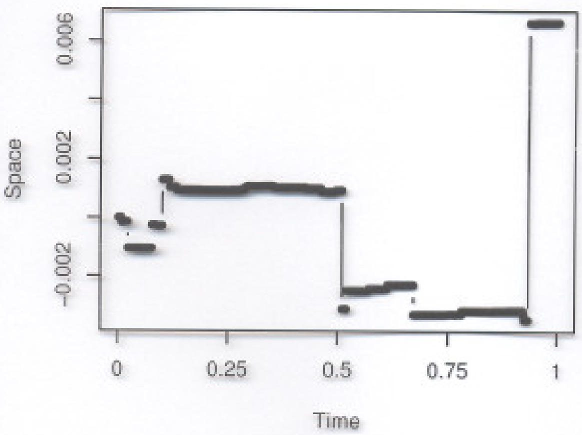
$$Au(x) = (-\psi\hat{u})^{\vee}(x) = -\int e^{-ix\xi}\hat{u}(\xi)\underbrace{\psi(\xi)}_{\text{symbol of } A}d\xi$$

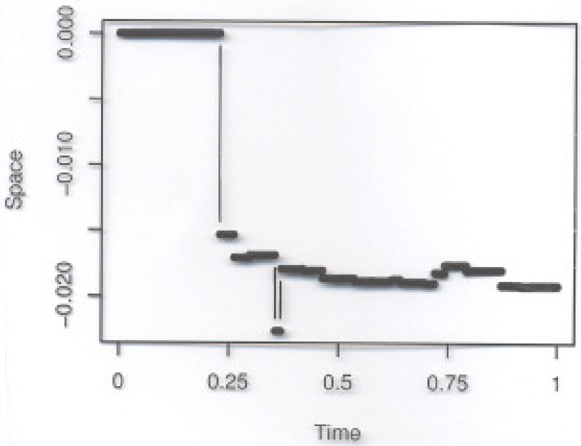
BUT symbols are non-classical

$\lvert\xi\rvert^2$	Brownian motion	$(-\Delta)$
$\lvert\xi\rvert^\alpha \quad 0<\alpha<2$	symm. α-stable	$+(-\Delta)^{\alpha/2}$
$\sqrt{\lvert\xi\rvert^\alpha + m} - \sqrt{m}$		$\sqrt{(-\Delta)^\alpha + m} - \sqrt{m}$
$\log(1+\lvert\xi\rvert^2)$	Gamma	$\log(1-\Delta)$
$1-e^{iy\xi}$	Poisson	$id - \tau_y$
$i\xi$	uniform	$-\frac{d}{dx}$

