RATIONAL CURVES IN THE FANO VARIETIES OF CUBIC 4-FOLDS AND GROMOV–WITTEN INVARIANTS

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ABSTRACT. We use Gromov–Witten theory to study rational curves in holomorphic symplectic varieties. We classify all rational curves in the primitive curve class of the Fano variety of lines in a very general cubic 4-fold, and prove the irreducibility of the corresponding moduli space. Our proof relies on a geometric construction of Voisin and Gromov–Witten calculations by the first author. A second proof via classical geometry is sketched.

We also present a numerical criterion for the existence of uniruled divisors swept out by rational curves in the primitive curve class of a very general holomorphic symplectic variety of $K3^{[n]}$ type.

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0. INTRODUCTION

0.1. **Overview.** Rational curves in $K3$ surfaces have been investigated for decades from various angles. However, not much is known about the geometry of rational curves in the higher-dimensional analogs of $K3$ surfaces—holomorphic symplectic varieties.[1] In this paper, we use *Gromov–Witten theory* (intersection theory of the moduli space of stable maps) together with classical methods to study these rational curves.

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[1] A nonsingular projective variety X is holomorphic symplectic if it is simply connected and $H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate holomorphic 2-form.

0.2. **Fano varieties of lines.** Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic 4-fold. By Beauville and Donagi [2], the Fano variety of lines in Y

$$F = \{l \in \mathrm{Gr}(2,6) : l \subset Y\}$$

is a holomorphic symplectic 4-fold. These varieties form a 20-dimensional family of polarized holomorphic symplectic varieties of $K3^{[2]}$ type.[2]

In [24], Voisin constructed a self-rational map

$$\varphi : F \dashrightarrow F \tag{1}$$

sending a general line l to its residual line with respect to the unique plane $\mathbb{P}^2 \subset \mathbb{P}^5$ tangent to Y along l. When Y is very general, the exceptional divisor associated to the resolution of φ

$$\begin{array}{ccc} D = \mathbb{P}(\mathcal{N}_{S/F}) & \xrightarrow{\phi} & F \\ \downarrow{\scriptstyle p} & & \\ S & & \end{array} \tag{2}$$

is a $\mathbb{P}^1$-bundle over a nonsingular surface $S \subset F$; see Amerik [1]. The image of each fiber

$$\phi(p^{-1}(s)) \subset F, \quad s \in S$$

is a rational curve lying in the primitive curve class in $H_2(F, \mathbb{Z})$.

The following theorem shows that every rational curve in the primitive curve class is of this form in a unique way.

Theorem 0.1. *Let F be the Fano variety of lines in a very general cubic 4-fold. Then for every rational curve $C \subset F$ in the primitive curve class, there is a unique $s \in S$ such that $C = \phi(p^{-1}(s))$.*

For very general F, we also show that S is connected and of general type; see Corollary 1.3. The next corollary is an immediate consequence.

Corollary 0.2. *For very general F, there is a unique irreducible uniruled divisor swept out by rational curves in the primitive curve class.*

The moduli space of rational curves in the primitive curve class of a very general $K3$ surface always has more than one irreducible component. The corollary indicates a difference between rational curves in $K3$ surfaces and in higher-dimensional holomorphic symplectic varieties.

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[2] A variety is of $K3^{[n]}$ type if it is deformation equivalent to the Hilbert scheme of n points on a $K3$ surface.

0.3. **Idea of the proof.** Let $\overline{M}_{0,m}(F,\beta)$ be the moduli space of genus 0 and m-pointed stable maps $f : C \to F$ in curve class

$$f_*[C] = \beta \in H_2(F,\mathbb{Z}).$$

If F is very general and β is the primitive curve class, the moduli space $\overline{M}_{0,0}(F,\beta)$ is pure of the expected dimension 2. The surface S in (2) is an irreducible component of $\overline{M}_{0,0}(F,\beta)$; see Proposition 3.1. Hence the universal map admits a decomposition

$$\overline{M}_{0,1}(F,\beta) = D \cup M',$$

where M' is the union of the other components. It remains to show $M' = \emptyset$.

