ELLIPTIC CURVES IN HYPER-KÄHLER VARIETIES

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Abstract. We show that the moduli space of elliptic curves of minimal degree in a general Fano variety of lines of a cubic fourfold is a non-singular curve of genus 631. The curve admits a natural involution with connected quotient. We find that the general Fano contains precisely 3780 elliptic curves of minimal degree with fixed (general) j-invariant.

More generally, we express (modulo a transversality result) the enumerative count of elliptic curves of minimal degree in hyper-Kähler varieties with fixed j-invariant in terms of Gromov–Witten invariants. In $K3[2]$-type this leads to explicit formulas of these counts in terms of modular forms.

1. Introduction

1.1. **Moduli of elliptic curves.** A non-singular complex projective variety X is *hyper-Kähler* if it is simply-connected and $H^0(X, \Omega_X^2)$ is spanned by a non-degenerate holomorphic 2-form. Let (X, H) be a very general (primitively) polarized hyper-Kähler variety. Since the Picard group of X is of rank 1, there exist a unique curve class of minimal degree,

$$H_2(X, \mathbb{Z})_{\mathrm{alg}} = \mathbb{Z}\beta.$$

Let $M_{g,n}(X, \beta)$ be the moduli space of stable maps $f : C \to X$ from non-singular curves of genus g with n distinct markings representing the class β. Let $\overline{M}_{g,n}(X, \beta)$ be its Deligne-Mumford compactification parametrizing stable maps from nodal curves. The moduli spaces $M_{g,n}(X, \beta)$ and $\overline{M}_{g,n}(X, \beta)$ both have expected dimension

$$\mathsf{vd} := (\dim X - 3)(1 - g) + n + 1.$$

In genus 1 the moduli space $\overline{M}_{1,0}(X, \beta)$ has expected dimension 1 and parametrizes two types of maps $f : C \to X$: Either C is irreducible of arithmetic genus 1, or C has an elliptic tail which is contracted by f. We expect the following non-degeneracy result for maps from non-singular curves:

Conjecture 1.1. *Let X be a very general polarized hyper-Kähler variety with primitive curve class β. Then $M_{1,0}(X, \beta)$ is pure of dimension* 1.

Date: July 25, 2019.

In the conjecture the moduli space $M_{1,0}(X,\beta)$ is allowed to be empty. This happens for example on deformations of generalized Kummer fourfolds of an principally polarized abelian surface, see Section 4.

For a K3 surface X the moduli space $M_{1,0}(X,\beta)$ is always non-empty and smooth of dimension 1, see Section 2.2. Hence Conjecture 1.1 holds for K3 surfaces. In genus 0 the moduli space of stable maps $\overline{M}_{0,0}(X,\beta)$ is pure of the expected dimension, see for example [12, Prop.2.1].

1.2. **Counting elliptic curves.** An *elliptic curve* in X is an irreducible curve $C \subset X$ of geometric genus 1. We want to count elliptic curves in class β with normalization having a fixed j-invariant. Since the family of elliptic curves in X has expected dimension 1 and fixing the j-invariant is a codimension 1 condition we expect a finite count.

More precisely, define $n_{X,j_0} \in \mathbb{N}_0 \cup \{\infty\}$ to be the number of elliptic curves in class β with j-invariant of the normalization equal to $j_0 \in \overline{M}_{1,1}$,

$$n_{X,j_0} = \#\{C \subset X \mid [C] = \beta,\, j(\tilde{C}) \cong j_0\}.$$

If the set on the right hand side is infinite, we set $n_{X,j_0} = \infty$.

Below we will express, up to a non-degeneracy assumption, the count $n_{X,j}$ in terms of the Gromov–Witten invariants of X. In several cases these are known and this will lead to explicit formulas for $n_{X,j}$.

1.3. **Gromov–Witten theory.** The moduli space of stable maps $\overline{M}_{g,n}(X,\beta)$ carries a (reduced) virtual fundamental class

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} \in \mathrm{CH}_{\mathsf{vd}}(\overline{M}_{g,n}(X,\beta)),$$

see [9, 10, 11]. Gromov–Witten invariants of X are defined by integrating the virtual class against classes pulled back along the natural maps

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \to X, i = 1,\dots,n, \quad p : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$$

which evaluate a stable map at the i-th marking and forget the map respectively. We will need two particular Gromov–Witten invariants. The first is the virtual analog of the count $n_{X,j}$. Let

$$N_X = \int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(D) \cup p^*[(E,0)]$$

where $(E,0) \in \overline{M}_{1,1}$ is a point and $D \in H^2(X,\mathbb{Q})$ is an arbitrary divisor class with intersection pairing $\langle \beta, D\rangle = 1$. The second invariant is

$$C_X = \int_{[\overline{M}_{0,1}(X),\beta)]^{\mathrm{vir}}} \frac{\mathrm{ev}_1^*(c(T_X))}{1-\psi_1}.$$

(β,β)	<0	0	2	4	6	8	10	12
n_X	0	24	648	9600	102600	881280	6442320	41513472

TABLE 1. First values of $n_{X,j}$ for a K3 surface X and general j.

(β,β)	<0	0	$\frac{3}{2}$	2	$\frac{7}{2}$	4	$\frac{11}{2}$	6
n_X	0	648	3780	23760	129600	470880	2396520	6629040

TABLE 2. First values of $n_{X,j}$ in $K3^{[2]}$-type for general j, assuming $M_{1,0}(X,\beta)$ is generically reduced of dimension 1.

where $c(T_X)$ is the total Chern class of X and ψ_1 is the cotangent line class at the first marking. The denominator is expanded formally:

$$\frac{1}{1-\psi_1} = 1 + \psi_1 + \psi_1^2 + \dots .$$

The following result relates the enumerative and virtual counts.

Proposition 1.2. *Let X be a hyper-Kähler variety with irreducible curve class β. If every irreducible component of $M_{1,0}(X,\beta)$ is generically reduced of dimension* 1*, then*

$$n_{X,j} = \frac{1}{2}(N_X - C_X) \tag{1}$$

for a general $j \in M_{1,1}$.

The factor $1/2$ arises since we do not identify conjugate maps on the right.

If $M_{1,0}(X,\beta)$ is of dimension 1 but not necessarily generically reduced, then the right hand side of (1) computes the length of the fiber of the forgetful map $M_{1,1}(X,\beta) \to M_{1,1}$ over the point $(E,0)$; the right hand side then counts the elliptic curves $C \subset X$ with multiplicities.

If X is a K3 surface or a fourfold of $K3^{[2]}$ type[1] then the Gromov–Witten theory of X is known in all primitive curve classes, see [9] and [10]. In this case we compute the right hand side of Proposition 1.2 in terms of modular forms in Section 2.1. By deformation invariance the Gromov–Witten invariant only depend on the Bauville-Bogomolov norm[2] of the class β, denoted (β,β). The first expected values of $n_{X,j}$ are listed in Tables 1 and 2.

We discuss the first cases. The number 24 in Table 1 is the number of nodal fibers in a general elliptic K3 surface. The degree zero case in $K3^{[2]}$-type, 648, is the virtual number of elliptic curves of fixed complex structure

[1] A hyper-Kähler is of $K3^{[n]}$-type if it is deformation equivalent to the Hilbert scheme of n points on a K3 surface.