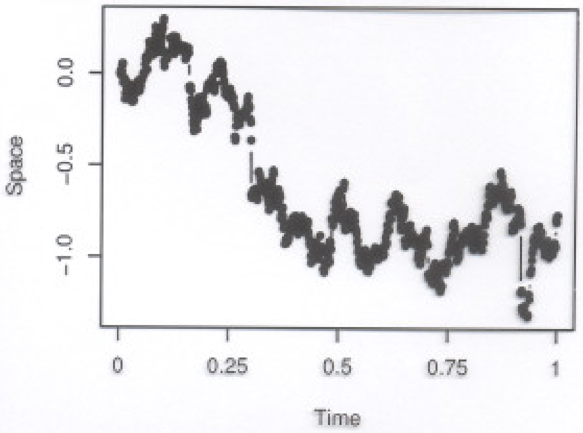
$|\xi|^{0.3} = \psi(\xi)$

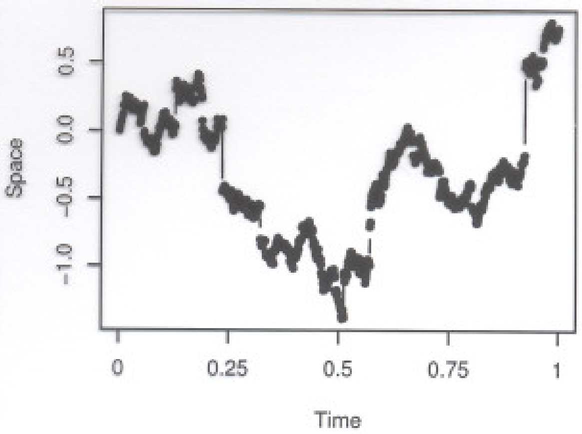


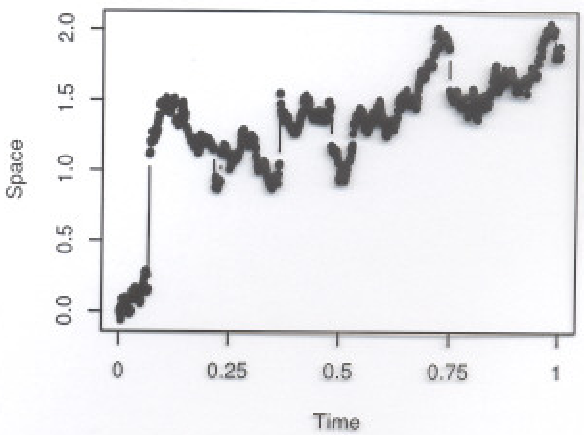




$|\xi|^{1.7} = \psi(\xi)$