The key observation is that $M' = \emptyset$ can be detected by the push-forward of the fundamental classes

$$\begin{aligned} \mathrm{ev}_*[\overline{M}_{0,1}(F,\beta)] &\in H^2(F,\mathbb{Q}), \\ \mathrm{ev}_{12*}[\overline{M}_{0,2}(F,\beta)] &\in H^8(F\times F,\mathbb{Q}), \end{aligned} \tag{3}$$

where ev and ev_{12} are the evaluation maps. Since $\overline{M}_{0,m}(F,\beta)$ is pure of the expected dimension, its fundamental class coincides with the (reduced) virtual fundamental class [3, 16],

$$[\overline{M}_{0,m}(F,\beta)] = [\overline{M}_{0,m}(F,\beta)]^{\mathrm{vir}}.$$

Hence both classes in (3) are determined by the Gromov–Witten invariants of F. By deformation invariance, the Gromov–Witten invariants of F can be calculated on a special model given by the Hilbert scheme of 2 points on a particular $K3$ surface; see [21] and Appendix A. This completes the proof.

Our proof of Theorem 0.1 is intersection-theoretic. Gromov–Witten theory effectively controls the rational curves in a very general fiber of the family via intersection theory on a special fiber. Another instance of this strategy is Corollary A.3 on the existence of uniruled divisors.

In Appendix B, we also sketch a second proof of Theorem 0.1 using a series of classification results in classical projective geometry.[3]

0.4. **General cases.** Let (X,H) be a very general (primitively) polarized holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X,\mathbb{Z})$ be the primitive curve class. The moduli space $\overline{M}_{0,0}(X,\beta)$ is pure of the expected dimension $2n-2$. Moreover, the universal map admits a decomposition

$$\overline{M}_{0,1}(X,\beta) = M^0 \cup M^1 \cup \cdots \cup M^{n-1} \tag{4}$$

[3]The proof in Appendix B was found only after a first version of this article appeared online. While Theorem 0.1 can be proven classically, the quantitative information obtained from Gromov–Witten theory was essential for us to find the statement.

such that a general fiber of the restricted evaluation map

$$\mathrm{ev} : M^i \to \mathrm{ev}(M^i) \subset X$$

is of dimension i; see Proposition 2.1 (i-ii).

In [19, Conjecture 4.3], Mongardi and Pacienza conjectured the nonemptiness of M^i for all i, which would lead to a geometric construction of algebraically coisotropic subvarieties in X in the sense of Voisin [25]; see Proposition 2.1 (iii). However, we disprove this conjecture by the following two types of counterexamples.

(i) Theorem 0.1 implies $M^1 = \emptyset$ for a very general holomorphic symplectic variety of $K3^{[2]}$ type of degree 6 and divisibility 2.

(ii) Corollary A.3 provides an example of a very general pair (X, H) of $K3^{[8]}$ type satisfying $M^0 = \emptyset$.

In fact, for very general (X, H) of $K3^{[n]}$ type, the class

$$\mathrm{ev}_*[M^0] \in H^2(X, \mathbb{Q})$$

was computed in [21] via Gromov–Witten theory; see also Appendix A.4. We thus obtain a necessary and sufficient numerical condition for $M^0 \neq \emptyset$ in the $K3^{[n]}$ case; see Corollary A.3. In particular, this yields uniruled divisors on all holomorphic symplectic varieties of $K3^{[n]}$ type with $n \leq 7$.

0.5. **Conventions.** We work over the complex numbers. A statement holds for a *very general* polarized projective variety (X, H) if it holds away from a countable union of proper Zariski-closed subsets in the corresponding component of the moduli space.

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1. Uniruled divisor

In this section, let F be the Fano variety of lines in a very general cubic 4-fold Y. We study the geometry of the uniruled divisor (2) in detail.

1.1. **Degeneracy locus.** The variety F is naturally embedded in the Grassmannian $\mathrm{Gr}(2,6)$. Let $\mathcal{U}$ and $\mathcal{Q}$ be the tautological bundles of ranks 2 and 4 with the short exact sequence

$$0 \to \mathcal{U} \to \mathbb{C}^6 \otimes \mathcal{O}_{\mathrm{Gr}(2,6)} \to \mathcal{Q} \to 0.$$

We use $\mathcal{U}_F, \mathcal{Q}_F$ to denote the restriction of $\mathcal{U}, \mathcal{Q}$ on F. Let $H = c_1(\mathcal{U}_F^*)$ be the hyperplane class on F with respect to the Plücker embedding. By [2], the primitive curve class $\beta \in H_2(F, \mathbb{Z})$ is characterized by $\int_\beta H = 3$.