[2] The pairing is induced from the Beauville-Bogomolov form on $H^2(X,\mathbb{Z})$ via Poincaré duality. The pairing is $\mathbb{Q}$-valued in general.

in a general Lagrangian fibration $X \to \mathbb{P}^2$ (the count is virtual since the moduli space is not of expected dimension). Indeed by a result of Markman the Lagrangian fibration is (a twist of) the relative Jacobian fibration of a genus two K3 surface (S, L). Under the Lagrangian fibration the elliptic curves in X map to points on the discriminant, and the j-invariant of that curve is precisely the j-invariant of the corresponding nodal elliptic curve in the genus 2 K3. This explains the equality

$$n_{K3[2],(\beta,\beta)=0} = n_{K3,(\beta,\beta)=2}.$$

The number 23760 computes (modulo our assumption) the number of elliptic curves with fixed complex structure on the double covers of EPW sextics. In $K3[2]$-type the case $(\beta, \beta) = 3/2$ corresponds to the Fano variety of lines that we will consider below.

If X is of $K3^{[n]}$-type explicit conjectural formulas for the right hand side of Proposition 1.2 can be obtained from conjectures made in [10].

1.4. **The Fano variety.** Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic 4-fold and let

$$F = \{\, l \in \mathrm{Gr}(2, 6) : l \subset Y \}$$

be the Fano variety of lines in Y. The Fano varieties F form a 20-dimensional family of polarized hyper-Kähler fourfolds of $K3^{[2]}$-type [3]. Let H_F be the Plückar hyperplane section on F and consider the primitive integral class

$$\beta = \frac{1}{36} H_F^3.$$

The class is of degree $\langle \beta, H \rangle = \frac{1}{36} \deg_{H_F}(X) = 3$ with respect to H_F. If F is very general, then β is the unique primitive curve class.

Consider the projective embedding

$$F \subset \mathrm{Gr}(2, 6) \hookrightarrow \mathbb{P}^{14}$$

defined by H_F and assume that β is irreducible (this is a general condition). If $C \subset F$ is a curve of class β then it is of degree 3 in $\mathbb{P}^{14}$. It spans a three-plane in $\mathbb{P}^{14}$ if and only if it is a rational normal curve. Otherwise the curve spans a $\mathbb{P}^2$ in which case it is a plane cubic and hence of arithmetic genus 1. Moreover, since $\mathbb{P}^2 \cap \mathrm{Gr}(2, 6)$ contains a cubic curve we see that in fact $\mathbb{P}^2 \subset \mathrm{Gr}(2, 6)$.[3] Every $\mathbb{P}^2 \subset \mathrm{Gr}(2, 6)$ corresponds to lines passing through a fixed point $v \in \mathbb{P}^5$. Hence the curves in class β of arithmetic genus 1 correspond to surfaces in Y which are cones.

Let $\mathrm{CH}_\beta(F)$ be the Chow variety of curves in F in class β. Define

$$\Sigma \subset \mathrm{CH}_\beta(F)$$

to be the (reduced) locus of curves of arithmetic genus 1.

[3] We thank C. Voisin for pointing out this approach to elliptic curves in F.

Theorem 1.3. *Let F be a general Fano of lines of a cubic fourfold. Then Σ is a non-singular curve with the following properties:*

a) *Σ has at most two connected components.*
b) *Σ admits a natural involution with connected quotient.*
c) $g(\Sigma) = 631$.
d) *If the curve $C \in \Sigma$ is singular, then $C \subset F$ is a rational nodal curve.*
e) *The map $\Sigma \to \mathbb{P}^1$ taking a point $[C] \in \Sigma$ to its j-invariant (in the coarse quotient of $\overline{M}_{1,1}$) is of degree* 3780.

Here the genus of a (possibly disconnected) curve is $g(\Sigma) = 1 - \chi(\mathcal{O}_\Sigma)$.

Let F be a general Fano variety of lines. Let

$$S \subset \mathrm{CH}_\beta(F)$$

be the locus of rational curves. Then S is a smooth connected surface isomorphic to the locus of lines of second type [1, 12]. We then have

$$\mathrm{CH}_\beta(F) = S \cup \Sigma$$

and the intersection $S \cap \Sigma$ consists of finitely many (at most 3780) points corresponding precisely to the nodal rational curves.

Finally we consider the moduli space of stable maps.

Proposition 1.4. *$M_{1,0}(X, \beta)$ is a non-singular curve and isomorphic to the open subset of Σ parametrizing smooth elliptic curves.*

By using Theorem 1.3 directly, or combining Propositions 1.2 and 1.4, we have thus obtained two proofs of the following:

Corollary 1.5. *A general Fano variety of lines contains precisely* 3780 *elliptic curves of minimal degree and of fixed general j-invariant.*

1.5. **Plan of the paper.** In Section 2.1 we prove Proposition 1.2 and recall the Gromov–Witten calculations for the K3 and $K3^{[2]}$ case. In Section 3 we discuss the geometry of the Fano and prove Theorem 1.3 and Proposition 1.4. In Section 4 we give an example where $M_{1,0}(X, \beta)$ is empty.

1.6. **Acknowledgements.** We would like to thank C. Voisin for suggesting the approach to calculate the genus of the curve Σ. We are also grateful to H.-Y. Lin who pointed out the generalized Kummer example. We also thank L. Battistella, D. Huybrechts, R. Mboro, J. Schmitt, J. Shen, and Q. Yin for useful discussions and their interest.

2. Gromov–Witten theory

2.1. **Proof of Proposition 1.2.** Let $(E, 0)$ be a general non-singular elliptic curve and let $\overline{M}_{E,n}(X, \beta)$ be the moduli space of n-pointed stable maps from E to X in class β. The points of $\overline{M}_{E,n}(X, \beta)$ correspond to stable maps from nodal, n-pointed degenerations of E to $\mathbb{P}^1$, see [13] for a definition[4]. The moduli space $\overline{M}_{E,n}(X, \beta)$ carries a perfect obstruction theory and its reduced virtual class [9, 10, 11] satisfies

$$[\overline{M}_{E,n}(X, \beta)]^{\mathrm{vir}} = p^*[(E, 0)] \cap [\overline{M}_{1,n}(X, \beta)]^{\mathrm{vir}}.$$

Using this and the divisor equation we obtain

$$N_X = \int_{[\overline{M}_{E,1}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(D) = \int_{[\overline{M}_{E,0}(X,\beta)]^{\mathrm{vir}}} 1.$$

Since β is irreducible the moduli space $\overline{M}_{E,0}(X, \beta)$ parametrizes the following two types of stable maps $f : C \to X$:

(1) C is non-singular, isomorphic to E, and $f : C \to X$ is birational onto an elliptic curve in X, or
(2) $C = \mathbb{P}^1 \cup_x E$ and the degree of $f|_E$ is zero.