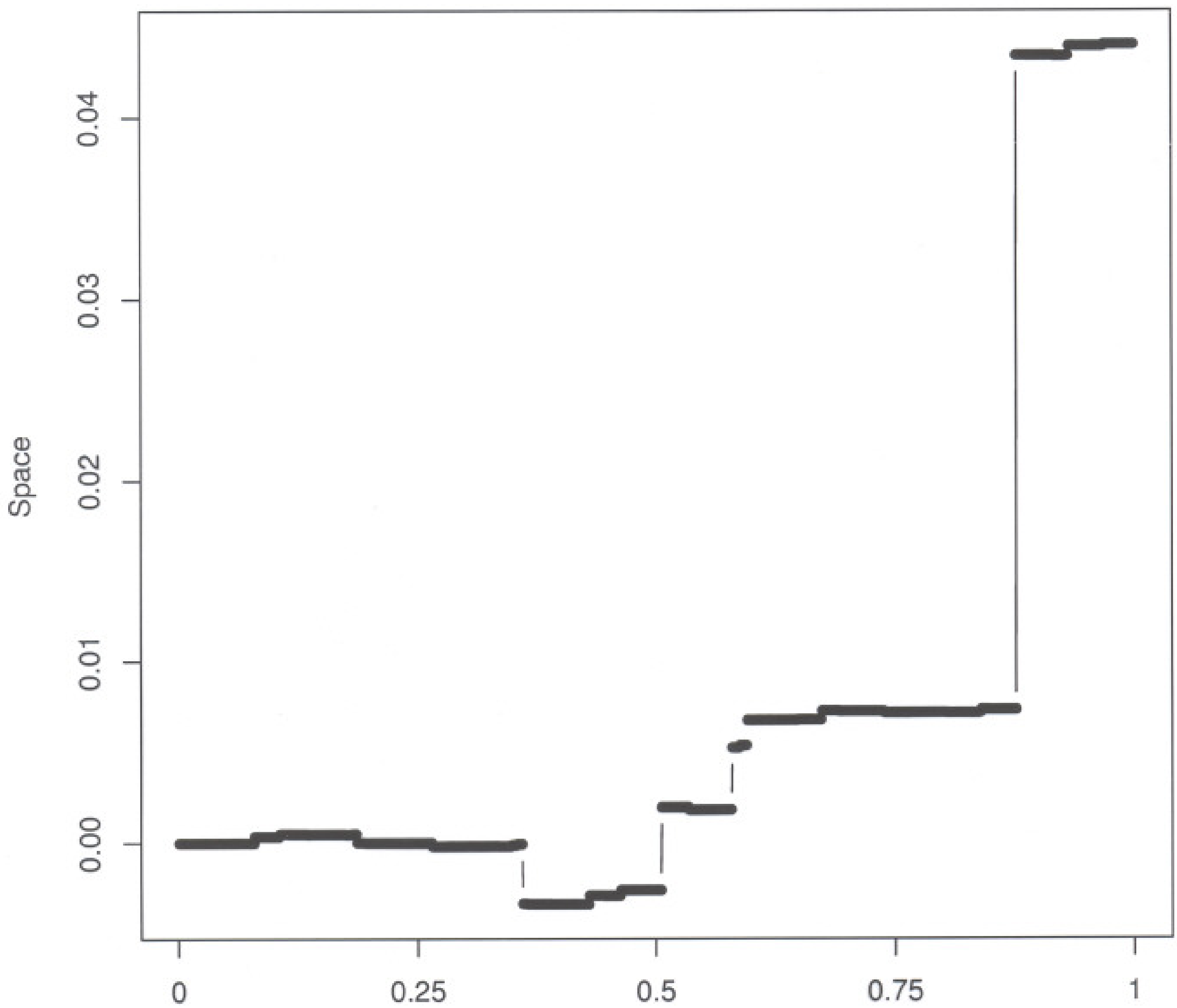




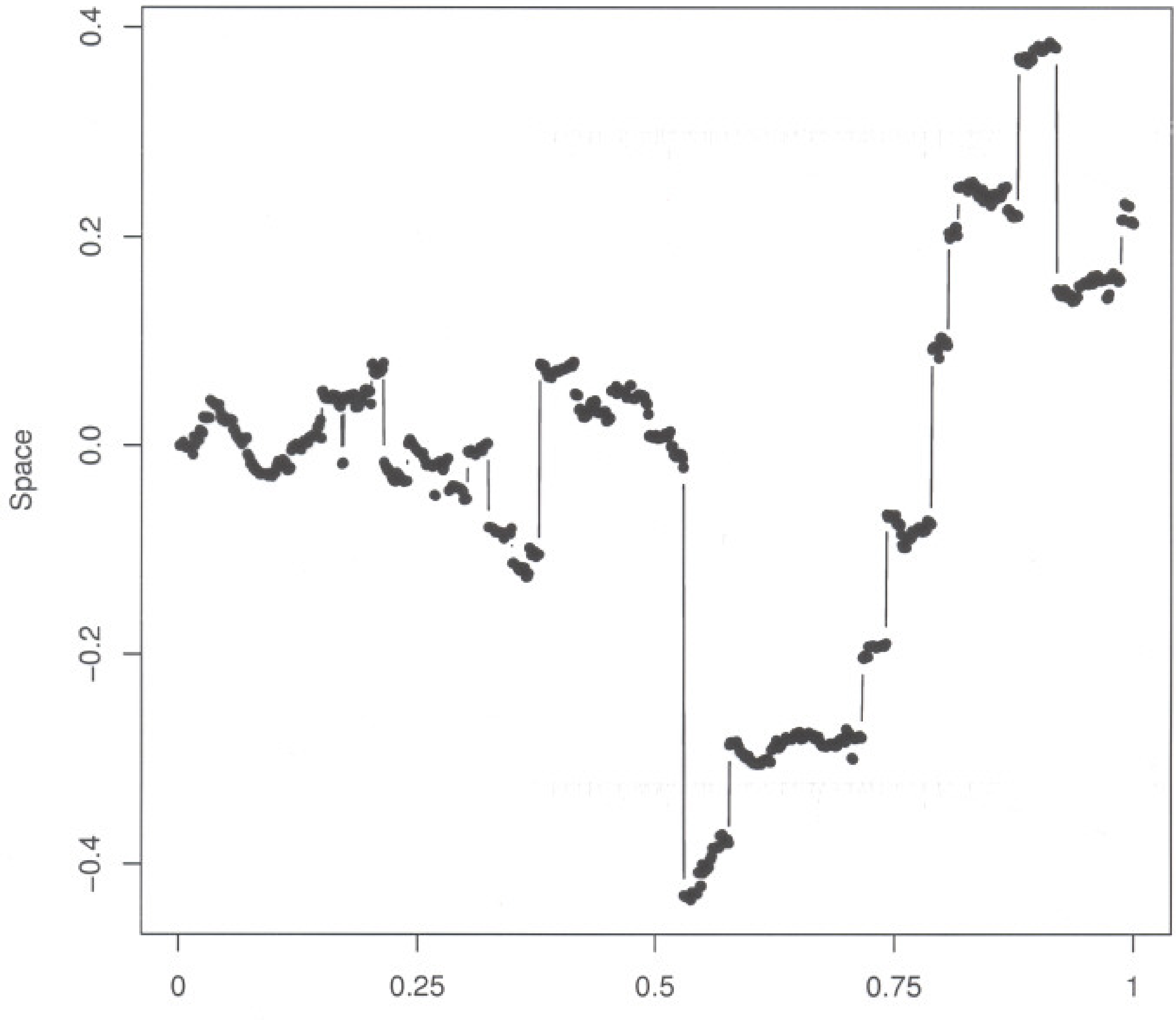


"typische" Lévy Pfade $(X_0 = 0)$

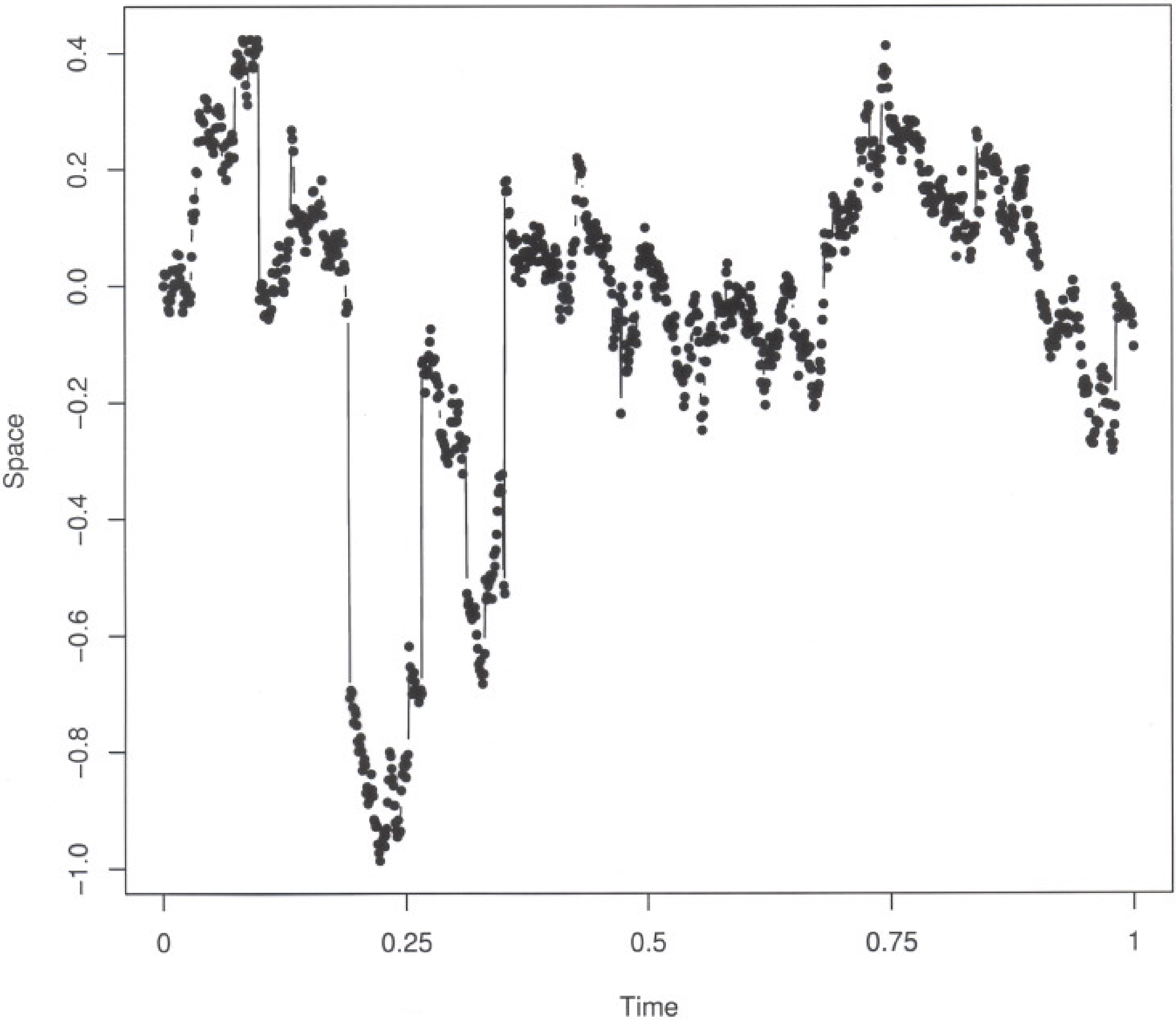
$\psi(\xi) = |\xi|^{0.3}$ ($S\alpha S$, $\alpha = 0.3$)