The indeterminacy locus S of the rational map (1) consists of lines $l \subset Y$ with normal bundle

$$\mathcal{N}_{l/Y} = \mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)^{\oplus 2}.$$

For every line $l \subset Y$ corresponding to $s \in S$, there is a pencil of planes tangent to Y along l. The residual lines of this pencil form the rational curve $\phi(p^{-1}(s)) \subset F$. By [1, Proposition 6], we have

$$\int_{[\phi(p^{-1}(s))]} H = 3.$$

Hence the curve $\phi(p^{-1}(s))$ lies in the primitive curve class β. Moreover, by the calculations in [1, Theorem 8], we find

$$\phi_*[D] = 60H \in H^2(F, \mathbb{Q}). \tag{5}$$

In [1], the surface S is shown to be nonsingular, and is expressed as the degeneracy locus of the (sheafified) Gauss map

$$g : \mathrm{Sym}^2(\mathcal{U}_F) \to \mathcal{Q}_F^*$$

associated to the cubic Y. Let $\pi : \mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F) \to F$ be the $\mathbb{P}^2$-bundle and let h be the relative hyperplane class. Then S is isomorphic to the zero locus S' of a section of the rank 4 vector bundle $\pi^*\mathcal{Q}_F^* \otimes \mathcal{O}(h)$ on $\mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F)$. Let $H_{S'}, h_{S'}$ be the restrictions of the divisor classes π^*H, h on S'. There is the following calculation of intersection numbers.

Lemma 1.1. *We have*

$$\int_{S'} H_{S'}^2 = \int_{S'} H_{S'} h_{S'} = \int_{S'} h_{S'}^2 = 315.$$

Proof. Let $c = c_2(\mathcal{U}_F^*) \in H^4(F, \mathbb{Q})$. Since $S' \subset \mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F)$ is the zero locus of a section of the vector bundle $\pi^*\mathcal{Q}_F^* \otimes \mathcal{O}(h)$, a direct calculation yields

$$[S'] = c_4(\mathcal{Q}_F^* \otimes \mathcal{O}(h)) = 5(\pi^*H^2 - \pi^*c)h^2 - \frac{35}{6}\pi^*H^3 \cdot h + \frac{10}{3}\pi^*H^4 \\ \in H^8(\mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F), \mathbb{Q}).$$

The lemma follows from the projection formula, the intersection numbers calculated in [1, Lemma 4], and the projective bundle formula associated to $\pi : \mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F) \to F$,

$$h^3 = 3\pi^* H \cdot h^2 - (2\pi^* H^2 + 4\pi^* c)h + \frac{5}{3}\pi^* H^3 \in H^6(\mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F), \mathbb{Q}). \quad \square$$

1.2. **Connectedness.** Now we prove that S is connected and calculate its first Chern class.

Let $\mathbb{G}$ be the total space of the projective bundle $\mathbb{P}\mathrm{Sym}^2(\mathcal{U})$ over the Grassmannian $\mathrm{Gr}(2,6)$, and let

$$\tilde{\pi} : \mathbb{G} \to \mathrm{Gr}(2,6)$$

be the projection. For convenience, we also write H for the hyperplane class on $\mathrm{Gr}(2,6)$, and h for the relative hyperplane class of $\tilde{\pi}$. We define

$$\mathcal{V} = \tilde{\pi}^*\mathrm{Sym}^3(\mathcal{U}^*) \oplus \tilde{\pi}^*\mathcal{Q}^* \otimes \mathcal{O}(h)$$

to be the rank 8 tautological vector bundle on $\mathbb{G}$. Then S is isomorphic to the zero locus of a section of $\mathcal{V}$. We consider the universal zero locus of all sections of $\mathcal{V}$,

$$W = \{(s,x) : s(x) = 0\} \subset \mathbb{P}H^0(\mathbb{G}, \mathcal{V}) \times \mathbb{G}$$

together with the two projections

$$\begin{array}{ccc} W & \xrightarrow{\quad \iota \quad} & \mathbb{G} \\ \big\downarrow q & & \\ \mathbb{P}H^0(\mathbb{G}, \mathcal{V}). & & \end{array}$$

Since the morphism q has a fiber isomorphic to the surface S, a general fiber $W_s \to s \in \mathbb{P}H^0(\mathbb{G}, \mathcal{V})$ is also of dimension 2 by upper semi-continuity.

Proposition 1.2. *For $s \in \mathbb{P}H^0(\mathbb{G}, \mathcal{V})$ very general, the surface W_s is nonsingular of Picard rank* 1.