Since by assumption every irreducible component of $M_{1,0}(X, \beta)$ is generically reduced of dimension 1, it is smooth on a dense open subset U. Therefore the map $M_{1,0}(X, \beta) \supset U \to \mathbb{P}^1$ which sends a stable map to the j-invariant of its source is generically smooth.[5] Since E is general, it follows that there are only finitely many reduced points in $\overline{M}_{1,0}(X, \beta)$ parametrizing maps of the first kind. On the other hand, the locus of maps from reducible curves is a closed subset of $\overline{M}_{1,0}(X, \beta)$. Hence we have the disjoint union:

$$\overline{M}_{E,0}(X, \beta) = M_1 \sqcup M_2$$

where M_i, $i = 1, 2$ is parametrizing maps of first and second time respectively. Splitting the contribution from the virtual class hence yields

$$N_X = \int_{[M_1]^{\mathrm{vir}}} 1 + \int_{[M_2]^{\mathrm{vir}}} 1.$$

We now isolate the contributions from each component. We consider M_1 first which consists of finitely many reduced points. Since the virtual fundamental class is of dimension 0 it coincides with the usual fundamental class and thus its integral is the number of these reduced points. Since

[4] In our case we do not identify maps that differ by a conjugation on E

[5] Here $\mathbb{P}^1$ is the coarse quotient of $\overline{M}_{1,1}$. After an étale cover the universal curve over U admits a section and defines a map to $\overline{M}_{1,1}$ and hence $\mathbb{P}^1$. Since $\mathbb{P}^1$ is a scheme these morphisms glue to an actual map $U \to \mathbb{P}^1$.

conjugate maps yield the same elliptic curve in X we conclude

$$n_{X,j_{\mathrm{gen}}} = \frac{1}{2}|M_1| = \frac{1}{2}\int_{[M_1]^{\mathrm{vir}}} 1.$$

For the contribution from the second component consider the isomorphism

$$M_2 \cong M_{0,1}(X,\beta)$$

obtained by sending the stable map $f : \mathbb{P}^1 \cup_x E \to X$ to the pair $(f|_{\mathbb{P}^1} : \mathbb{P}^1 \to X, x)$. Since all smoothings are obstructed for maps of second type, the deformation spaces are the same on both sides of the isomorphism. The obstruction sheaf however differs by the cokernel of the inclusion

$$T_x\mathbb{P}^1 \otimes T_xE \to H^1(C, f^*T_X) \to H^1(E, f|_E^*T_X) = H^1(E, \mathcal{O}_E) \otimes \mathrm{ev}_x^* T_X.$$

We conclude

$$\begin{aligned} \int_{[M_2]^{\mathrm{vir}}} 1 &= \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \left[\frac{\mathrm{ev}_1^* c(T_X)}{1-\psi_1}\right]_{2d-1} \\ &= \int_{[\overline{M}_{0,1}(X),\beta)]^{\mathrm{vir}}} \frac{\mathrm{ev}_1^*(c(T_X))}{1-\psi_1} \\ &= C_X. \end{aligned}$$

Putting everything together we find $N_X = 2n_{X,j_{\mathrm{gen}}} + C_X$. □

2.2. **Calculations: K3 surfaces.** Let X be a very general K3 surface with primitive curve class $\beta \in H_2(X,\mathbb{Z})$ of self-intersection

$$\langle \beta, \beta \rangle = 2h - 2, \quad h \in \mathbb{Z}_{\geq 0}.$$

By the deformation-obstruction sequence the moduli space $M_{1,0}(E,\beta)$ is a smooth curve.[6] Hence the assumption of Proposition 1.2 are satisfied. By deformation invariance N_X, C_X only depend on h, hence by Proposition 1.2 so does n_X. We write

$$N_X = N_{K3,h}, \quad C_X = C_{K3,h}, \quad n_{X,j_{\mathrm{gen}}} = n_{K3,h}.$$

By the Yau-Zaslow formula (proven in [2, 4]) and a basic Gromov–Witten calculation we have

$$N_{K3}(q) = \sum_{h\geq 0} N_{K3,h} q^{h-1} = 2q\frac{d}{dq}\frac{1}{\Delta(q)}$$

$$C_{K3}(q) = \sum_{h\geq 0} C_{K3,h} q^{h-1} = -2\frac{1}{\Delta(q)}$$

[6] We have $0 \to T_E \to f^*T_X \to \mathcal{L} \to 0$ for a degree 0 line bundle $\mathcal{L}$ on the elliptic curve E. Hence the tangent space to the moduli space at the point $[f : E \to X]$, which is $H^0(E,\mathcal{L})$, is of dimension ≤ 1 (in fact, exactly 1-dimensional since there is a non-trivial cosection of the obstruction sheaf, so the obstruction sheaf is non-trivial, see [9].)

where $\Delta(q) = q\prod_{m\geq 1}(1-q^m)^{24}$ is the discriminant modular form. Hence

$$\sum_{h=0}^{\infty} n_{K3,h} q^{h-1} = \frac{1}{\Delta(q)} + q\frac{d}{dq}\frac{1}{\Delta(q)}.$$

2.3. **Calculations: K3[2] type.** Let X be a hyper-Kähler variety of $K3[2]$ type. By deformation invariance the Gromov–Witten invariants in primitive class β only depends on the Bauville-Bogomolov-square of the class β,

$$(\beta,\beta) = \frac{s}{2}, \quad s\in\mathbb{Z}, s\equiv 0,3 (\mathrm{mod}\ 4).$$

As before we write $N_X = N_{K3[2],s}$, etc., and form the generating series

$$N_{K3[2]}(q) = \sum_s N_{K3[2],s} q^s, \quad C_{K3[2]}(q) = \sum_s C_{K3[2],s} q^s.$$

Assuming the conditions of Proposition 1.2 let us write also $n_{K3[2],s} = n_X$ and form the series

$$n_{K3[2]}(q) = \sum_s n_{K3[2],s} q^s.$$

The series $N_{K3[2]}(q)$ is determined by [10, Prop.1] and the correspondence [6, Thm.5.4] between Jacobi forms of index 1 and modular forms. The series $C_{K3[2]}(q)$ can be computed by removing the ψ-classes via topological recursions and applying the formalism of [10, Thm.10], see also [12, App. A]. The result of this calculation is

$$\begin{aligned} & n_{K3[2]}(-q) \\ & = \frac{1}{F(q)\Theta(q)\Delta(q^4)}\left[-\frac{9}{4}(\Theta^4+4F)(G_2-1) - \frac{3}{8}G_4 + \frac{9}{8}G_2^2 - \frac{15}{4}G_2 + 3\right], \end{aligned}$$

where we have suppressed the argument of q in the bracket on the right, and

$$\Theta(q) = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad F(q) = \sum_{\substack{\text{odd } n>0 \\ d|n}} dq^n, \quad G_k(q) = 1 - \frac{2k}{B_k}\sum_{\substack{n\geq 1\\ d|n}} d^{k-1}q^{4n}$$

are (quasi-)modular forms for $\Gamma_0(4)$ with B_k the Bernoulli numbers.

3. The Fano of lines

3.1. **Overview.** Let F be the Fano variety of lines on a smooth cubic fourfold Y. Consider the incidence correspondence

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{q_Y} & Y \\ {\scriptstyle q_F}\downarrow & & \\ F & & \end{array}$$

Every curve $C\subset F$ corresponds via the correspondence to a surface in Y,

$$S_C := q_Y(q_F^{-1}(C)).$$

If the curve C is of class β, the surface S_C is of degree $\langle \beta, H_F \rangle = 3$ and thus has cohomology class $H_Y^2 \in H^4(Y, \mathbb{Z})$, where H_Y is the hyperplane class on Y, see [3]. In particular, if C is of class β and reducible, then Y contains a $\mathbb{P}^2$ and hence Y lies in a divisor in the moduli space of cubic fourfolds [8]. Since we are interested here in the general case, we may assume that Y does not contain a plane and hence that β is irreducible.