$\psi(\xi) = |\xi|^{1.0}$ (Cauchy-Prozess)



$$\psi(\xi) = |\xi|^{1.7}$$



11d

Positive maximum principle – structure results

<u>Theorem</u> (v. Waldenfels, Courrège) Any Feller generator A such that $C_c^\infty \subset \mathcal{D}(A)$ is of the form

$$Au(x) = -\int p(x,\xi)\, \hat{u}(\xi)\, e^{-ix\xi}\, d\xi$$

i.e. a pseudo-differential operator with symbol

$$p(x,\xi) = i\,l(x)\,\xi + \sum q_{jk}(x)\,\xi_j\xi_k + \int \left[1 - e^{iy\xi} + \frac{iy\xi}{1+|y|^2}\right] N(x,dy)$$

with characteristics $(l(x), q_{jk}(x), N(x,\cdot))$.

Lévy $\hat{=}$ constant "coefficients"

Feller $\hat{=}$ variable "coefficients"

e.g. $\quad -\Delta \quad \longleftrightarrow \quad -\sum_{j,k=1}^{d} q_{jk}(x)\,\partial_j\partial_k$

$\quad (-\Delta)^\alpha \quad \longleftrightarrow \quad (-\Delta)^{\alpha(x)}$

NOTICE

- pseudo-differential operators with negative definite symbols i.e. rough, non-classical symbols ($\leadsto$ 1st order symbolic calculus...)

(W. Hoh)

- only necessary conditions!

Levy:

$$\text{process} \xleftrightarrow{1:1} \psi(\xi) \xleftrightarrow{1:1} (\ell, q_{jk}, N(\cdot))$$

Feller:

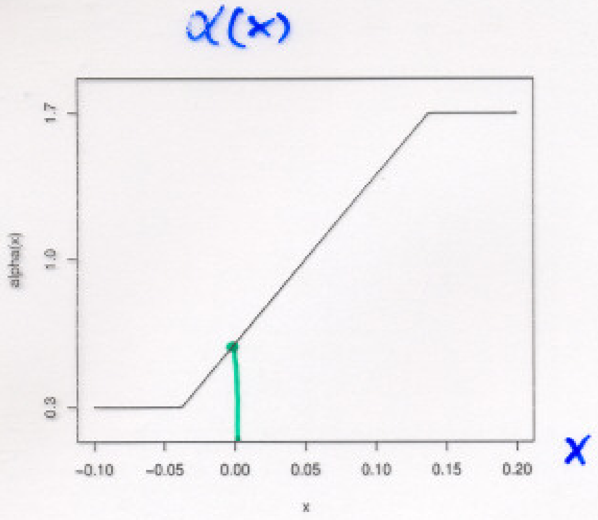
$$\text{process} \longrightarrow p(x,\xi) \xleftrightarrow{1:1} (\ell(x), q_{jk}(x), N(x,dy))$$

+ extra conditions (N. Jacob, W. Hoh, F. Baldus, R. Bass, Komatsu, Tsuchiya)