Proof. Over a point $x \in \mathbb{G}$, the fiber of ι is the projective space

$$\mathbb{P}H^0(\mathbb{G}, \mathcal{V} \otimes \mathcal{I}_x)$$

where $\mathcal{I}_x$ is the ideal sheaf of x. By the projection formula, we have

$$H^0(\mathbb{G}, \mathcal{V} \otimes \mathcal{I}_x) = H^0(\mathrm{Gr}(2,6), \mathrm{Sym}^3(\mathcal{U}^*) \otimes \tilde{\pi}_*\mathcal{I}_x \oplus \mathcal{Q}^* \otimes \tilde{\pi}_*\mathcal{I}_x(h)).$$

In particular, the dimension of $H^0(\mathbb{G}, \mathcal{V} \otimes \mathcal{I}_x)$ only depends on the projection $\tilde{\pi}(x) \in \mathrm{Gr}(2,6)$. The homogeneity of $\mathrm{Gr}(2,6)$ implies that $\iota : W \to \mathbb{G}$ is a projective bundle.

Since W is nonsingular, a general fiber W_s is also nonsingular. For W_s very general, an identical argument as in [23, Lemma 2.1] yields

$$\mathrm{Pic}(W_s)_{\mathbb{Q}} = \mathrm{Im}\left(\iota^* : \mathrm{Pic}(\mathbb{G})_{\mathbb{Q}} \to \mathrm{Pic}(W_s)_{\mathbb{Q}}\right).$$

Hence the Picard group $\mathrm{Pic}(W_s)_{\mathbb{Q}}$ is spanned by $\tilde{\pi}^*H$ and h. The calculation in Lemma 1.1 and the Hodge index theorem imply that

$$(\tilde{\pi}^*H - h)|_{W_s} = 0 \in H^2(W_s, \mathbb{Q}).$$

Hence the classes $\tilde{\pi}^*H$ and h coincide in the Néron–Severi group of W_s. □

Corollary 1.3. *The surface S in* (2) *is connected. If H_S is the restriction of H to S, then we have*

$$c_1(S) = -3H_S \in H^2(S, \mathbb{Q}).$$

Proof. The surface S is isomorphic to the zero locus S' of a section of $\mathcal{V}$ via the natural projection $\pi|_{S'} : S' \xrightarrow{\sim} S$. This isomorphism identifies the divisor classes $H_{S'}$ and H_S.

Since S is nonsingular, its connectedness follows from Proposition 1.2 by specialization. Moreover, Proposition 1.2 implies that $c_1(S)$ is proportional to H_S in $H^2(S, \mathbb{Q})$. The coefficient is determined by a calculation of intersection numbers; see [1, Remark in Section 2]. □

2. Moduli spaces of stable maps

In this section, we discuss some properties of the moduli spaces of stable maps into holomophic symplectic varieties, and introduce tools from Gromov–Witten theory.

2.1. **Dimensions.** Let X be a holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be an *irreducible* curve class. We show that the moduli space $\overline{M}_{0,1}(X, \beta)$ of genus 0 pointed stable maps into X in curve class β is pure of the expected dimension.

Let M be an irreducible component of $\overline{M}_{0,1}(X, \beta)$. We know *a priori*

$$\dim M \geq \int_\beta c_1(X) + \dim X - 1 = 2n - 1.$$

Consider the restriction of the evaluation map to M,

$$\mathrm{ev} : M \to Z = \mathrm{ev}(M) \subset X. \tag{6}$$

Proposition 2.1. *If a general fiber of* (6) *is of dimension $r - 1$, then*

(i) $\dim Z = 2n - r$, *so that* $\dim M = 2n - 1$;
(ii) $r \leq n$;
(iii) *a general fiber of the MRC fibration*[4] $Z \dashrightarrow B$ *is of dimension r.*

[4] We refer to [10] for the definition and properties of the maximal rationally connected (MRC) fibration.

Proof. Since the curve class β is irreducible, the family of rational curves $M \to T \subset \overline{M}_{0,0}(X,\beta)$ viewed as in X is unsplit in the sense of [14, IV, Definition 2.1]. Given a general point $x \in Z$, let $T_x \subset T$ be the Zariski-closed subset parametrizing maps passing through x. Consider the universal family $\mathcal{C}_x \to T_x$ and the restricted evaluation map

$$\mathrm{ev} : \mathcal{C}_x \to V_x = \mathrm{ev}(\mathcal{C}_x) \subset Z. \tag{7}$$

By [14, IV, Proposition 2.5], we have

$$\dim T = \dim Z + \dim V_x - 2.$$

Hence $\dim V_x = \dim M - \dim Z + 1 = r$. In other words, rational curves through a general point of Z cover a Zariski-closed subset of dimension r.