Following a discussion of Amerik [1] we give a second description on the elliptic curves in class β. For every point $y \in Y$ let $Z_y \subset Y$ be the cone spanned by the lines in Y incident to y. The cone Z_y is the intersection of the cubic fourfold $Y = V(f)$, the tangent plane $T_yY := V(\sum X_i \partial_i f(y))$ and the polar quadric $R_yY := V(\sum y_i \partial_i f)$,

$$Z_y = T_yY \cap R_yY \cap Y. \tag{2}$$

The base B_y of the cone is obtained from Z_y by intersecting with a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$ which does not contain y. Since the base is in one-to-one correspondence with the lines through y we have $B_y \subset F$.

If the hypersurfaces in (2) intersect properly then the base is a $(2,3)$-complete intersection curve in

$$\mathbb{P}^3 = T_yY \cap \mathbb{P}^4.$$

By a result of Amerik [1] this curve is of class 2β in F. However, assume that the quadric $T_yY \cap R_yY$ degenerates to a union of two distinct planes. Since Y does not contain a plane, the base splits as a union

$$B_y = E_1 \cup E_2 \tag{3}$$

of two distinct planar cubics E_i meeting each other in three points. Each E_i is an arithmetic genus 1 curve of class β in F.

We consider other possible degenerate intersections of (2). Since Y does not contain a plane it also does not contain a quadric surface, so it always intersects the quadric $R_yY \cap T_yY$ properly. Hence the only other degenerate intersection is when $R_y \cap T_yY$ degenerates further, either to a double plane or to a 3-plane. In Lemma 3.2 we prove that both cases are ruled out for the general Fano. By the discussion in Section 1.4 we know that every curve in F of arithmetic genus 1 and of class β comes from a cone. We conclude that they must be one of the E_i in (3).[7]

We now proceed as follows. Define the locus where the cone is degenerate:

$$\mathcal{M}_Y = \{y \in Y \mid T_yY \cap R_yY \text{ is not integral or of dimension} \geq 3\}. \tag{4}$$

[7] For an alternative proof one can also use the curve-surface correspondence and classification of cubic surfaces as in [12, App.B].

By the above we have a morphism

$$\Sigma \to \mathcal{M}_Y$$

sending an elliptic curve in F to the vertex of the corresponding cone in Y. In Section 3.2 we show that $\mathcal{M}_Y$ is non-empty and connected, and for the general Fano smooth of dimension 1. We also rule out the more degenerate cases when $T_yY \cap R_yY$ is a double-plane or 3-dimensional. This will prove the first two parts of Theorem 1.3. In Section 3.3 we then express Σ directly as the section of a vector bundle in a homogeneous space.

3.2. **The locus $\mathcal{M}_Y$.** We prove connectivity and non-emptiness of $\mathcal{M}_Y$.

Lemma 3.1. *Let Y be a smooth cubic fourfold. Then $\mathcal{M}_Y$ is connected and non-empty.*

Proof. Assume $y = [1 : 0 : \ldots : 0]$ and $T_yY = V(X_1)$. If $Y = V(f)$ for a cubic homogeneous polynomial f, then

$$f = X_0^2 X_1 + X_0 f' + f''$$

where $f', f'' \in \mathbb{C}[X_1, ..., X_5]$ are of degree 2 and 3 respectively. Hence

$$T_yY \cap R_yY = V(X_1) \cap V(f' + 2X_1X_0) = V(f'|_{X_1=0}).$$

Since this locus is cut out by a quadratic polynomial, we have $y \in \mathcal{M}_Y$ if and only if the symmetric matrix $(f'_{i,j})_{i,j\geq 2}$ has rank ≤ 2. Here $f'_{i,j}$ is the coefficient of the monomial X_iX_j in f'.

The Jacobian $(\partial_i \partial_j f(y))_{i,j}$ of f evaluated at y takes the form

$$\begin{vmatrix} 0 & 2 & 0 & \ldots & 0 \\ 2 & f'_{1,1} & f'_{1,2} & \cdots & f'_{1,5} \\ 0 & f'_{2,1} & \ddots & & \iddots \\ \vdots & \vdots & & (f')_{i,j\geq 2} & \\ 0 & f'_{5,1} & \iddots & & \ddots \end{vmatrix}$$

The first two rows of the matrix are linearly independent, so it follows that

$$\operatorname{rk}(\partial_i \partial_j f(y))_{i,j} \;=\; 2 + \operatorname{rk}(f'_{i,j})_{i,j\geq 2}. \tag{5}$$

The Jacobian of f can be upgraded to a global symmetric form of a trivial vector bundle $\mathcal{O}_Y^6$ with values in $\mathcal{O}(1)$,

$$J(f) : \mathcal{O}_Y^6 \otimes \mathcal{O}_Y^6 \to \mathcal{O}(1),$$

described as follows. For a point $y \in Y$ consider vectors $w_1, w_2 \in \mathcal{O}_y^6$ and $\tilde{y} \in \mathcal{O}(-1)_y$ in the fibers above the point y, then we define

$$J(f)_y(w_1, w_2)(\tilde{y}) := (D_{w_1} D_{w_2} f)(\tilde{y}).$$

We have obtained the following global description of $\mathcal{M}_Y$,

$$\mathcal{M}_Y = \{y \in Y | \mathrm{rk}(J(f)_y) \leq 4\}.$$

The expected dimension of this degeneracy locus is 1. By the main result of [14] ampleness of $\mathrm{Sym}^2(\mathcal{O}^6) \otimes \mathcal{O}(1)$ implies connectedness of the associated degeneracy locus $\mathcal{M}_Y$. In turn, by the main result of [7], ampleness of $\mathrm{Sym}^2(\mathcal{O}^6) \otimes \mathcal{O}(1)$ implies also that $\mathcal{M}_Y$ is non-empty. □

Lemma 3.2. *Let Y be a general cubic fourfold. Then $\mathcal{M}_Y$ is a smooth curve. Moreover, there exists no $y \in Y$ such that $T_yY \subset R_yY$ is a double plane or 3-dimensional.*

Proof. Define the locally-closed subscheme

$$\Omega = \{(y, Y) \,|\, Y \text{ smooth cubic fourfold}, y \in \mathcal{M}_Y\} \subset \mathbb{P}^5 \times \mathbb{P}^{55}.$$

The projections from Ω to both factors yield a correspondence

$$\begin{array}{ccc} \Omega & \xrightarrow{q} & \mathbb{P}^{55} \\ {\scriptstyle p}\big\downarrow & & \\ \mathbb{P}^5 & & \end{array}$$

such that $p^{-1}(y) = \{Y | y \in \mathcal{M}_Y\}$ and $q^{-1}(Y) = \mathcal{M}_Y$. A change of coordinates identifies different fibers of p. Hence Ω is a fiber space over $\mathbb{P}^5$.