strategy: check in Hille-Yosida-Ray

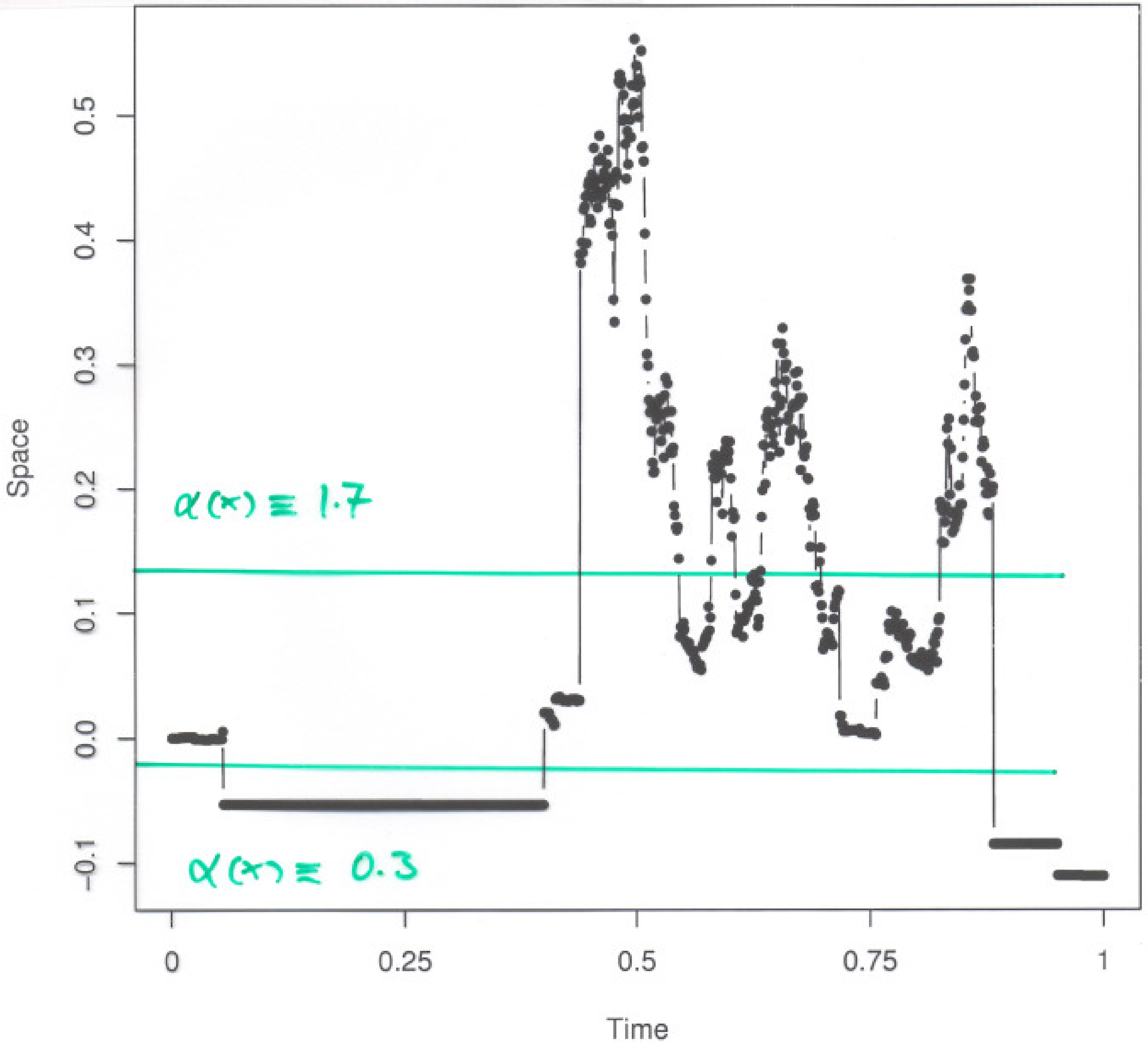
$$\text{range}(\lambda + p(x,D)) \underset{\text{dense}}{\hookrightarrow} C_\infty$$

typical: $x \longmapsto p(x,\xi)$ diff'ble (C^{3d})