Assume $y = [1 : 0 : \ldots : 0]$ and $T_yY = V(X_1)$. Hence as before Y is defined by a polynomial

$$f = X_0^2 X_1 + X_0 f' + f''$$

where f', f'' do not depend on X_0. By the previous section, $y \in \mathcal{M}_Y$ if and only if $\mathrm{rk}((f'_{i,j})_{i,j\geq 2}) \leq 2$. This condition defines a 12-dimensional locus $\mathbb{D}_2 \subset \mathbb{C}[X_1, ..., X_5]_2$, which is smooth at quadrics with $\mathrm{rk}((f'_{i,j})_{i,j\geq 2}) = 2$ and singular along the codimension 3 locus of quadrics with $\mathrm{rk}((f'_{i,j})_{i,j\geq 2}) \leq 1$. The codimension 3 locus corresponds precisely to the cubics Y such that $T_yY \cap R_yY$ is a double plane or 3-dimensional. As there are no conditions on f'' or on f''' it follows that $\{Y | y \in \mathcal{M}_Y\}$ is the (non-empty) open subset of the projective space

$$\mathbb{P}\left(k[X_1, ..., X_5]_3 \times \mathbb{D}_2 \times k[X_1, ..., X_5]_1\right)$$

corresponding to smooth cubic fourfolds. Adding up dimensions we obtain $\dim p^{-1}(y) = 51$. Therefore Ω is 56-dimensional. Moreover, we also conclude that Ω is smooth outside the codimension 3 locus of lower rank quadrics.

By Lemma 3.1 the locus $\mathcal{M}_Y$ is non-empty for all smooth cubic 4-folds. In particular $p : \Omega \to \mathbb{P}^{55}$ is dominant. Therefore a general fiber $q^{-1}(Y) = \mathcal{M}_Y$ of q is smooth and 1-dimensional. The singular locus does not dominate. □

We record the following which proves parts a) and b) of Theorem 1.3.

Corollary 3.3. *Let F be a general Fano. Then the vertex-assignment $\Sigma \to \mathcal{M}_Y$ is a degree 2 unramified cover of the smooth and connected curve $\mathcal{M}_Y$. In particular, Σ is non-singular of dimension 1.*

Proof. By Lemma 3.2 every fiber of $\nu : \Sigma \to \mathcal{M}_Y$ consists of precisely two points. It follows that ν is finite and since $\mathcal{M}_Y$ is smooth also flat. Therefore every fiber consists of two reduced points, so the map is unramified. □

Remark 3.4. Consider the Gauss map of a cubic fourfold Y,

$$\gamma : Y \to \mathrm{Gr}(5,6),\ y \mapsto T_y Y$$

The derivative of the map can be expressed as a symmetric form on the tangent bundle TY with values in $\mathcal{O}(1)$,

$$\mathrm{II}_Y : TY \otimes TY \to \mathcal{O}(1).$$

Let f be a defining equation of Y in a standard affine chart on $\mathbb{P}^5$. Then locally II_Y is the Jacobian of f,

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}.$$

Hence in the notation of the proof of Lemma 3.2 we have

$$\operatorname{rank} \mathrm{II}_{Y,y} = \operatorname{rank}\,(f'_{i,j})_{i,j\geq 2}.$$

A point $y \in Y$ is called *r-Eckardt* if $\operatorname{rank} \mathrm{II}_{Y,y} \leq r$. If $r = 0$, then y is simply called *Eckardt*. The name originates from Eckardt points on cubic surfaces, which are by definition the points of intersections of its lines.

We see that Eckardt points correspond to points where $T_y Y \subset R_y Y$. These give a 3-dimensional family of elliptic curves in class β. The 1-Eckardt points give ramification points of the map $\Sigma \to \mathcal{M}_Y$ (corresponding to double planes). Finally the 2-Eckardt points correspond to the pairs of elliptic curves $E \cup E'$.

A cubic fourfold Y containing an Eckardt point also contains a plane, hence, in Hassett's notation [8], $Y \in \mathcal{C}_8$. We expect Y containing 1-Eckardt points or a locus of 2-Eckardt points of bigger dimension to be special too.

3.3. **Parametrizing cubic cones.** Let Y be a general cubic fourfold. We know that every curve $C \subset F$ of arithmetic genus 1 corresponds to a surface S_C which is an intersection $Z = \mathbb{P}^3 \cap Y$ such that Z is a cone. We describe a 20-dimensional homogeneous space that parametrizes (birationally) all pairs (P, Z) of a 3-plane $\mathbb{P}^3 \cong P \subset \mathbb{P}^5$ together with a cone cubic surface $Z \subset P$. We express the curve Σ (of cones contained in Y) as the zero locus of a globally generated rank 19 vector bundle on this homogeneous space. This gives a second proof that Σ is non-singular and via adjunction also yields a formula for its genus. Finally we obtain the degree over $\mathbb{P}^1$ by intersecting Σ with the divisor of cones over singular plane cubics.

Let $G = \mathrm{Gr}(4,6)$ be the Grassmannian of 3-planes $\mathbb{P}^3 \subset \mathbb{P}^5$. Let

$$0 \to \mathcal{K} \to \mathcal{O}_G \otimes \mathbb{C}^6 \to Q \to 0$$

be the universal sequence with $\mathcal{K}$ the universal rank 4 subbundle. A point on the projective bundle

$$\mathbb{P}(\mathcal{K}) = \operatorname{Proj} \mathrm{Sym}^{\bullet} \mathcal{K}^{\vee}, \quad p : \mathbb{P}(\mathcal{K}) \to G$$

corresponds to a 3-plane $\mathbb{P}^3 \subset \mathbb{P}^5$ together with a point $v \in \mathbb{P}^3$.

Given a vector space V and a line $\ell \subset V$, the cones of degree d with vertex $[\ell] \in \mathbb{P}(V)$ are canonically parametrized by the following subspace of degree d polynomials:

$$\mathrm{Sym}^d (V/\ell)^* \subset \mathrm{Sym}^d V^*.$$

Hence consider the universal sequence on $\mathbb{P}(\mathcal{K})$,

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{K})}(-1) \to p^* \mathcal{K} \to \tilde{Q} \to 0 \tag{6}$$

and let

$$\mathcal{E} = \mathrm{Sym}^3(\tilde{Q}^*).$$

The projective bundle

$$q : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{K})$$

parametrizes triples (Z, v, P) where $P \subset \mathbb{P}^5$ is the 3-plane and $Z \subset P$ is a cone with vertex v. The map from $\mathbb{P}(\mathcal{E})$ to the space of cubic cones in $\mathbb{P}^5$ is an isomorphism away from the locus of cubic cones with more than one vertex (the 3-plane is always uniquely determined). Since these are given by cones over the union of three lines (or more degenerate configurations) the curve Σ never intersects this locus. Hence $\Sigma \subset \mathbb{P}(\mathcal{E})$.

We write now Σ as the zero locus of a section on $\mathbb{P}(\mathcal{E})$. Consider the universal subbundle

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \hookrightarrow q^* \mathcal{E} \hookrightarrow (p \circ q)^* \mathrm{Sym}^3 \mathcal{K}^* \tag{7}$$

that over a point describes the (1-dimensional span of the) equation cutting out the cone in the $\mathbb{P}^3$. Here the second inclusion is the natural one obtained from (6). Let $\mathcal{F}$ be the cokernel of the composition (7).

The section $f \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))$ defining the cubic Y defines via the correspondence of the Grassmannian a section

$$s_f \in H^0(G, \mathrm{Sym}^3 \mathcal{K}^*)$$

with fiber $f|_{\mathbb{P}^3} \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ over the moduli point $[\mathbb{P}^3 \subset \mathbb{P}^5] \in G$. We pullback s_f to $\mathbb{P}(\mathcal{E})$. Then the composition

$$\hat{s_f} : \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to (p \circ q)^* \mathrm{Sym}^3 \mathcal{K}^* \to \mathcal{F}$$

vanishes at a point $(\mathbb{P}^3, v, Z)$ if and only if $V(f) \cap \mathbb{P}^3$ is the cone Z. Hence

$$\Sigma = V(\hat{s_f}).$$

Lemma 3.5. $g(\Sigma) = 631$.

Proof. By the adjunction formula we have

$$\omega_\Sigma = \Big(\omega_{\mathbb{P}(\mathcal{E})} \otimes \det(\mathcal{F})\Big)|_\Sigma$$

and hence taking degree

$$2g(\Sigma) - 2 = \int_{\mathbb{P}(\mathcal{E})} c_{19}(\mathcal{F}) \cup (K_{\mathbb{P}(\mathcal{E})} + c_1(\mathcal{F})).$$

This calculation can be performed using the SAGE package 'Chow' [5] using the following code:

```
G=Grass(4,6)
K=G.sheaves["universal_sub"]
PK=ProjBundle(K.dual(), 'y', name='PK')
E=PK.sheaves['universal_sub'].symm(3)
PE = ProjBundle(E.dual(), 'z', name='PE')
C1 = K.dual().symm(3).chern_character()
C2 = PE.sheaves['universal_quotient'].dual().chern_character()
F=Sheaf(PE, ch=C1-C2)
(F.chern_classes()[19] * (PE.canonical_class() + F.chern_classes()[1])).integral()
```

□

We finally compute the degree over the coarse space of $\overline{M}_{1,1}$ by counting how many of the points in Σ correspond to cones over singular plane cubics. For this we need the following lemma:

Let X be a smooth variety, let V be a 3-dimensional vector bundle and let $W = \mathrm{Sym}^3(V^*)$. The points of $\mathbb{P}(W)$ lying over $x \in X$ canonically parametrize the cubic curves in $\mathbb{P}(V_x) \cong \mathbb{P}^2$.

Lemma 3.6. *The divisor $D \subset \mathbb{P}(W)$ parametrizing singular cubic curves has class $12z - 12c_1(\mathcal{V})$ where $z = c_1(\mathcal{O}_{\mathbb{P}(W)}(1))$.*

Proof. We sketch the proof for a lack of reference. Let $P = \mathbb{P}(W) \times_X \mathbb{P}(V)$ and let p_1, p_2 be the projection to the first and second factor and $q : P \to X$ the projection to X. Taking the derivative gives the $\mathcal{O}_X$-derivation

$$d : \mathrm{Sym}^3 V^* \to (\mathrm{Sym}^2 V^*) \otimes V^*.$$

Consider the morphism obtained by pulling back to P and precomposing with the universal subbundle and post-composing with the canonical quotient bundle,

$$\psi : p_1^*\mathcal{O}_{\mathbb{P}(W)}(-1) \to q^*\mathrm{Sym}^3 V^* \xrightarrow{q^*d} q^*\mathrm{Sym}^2(V^*) \otimes V^* \to p_2^*\mathcal{O}_{\mathbb{P}(V)}(2) \otimes q^*V^*$$

By the Jacobi criterion ψ, vanishes precisely on the locus of pairs (C, y) where y is a singular point of the cubic C. The class of the vanishing locus of ψ is the Euler class of the bundle

$$\mathcal{O}_P(1,2) \otimes V^*.$$

Pushing forward the Euler class by p_1 yields the claim. □

Lemma 3.7. *Let $D \subset \mathbb{P}(\mathcal{E})$ be the divisor parametrizing singular cubic curves in $\mathbb{P}(\tilde{Q})$. Then*

$$D \cdot \Sigma = 3780.$$

Proof. This follows from Lemma 3.6 and the following submission to 'Chow':

```
c1,c2 = G.gens(); y=PK.gen(); z=PE.gen()
D = 12*z - 12*y + 12*c1
(F.chern_classes()[19]*D).integral()
```

□

Lemma 3.8. *Let F be general. Then there are only finitely many singular rational curves $C \subset F$ and all of them are nodal.*

Proof. By a dimension count as in Lemma 3.2 the locus

$$\Omega = \{(Y, P) \in \mathbb{P}^{55} \times \mathrm{Gr}(4,6) | Y \text{ smooth}, P \cap Y \text{ cone over a singular cubic}\}$$

is of dimension 55. By Lemma 3.8 the map $\Omega \to \mathbb{P}^{55}$ is dominant so the fibers are generically finite. The argument for cuspidal curves is parallel. □

Lemma 3.9. *The degree of $\Sigma \to \mathbb{P}^1$ sending a curve C to its j-invariant is of degree* 3780*.*

Proof. Let $D \subset \mathbb{P}(\mathcal{E})$ be the divisor parametrizing singular cubics. Since Σ does not parametrize any nodal cubics and is non-singular, the restriction of D to Σ is the pullback of the class of a point from the natural map $\Sigma \to \mathbb{P}^1$, where $\mathbb{P}^1$ is the coarse space of $\overline{M}_{1,1}$. Hence it is enough to compute the intersection pairing of D with Σ, which we have done in Lemma 3.8. □

3.4. **Moduli of stable maps.** Let F be a general Fano of lines and let

$$f : E \to F$$

be a map from a non-singular smooth elliptic curve in class β. The following implies Proposition 1.4.

Proposition 3.10. *Let $N_{E/F} = f^*T_F/T_E$ be the normal bundle to f. Then*

$$h^0(E, N_{E/F}) = 1.$$

As explained in the introduction every map $f : E \to F$ of class β from a curve of arithmetic genus 1 is a closed immersion. Moreover it is given by a family of lines $\ell \subset Y$ through a fixed vertex $v \in Y$. The idea of the proof is to compare arbitrary deformations of ℓ with those which remain incident to v. This is facilitated by the following lemma.

Lemma 3.11. *Let X be a smooth variety, let $g : \tilde{X} \to X$ be the blow-up at a point $v \in X$ and let $\iota : D = \mathbb{P}(T_{X,x}) \hookrightarrow \tilde{X}$ be the exceptional divisor. We have an exact sequence*

$$0 \to T_{\tilde{X}} \to g^*T_X \to \iota_* T_D(-1) \to 0.$$

Proof. Since X is smooth and g is birational we have an exact sequence

$$0 \to g^*\Omega_X \to \Omega_{\tilde{X}} \to \Omega_{X/\tilde{X}} \to 0.$$

Dualizing we get

$$0 \to T_{\tilde{X}} \to g^*T_X \to \mathscr{E}xt^1_{\mathcal{O}_X}(\Omega_{X/\tilde{X}}, \mathcal{O}_{\tilde{X}}) \to 0.$$

By a direct check $\Omega_{X/\tilde{X}} = \iota_*\Omega_D$ and by Grothendieck-Verdier duality

$$\mathscr{E}xt^1_{\mathcal{O}_X}(\iota_*\Omega_D, \mathcal{O}_{\tilde{X}}) = \iota_*\mathscr{H}om_{\mathcal{O}_D}(\Omega_D, \omega_D \otimes \omega_{\tilde{X}}^{-1}|_D).$$

Using $\omega_{\tilde{X}} = g^*\omega_X((n-1)D)$, $n = \dim X$ yields the claim. ∎