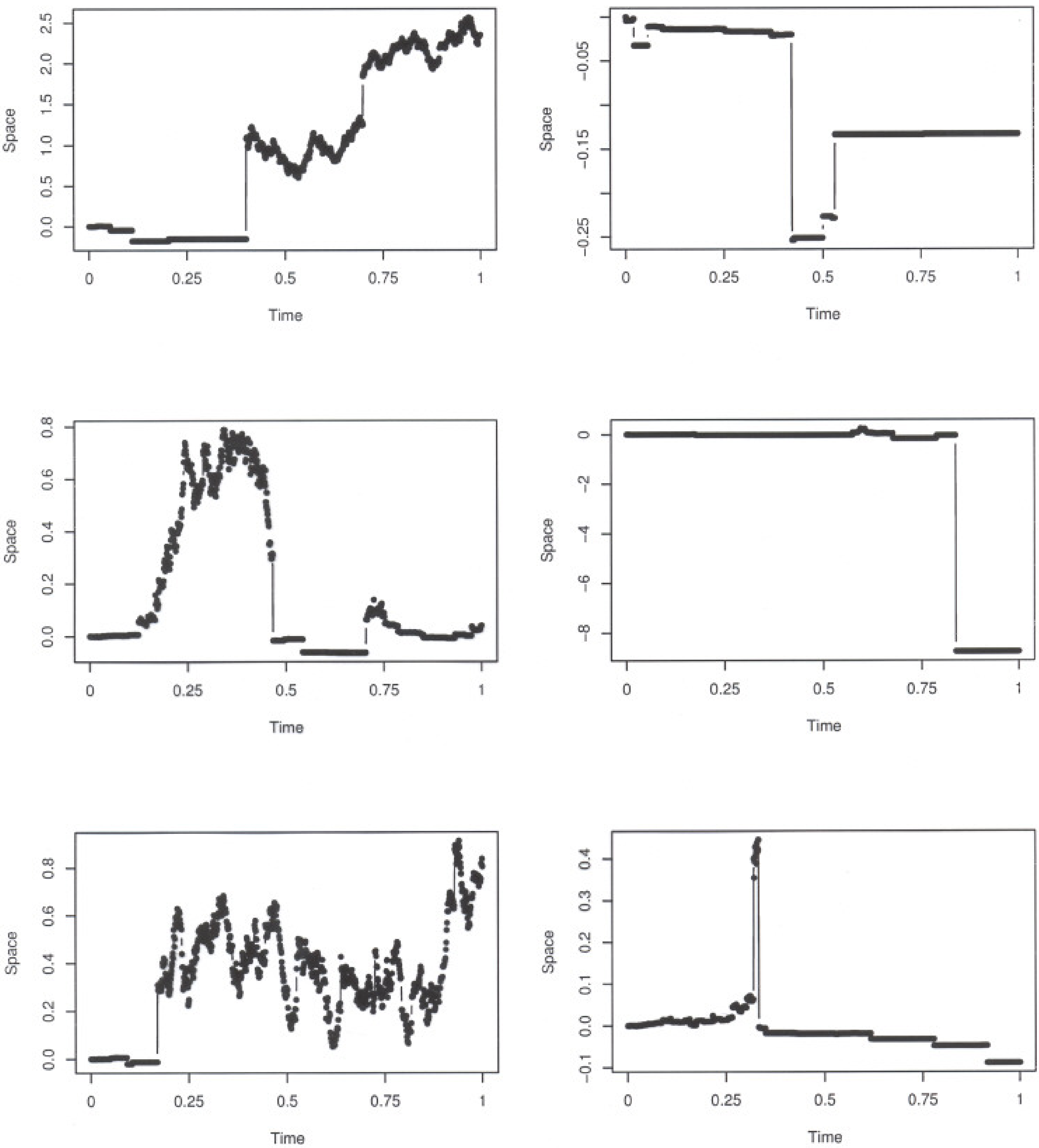
$$p(x,\xi) = |\xi|^{\alpha(x)}$$

Note the qualitatively different path properties depending on $\alpha(x)$!



algorithm + realisation: B. Böttcher (Marburg)

$p(x,\xi) = |\xi|^{\alpha(x)}$ ($\alpha(x)$ as before)



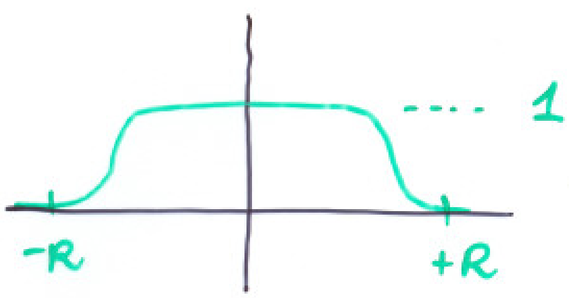
Sample path properties – structural results

Theorem (Lévy canonical form) A Lévy process X_t satisfies

$$X_t = \boxed{B_t^Q + \int x\,\chi(x)\,\big(\mu_t(\cdot,dx) - t\,N(dx)\big)} \quad \text{martingale}$$

$$\boxed{\underbrace{c^{\ell,\chi}\,t}_{\text{continuous}} + \underbrace{\sum_{0<s\le t} \Delta X_s \cdot \big(1-\chi(\Delta X_s)\big)}_{\text{jump part}}} \quad \text{bdd. variation}$$

$\chi(x)$ = cut-off function



$\mu_t(\omega, dx)$ = Poisson Point Process (jumps!)

$t\,N(dx)$ = intensity measure

Theorem Any Feller process X_t admitting a symbol $p(x,\xi)$ with bounded coefficients is a semi-martingale (even: homogeneous diffusion with jumps). The semimartingale characteristics are

$$\left(\int_0^t \ell(X_s)\,ds,\ 2\int_0^t q_{jk}(X_s)\,ds,\ N(X_{s-},dy)\,ds \right)$$

where $(\ell(\cdot), Q(\cdot), N(\cdot,dy)) \xleftrightarrow{1:1} p(x,\xi)$.

Moreover:

$$X_t = X_t^{c,Q} + \int_0^t y\chi(y)\left(\mu^X(\cdot,ds,dy) - N(X_{s-},dy)\,ds\right)$$

$$+ \int_0^t c^{\chi,\ell(\cdot)}(X_{s-})\,ds + \sum_{s\le t} \Delta X_s\,(1-\chi(\Delta X_s))$$

Here: $\mu^X(\cdot,ds,dy) = \sum_{s\le t} \delta_{(s,\Delta X_s)}(ds,dy)$

Corollary (+ Uemura) The above Theorem holds for any HUNT process admitting a symbol $p(x,\xi)$ with bounded coefficients.

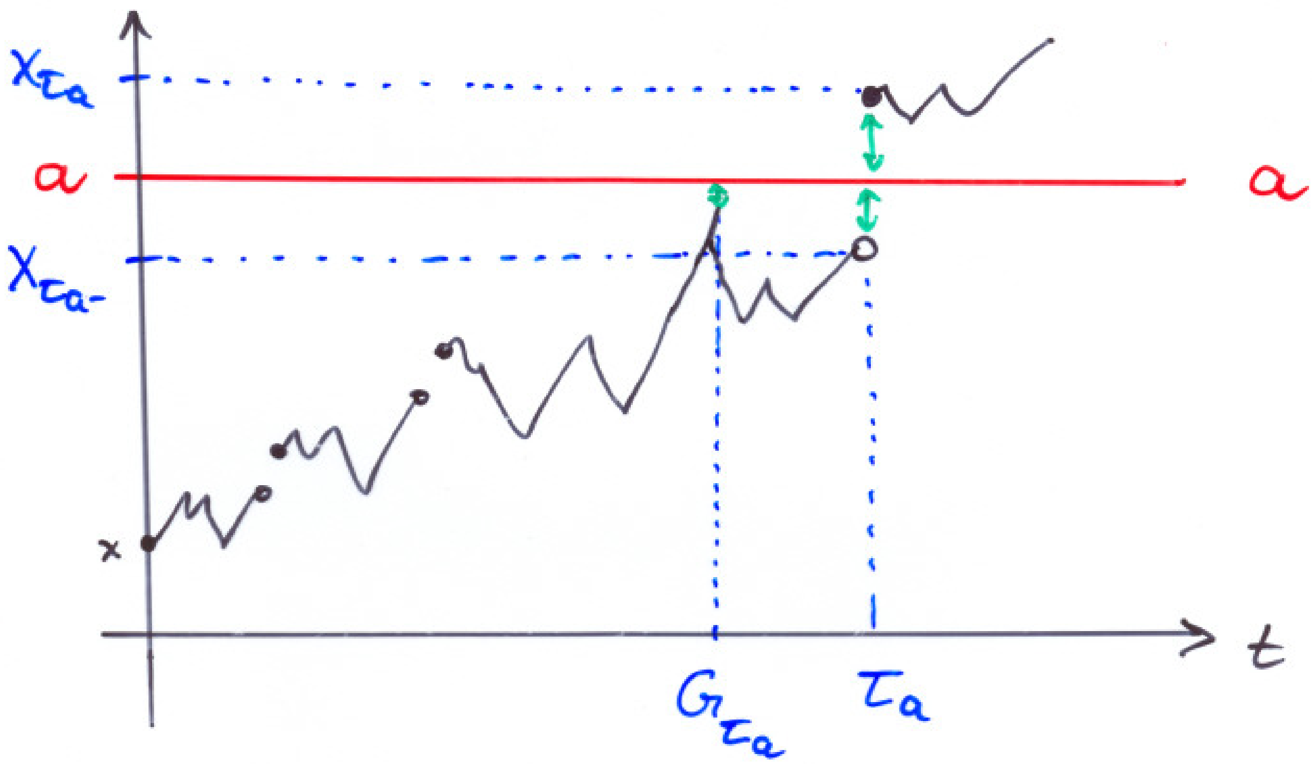
Some open questions

- characterise all [Hunt] processes having a symbol $p(x,\xi)$

$d=1$

- develop a fluctuation theory for real X_t (e.g. a "quintuplet law":

$$(\tau_a - G_{\tau_a -},\ G_{\tau_a -},\ X_{\tau_a} - a,\ a - X_{\tau_a -},\ a - \sup_{s\le\tau_a} X_t)$$



⇝ Lévy case: Doney, Vigon, Kyprianou

⇝ Feller case: Lévy systems (Watanabe)

équations amicales (Vigon)

Further path properties

- → short-time asymptotics
- → long-time behaviour
- → dimension of range / occupied sites
- → smoothness of trajectories
- → upper + lower functions
- → multiple points
- → recurrence, transience
- → local times

⋮

a lot is known for $\boxed{\text{Lévy processes}}$

main tool $\psi(\xi)$ resp. (ℓ, q_{jk}, N)

$\boxed{\text{Programme:}}$

- use $p(x,\xi)$ in Feller case
- compare / control $p(x,\xi)$ by a "benchmark" $\psi(\xi)$

Probability estimates

Theorem Let X_t be a Feller process with symbol $p(x,\xi)$. If

- $\sup_x |p(x,\xi)| \leq c\cdot(1+|\xi|^2)$ "bounded coeff."
- $|\operatorname{Im} p(x,\xi)| \leq c' \operatorname{Re} p(x,\xi)$ "sector cond."

then

- $$\mathbb{P}^x\Big(\sup_{s\leq t}|X_s-x|<R\Big) \leq \frac{c''}{t \inf\limits_{|x-y|\leq R}\sup\limits_{|\vec{e}|=1}|p(y,\tfrac{1}{R}\vec{e})|}$$

- $$\mathbb{P}^x\Big(\sup_{s\leq t}|X_s-x|\geq R\Big) \leq c'''\cdot t\cdot \sup_{|x-y|\leq R}\sup_{|\vec{e}|=1}|p(y,\tfrac{1}{R}\vec{e})|$$