Let $I_E = \{(\ell, y) \in E \times Y | y \in \ell\}$ be the incidence correspondence corresponding to E. We write

$$\pi : I_E \to E$$

for the projection to the Fano side. Via the projection to the second factor we may view I_E as the blow-up at the vertex of the cone Z of lines parametrized by E. Hence naturally $I_E \subset \tilde{Y}$, where $g : \tilde{Y} \to Y$ is the blow-up of Y at v. We consider the sequence

$$0 \to T_{\tilde{Y}} \to g^*T_Y \to T_D(-1) \to 0.$$

Restricting to I_E and quotienting out by $T_\pi \subset T_{\tilde{Y}}|_{I_E}$ yields

$$0 \to \tilde{\mathcal{M}} \to \mathcal{M} \to T_D(-1) \to 0 \tag{8}$$

where

$$\tilde{\mathcal{M}} = \left(T_{\tilde{Y}}|_{I_E}\right)/T_\pi, \quad \mathcal{M} = \left(g^*T_Y|_{I_E}\right)/T_\pi.$$

We want to compute the pushforward by π_* of the sequence (8). By construction $\pi_*\mathcal{M} = T_{F|E}$. Since

$$(R^1\pi_*\mathcal{M}) \otimes k(\ell) = H^1(\ell, N_{\ell/Y}) = 0$$

for every $\ell \in E$ we have $R^1\pi_*\mathcal{M} = 0$. On the other hand,

$$\pi_*\tilde{\mathcal{M}} = T_{F(\tilde{Y})}|_E$$

where $F(\tilde{Y})$ is the Fano variety of lines in $\tilde{Y}$. The Fano $F(\tilde{Y})$ is cut out from the Fano variety $F(\tilde{\mathbb{P}^5}) = \mathbb{P}(T_{\mathbb{P}^5,v}) = \mathbb{P}^4$ by the tangent space to Y, the polar quadric and Y, i.e.

$$F(\tilde{Y}) = \mathbb{P}^4 \cap T_v(Y) \cap R_v(Y) \cap Y = E \cup E'$$

where E' is the partner of E. Hence we find $\pi_*\tilde{\mathcal{M}} = T_E$. On the other hand

$$(R^1\pi_*\tilde{\mathcal{M}}) \otimes k(\ell) = H^1(\ell, N_{\ell/Y}(-1)) = \begin{cases} \mathbb{C} & \text{if } \ell \text{ is of second type} \\ 0 & \text{otherwise .} \end{cases}$$

Since the three points $E \cap E'$ are precisely the lines of second type we get

$$R^1\pi_*\tilde{\mathcal{M}} = \bigoplus_{\ell \in E \cap E'} \mathbb{C}_\ell$$

where $\mathbb{C}_\ell$ is the skyscraper sheaf at ℓ. (This may be seen also directly by cohomology and base change: The fiber $H^0(\ell, N_{\ell/F}(-1)) = T_{E\cup E',\ell}$ is 2-dimensional precisely at the intersection points $E \cap E'$). Therefore pushing forward (8) by π yields the exact sequence

$$0 \to T_E \xrightarrow{\varphi} T_F|E \to T_D(-1)|_E \to \oplus_{\ell \in E\cap E'}\mathbb{C}_\ell \to 0.$$

The first map φ is precisely the differential of $f : E \to F$, so its cokernel is the normal bundle. We hence obtain

$$0 \to N_{E/F} \to T_D(-1)|_E \to \bigoplus_{\ell \in E\cap E'} \mathbb{C}_\ell \to 0.$$

To understand the global sections of this sequence we need the following lemma.

Lemma 3.12. *Let $E \subset \mathbb{P}^3$ be an elliptic curve contained in a $\mathbb{P}^2$. Then*

$$H^0(E, T_{\mathbb{P}^3}(-1)|_E) = \mathbb{C}^4, \quad H^1(E, T_{\mathbb{P}^3}(-1)|_E) = \mathbb{C}.$$

Proof. Twisting the Euler sequence and restricting to E we get

$$0 \to \mathcal{O}_E(-1) \to \mathcal{O}_E^4 \to T_{\mathbb{P}^3}(-1)|_E \to 0.$$

Taking cohomology the induced map $\alpha : H^1(E, \mathcal{O}_E(-1)) \to H^1(E, \mathcal{O}_E^4)$ is Serre dual to

$$H^0(E, \mathcal{O}_E^4) \to H^0(E, \mathcal{O}_E(1)).$$

This sequence is obtained from taking the global section of the restriction of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)$ to E. Since its kernel is precisely the space of hyerplane that contain E which is of dimension 1, the map is surjective. We conclude that α is injective which gives the claim. □

It remains to show that

$$H^0(E, T_D(-1)|_E) = \mathbb{C}^4 \to \bigoplus_{\ell \in E \cap E'} H^0(E, \mathbb{C}_\ell) = \mathbb{C}^3 \tag{9}$$

is surjective. For that we need to analyse the map. Let $\ell \in E \cap E'$. The composition of (9) with the projection to the ℓ-th summand factors as

$$H^0(E, T_D(-1)|_E) \to T_D(-1) \otimes k(\ell) \xrightarrow{\rho} k_\ell = H^1(\ell, N_{\ell/Y}(-1)).$$

The map ρ is obtained from the long exact cohomology sequence of

$$0 \to N_{\ell/Y}(-v) \to N_{\ell/Y} \to N_{\ell/Y,v} \to 0$$

where we have written $N_{\ell/Y,v} = N_{\ell/Y} \otimes k(v)$ for the fiber at $v \in \ell$. Consider the decomposition

$$N_{\ell/Y} = \mathcal{O}_\ell(1)^2 \oplus \mathcal{O}_\ell(-1).$$

We can hence identify ρ with the projection

$$N_{\ell/Y,v} \to N_{\ell/Y,v}/(\mathcal{O}_\ell(1)_v \oplus \mathcal{O}_\ell(1)_v) = \mathcal{O}_\ell(-1)_v.$$

The normal directions spanned by the $\mathcal{O}_\ell(1)_v$ summands is the space spanned by the tangent spaces $T_{E,\ell}$ and $T_{E',\ell}$. We hence need to show that the following map is surjective:

$$H^0(E, T_D(-1)|_E) \to \bigoplus_{\ell \in E \cap E'} T_D(-1)_\ell / \left(T_{E,\ell} \oplus T_{E',\ell}\right). \tag{10}$$

To do so we pick coordinates. We can take the equation of the cubic fourfold to be

$$f = x_0^2 x_1 + x_0 x_1 f_2' + x_0 x_2 x_3 + f_3$$

where $f_2' \in k[x_1, \ldots, x_5]_1$ and $f_3 \in k[x_1, \ldots, x_5]_3$. Here $v = [1, 0, \ldots, 0]$. We set $x_0 = 1$ and consider the projectivization of the tangent space at v,

$$D = \mathbb{P}(T_{Y,v}) = \mathbb{P}^3_{x_2, x_3, x_4, x_5}.$$

Inside D we have the complete intersection

$$E \cap E' = \mathbb{P}(V(x_2, x_3, f_3)) = \{\ell_1, \ell_2, \ell_3\}.$$

By change of coordinates we may assume that the ℓ_i are distinct from $[0, 0, 1, 0]$ and hence write $\ell_i = (0, 0, a_i, 1)$ for some $a_i \in \mathbb{C}$. Let $e_2, \ldots, e_5$ be the basis of $H^0(T_D(-1)|_E)$ corresponding to the basis vectors. The sequence $\mathcal{O}_E(-1) \to T_D(-1)|_E$ is given by $1 \mapsto x_2 e_2 + \ldots + x_5 e_5$. Hence at ℓ_i its image is $a_i e_4 + e_5$. Write

$$T_{E,\ell_i} = \mathrm{Span}(e_2 + \alpha_i e_4), \quad T_{E',\ell_i} = \mathrm{Span}(e_3 + \beta_i e_4).$$