- $$\frac{\kappa}{\sup\limits_{|x-y|\leq R}\sup\limits_{\vec{e}}|p(y,\tfrac{1}{R}\vec{e})|} \leq \mathbb{E}^x\sigma_R^x \leq \frac{K}{\inf\limits_{|x-y|\leq R}\sup\limits_{\vec{e}}|p(y,\tfrac{1}{R}\vec{e})|}$$

where σ_R^x is the first passage time from $B_R(x)$

$$\sigma_R^x = \inf\{t\geq 0 : |X_t-x|\geq R\}$$

nicest case for the above Theorem:

$$p(x,\xi) = |\xi|^{\alpha} \rightsquigarrow \mathbb{E}^x \sigma_R^x \asymp R^{\alpha}$$

$$p(x,\xi) = |\xi|^{\alpha(x)} \rightsquigarrow R^{\underline{\alpha}} \lesssim \mathbb{E}^x \sigma_R^x \lesssim R^{\overline{\alpha}}$$

if $\alpha(x) \in [\underline{\alpha}, \overline{\alpha}]$

(and $R \geq 1$)

<u>consequences</u>

- Harnack inequality (uses methods developed by Bass-Levin, Song-Vondraček, Bass-Kassmann)

- Feller property for certain classes of Dirichlet forms.

Short-time asymptotics – Hölder conditions

Theorem Let X_t be a Feller process, symbol $p(x,\xi)$ with bdd coefficients and sector condition. Then

- $$\limsup_{t \downarrow 0} t^{-1/\lambda} \sup_{s \le t} |X_s - x| = \begin{cases} 0 & , \lambda > \beta_\infty^x \\ \infty & , \lambda < \delta_\infty^x \end{cases} \quad \text{a.s.}$$

with the Blumenthal-Getoor type indices ("at ∞")

$$\beta_\infty^x = \inf\left\{ \lambda > 0 : \lim_{|\xi| \to \infty} \frac{\sup_{|x-y| < 1/|\xi|} |p(y,\xi)|}{|\xi|^\lambda} = 0 \right\}$$

$$\delta_\infty^x = \inf\left\{ \lambda > 0 : \lim_{|\xi| \to \infty} \frac{\inf_{|x-y| < 1/|\xi|} |p(y,\xi)|}{|\xi|^\lambda} = 0 \right\}$$

- $0 \le \delta_\infty^x \le \beta_\infty^x \le 2$
- Lévy case $p(x,\xi) = \psi(\xi)$: $\beta_\infty^x = \delta_\infty^x = \beta$

  upper index
- stable-like-case: $p(x,\xi) = |\xi|^{\alpha(x)}$:

  $$\beta_\infty^x = \delta_\infty^x = \alpha(x)$$
- open question: multifractal analysis ?

Long-time asymptotics: growth behaviour

Theorem X_t Feller, symbol $p(x,\xi)$ with bdd coefficients and sector condition. Then

$$\limsup_{t\to\infty} t^{-1/\lambda} \sup_{s\le t} |X_t - x| = \begin{cases} 0 & , \lambda < \beta_0 \\ \infty & , \lambda > \delta_0 \end{cases}$$

with the generalised indices ("at zero")

$$\beta_0 = \sup\left\{ \lambda \geq 0 : \lim_{\xi\to 0} \frac{\sup_x |p(x,\xi)|}{|\xi|^\lambda} = 0 \right\}$$

$$\delta_0 = \sup\left\{ \lambda \geq 0 : \lim_{\xi\to 0} \frac{\inf_x |p(x,\xi)|}{|\xi|^\lambda} = 0 \right\}$$

- open questions: upper/lower functions

Dimension estimates (· + Y. Xiao)

$$\dim_P X(E,\omega) = \dim_P \{ X_t(\omega) : t \in E \}$$

packing dimension $\qquad\qquad \subset \mathbb{R}_+$

Let X_t be a Feller process such that for some $H > 0$

(C1) $\mathbb{P}^x \left(\sup_{s \le t} |X_s - x| \geq t^H |\log t|^\lambda \right) \le c \cdot |\log t|^{-\lambda\beta}$

(for some $\lambda, \beta > 0$, $\lambda\beta > 1$, all x, all small $t < h_0$)

(C2) $\mathbb{P}^x \left(|X_t - x| \le r \right) \le c' \left(\frac{r}{t^H} \right)^d$

(for all x, all $t \le 1$, all $r \le r_0$)

Theorem Let $E \subset [0,\infty)$ be closed. Under (C1) + (C2) we have a.s.

$$\dim_P X(E) = \frac{1}{H} \operatorname{Dim}_{Hd} E \qquad \text{(packing dim. profile)}$$

$$= \frac{1}{H} \dim_P E \qquad \text{(if } Hd \geq 1\text{)}$$

Note "$\le$" uses only (C1)
"$\geq$" uses only (C2)

Remarks

- X_t Lévy with $|\psi(\xi)| \leq c \cdot |\xi|^\alpha$ $(|\xi| \gg 1)$

  $\Longrightarrow$ (C1) with $H < 1/\alpha$

- X_t Lévy with $|\psi(\xi)| \geq c' |\xi|^\alpha$ $(|\xi| \gg 1)$

  $\Longrightarrow$ (C2) with $H > 1/\alpha$

- X_t Feller with symbol $p(x, \xi)$,

  bounded coefficients, sector condition

  $\Longrightarrow$ (C1) with $H < 1 / \sup_x \beta_\infty^x$