Then i-th factor of the map (10) is given by

$$e_2 \mapsto -\alpha_i e_4, \quad e_3 \mapsto -\beta_i e_4, \quad e_4 \mapsto e_4, \quad e_5 \mapsto -a_i e_4.$$

Hence (10) is represented by the matrix

$$\begin{pmatrix} -\alpha_1 & -\beta_1 & 1 & a_1 \\ -\alpha_2 & -\beta_2 & 1 & a_2 \\ -\alpha_3 & -\beta_3 & 1 & a_3 \end{pmatrix} \tag{11}$$

We need to check it is surjective. Set $x_5 = 1$ and let $g = f_3|_{x_1=0,x_5=1}$. Then

$$\alpha_i = -\frac{g_{x_4}(0,0,a_i)}{g_{x_2}(0,0,a_i)} = -\frac{\prod_{j\neq i}(a_j - a_i)}{g_{x_2}(0,0,a_i)},$$
$$\alpha_i = -\frac{g_{x_4}(0,0,a_i)}{g_{x_3}(0,0,a_i)} = -\frac{\prod_{j\neq i}(a_j - a_i)}{g_{x_3}(0,0,a_i)}.$$

Since $g_{x_2}(0,0,a_i), g_{x_3}(0,0,a_i), g_{x_4}(0,0,a_i)$ involve the monomials $x_2x_4^{\bullet}$, $x_3x_4^{\bullet}$ and $x_4^{\bullet}$ respectively, they can be choosen independently from each other. On the other hand, the locus of 4×3-matrices of rank ≤ 2 is of codimension 2 in the corresponding space of matrices. We conclude that the condition that (11) is not surjective is a codimension 2 condition on the function f_3.

The locus of cubic fourfolds with a point $v \in Y$ with $T_yY \cap R_yY$ non-integral and (11) not surjective is therefore of dimension

$$\underbrace{40}_{\text{Choice of } f_2, f_3} + \underbrace{17}_{\text{Choice of } v, T_vY, T_v\cap R_vY} + \underbrace{-2}_{\text{non-surjectivity}} + \underbrace{-1}_{\text{overall scaling}} = 54.$$

Since non-singular cubic fourfolds form an open subset in $\mathbb{P}^{55}$, this locus can not dominate this open subset. The proof is complete. □

4. Generalized Kummer fourfolds

In this section we present an example of a very general polarized hyper-Kähler fourfold X with primitive curve class β such that $M_{1,0}(X,\beta)$ is empty.

We begin with some generalities. First, if $[f : C \to X] \in M_{1,0}(X,\beta)$ is a stable map, then the image curve $f(C)$ is of arithmetic genus ≥ 1. Hence to show $M_{1,0}(X,\beta)$ is empty it is enough to show that the Hilbert scheme $\mathrm{Hilb}^0(X,\beta)$ of 1-dimensional subschemes $Z \subset X$ satisfying the numerical conditions

$$[Z] = \beta \in H_2(X,\mathbb{Z}), \quad \chi(\mathcal{O}_Z) = 0$$

is empty. Second, since $\mathrm{Hilb}^0(X,\beta)$ is projective, if for a deformation $\mathcal{X} \to C$ the Hilbert scheme of the very general fiber is non-empty, then the Hilbert scheme of the special fiber is non-empty as well. Hence it is enough to show that for a special pair (X,β) the Hilbert scheme $\mathrm{Hilb}^0(X,\beta)$ is empty.[8] This is done in the following example.[9]

[8] The Hilbert scheme also provides a possible pathway to proving Conjecture 1.1. Indeed, since the Hilbert scheme is of (reduced) expected dimension 1, to prove Conjecture 1.1 it is enough to show it is precisely of dimension 1 for a special pair (X,β).

[9] This example was pointed out to us by H. Y. Lin.

Let A be a simple principally polarized abelian surface of Picard rank 1 and let $\mathrm{Km}_2(A)$ the associated generalized Kummer fourfold defined as the fiber over the origin of the summation map

$$\mathrm{Hilb}^3(A) \to \mathrm{Sym}^3(A) \xrightarrow{+} A.$$

By the universal property of the Hilbert scheme every map $g : C \to \mathrm{Km}_2(A)$ from a smooth connected curve C corresponds to a curve $\tilde{C} \subset C \times A$ flat of degree 3 over C. Since A is of Picard rank 1 the projection of the class of $\tilde{C}$ is a multiple of the theta divisor. We let $d = d(C)$ be this multiple and call it the degree of C. We also let

$$k = k(C) = \chi(\mathcal{O}_{\tilde{C}}) - 3(1 - g).$$

where g is the genus of C. We define the class of a curve $C \subset \mathrm{Km}_2(A)$ to be the sum of $m_i \cdot (d(C_i), k(C_i)) \in \mathbb{Z}^2$ where C_i runs over the normalizations of the (reduced) irreducible components of C and m_i is the multiplicity of that component in C. The class of a curve only depends on the homology class of the curve, see [10, Lem.2] for an alternative definition. We write $\mathrm{Hilb}^0(\mathrm{Km}_2(A), (d, k))$ for the Hilbert scheme of curves of class (d, k) and Euler characteristic 0.

Proposition 4.1. *Every (possibly reducible, non-reduced) curve in* $\mathrm{Km}_2(A)$ *of class* $(1, -4)$ *is isomorphic to* $\mathbb{P}^1$*. In particular,*

$$\mathrm{Hilb}^0(\mathrm{Km}_2(A), (1, -4)) = \varnothing.$$

Proof. Let $C \subset \mathrm{Km}_2(A)$ be a curve in class $(1, -4)$. Then there exist an irreducible reduced component $C_0 \subset C$ with $d(C_0) = 1$. Since $d(C) = 1$, the curve C must be reduced at C_0. We claim that C_0 is isomorphic to $\mathbb{P}^1$ and of class $(1, -4)$, so $C = C_0$. For this let B be the normalization of C_0 and consider the corresponding universal family $\tilde{B} \subset B \times A$. Since $\tilde{B}$ maps to the abelian surface in degree 1, its image in A is precisely the genus 2 curve Σ whose Jacobian is A. Hence $\tilde{B} = \Sigma$ and thus B is of genus ≤ 2. By Riemann-Hurwitz B can not be of genus 2 and since A is simple it can not be of genus 1. Hence $B = \mathbb{P}^1$ and the map $\Sigma \to B$ has precisely 8 branch points so $k(C_0) = -4$. Finally, the map $\Sigma \to B$ is obtained from the complete linear system of an arbitrary degree 3 line bundle on Σ, so non of the fibers are the same. Hence $B \to \mathrm{Km}_2(A)$ is a closed immersion. $\square$

Remark 4.2. With a bit more work one can show that $\overline{M}_{0,0}(\mathrm{Km}_2(A), (1, -4))$ is a disjoint union of copies of the quotients $A/\pm 1$.

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