A general fiber of the MRC fibration $Z \dashrightarrow B$ is thus of dimension $\geq r$. By an argument of Mumford (see [25, Lemma 1.1]), this implies $\dim Z \leq 2n - r$ and $r \leq n$. On the other hand, since $\dim M \geq 2n-1$, we have

$$\dim Z = \dim M - (r-1) \geq 2n - r.$$

Hence there is equality $\dim Z = 2n - r$, and the dimension of a general fiber of $Z \dashrightarrow B$ is exactly r. $\square$

Proposition 2.1 shows that $\overline{M}_{0,1}(X,\beta)$ is pure of the expected dimension $2n-1$. It also justifies the decomposition (4).

Remark 2.2. Applying [5, Theorem 0.1], we further deduce that the normalization of a general V_x in (7) is isomorphic to the projective space $\mathbb{P}^r$.

2.2. **Gromov–Witten theory.** The proof of Theorem 0.1 uses *Gromov–Witten theory* [9]. We briefly recall the background that we need.

Let X be a holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X,\mathbb{Z})$ be an arbitrary curve class. By Li–Tian [16] and Behrend–Fantechi [3], the moduli space of stable maps $\overline{M}_{0,m}(X,\beta)$ carries a (reduced[5]) virtual fundamental class

$$[\overline{M}_{0,m}(X,\beta)]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(\overline{M}_{0,m}(X,\beta),\mathbb{Q}).$$

It has the following basic properties.

(a) *Virtual dimension.* The virtual fundamental class is of dimension

$$\mathrm{vdim} = 2n - 2 + m. \tag{8}$$

[5]Since X is holomorphic symplectic, the (standard) virtual fundamental class on the moduli space vanishes. The theory is nontrivial only after reduction; see [18, Section 2.2] and [21, Section 0.2]. The virtual fundamental class is always assumed to be reduced in this paper.

(b) *Expected dimension.* If $\overline{M}_{0,m}(X,\beta)$ is pure of the expected dimension (8), then the virtual and the ordinary fundamental classes agree:

$$[\overline{M}_{0,m}(X,\beta)]^{\mathrm{vir}} = [\overline{M}_{0,m}(X,\beta)].$$

(c) *Deformation invariance.* Let $\pi : \mathcal{X} \to B$ be a family of holomorphic symplectic varieties, and let $\beta \in H^0(B, R\pi_*^{4n-2}\mathbb{Z})$ be a class which restricts to a curve class in $H_2(X_b,\mathbb{Z})$ on each fiber.[6] Then there exists a class on the moduli space of relative stable maps

$$[\overline{M}_{0,m}(\mathcal{X}/B,\beta)]^{\mathrm{vir}} \in H_{2(\mathrm{vdim}+\dim B)}(\overline{M}_{0,m}(\mathcal{X}/B,\beta),\mathbb{Q})$$

such that for every fiber $X_b \hookrightarrow \mathcal{X}$, the inclusion $\iota_b : b \hookrightarrow B$ induces

$$\iota_b^![\overline{M}_{0,m}(\mathcal{X}/B,\beta)]^{\mathrm{vir}} = [\overline{M}_{0,m}(X_b,\beta)]^{\mathrm{vir}}.$$

In particular, intersection numbers of $[\overline{M}_{0,m}(X,\beta)]^{\mathrm{vir}}$ against cohomology classes pulled back from X via the evaluation maps

$$\mathrm{ev}_i : \overline{M}_{0,m}(X,\beta) \to X, \quad (f, x_1, \ldots, x_m) \mapsto f(x_i)$$

are invariant under deformations of (X,β) which keep β of Hodge type.

2.3. **Gromov–Witten correspondence.** Let X, β be as in Section 2.1. The evaluation maps from the 2-pointed moduli space

$$X \xleftarrow{\ \mathrm{ev}_1\ } \overline{M}_{0,2}(X,\beta) \xrightarrow{\ \mathrm{ev}_2\ } X$$

induce an action on cohomology:

$$\mathsf{GW}_\beta : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q}), \quad \gamma \mapsto \mathrm{ev}_{2*}(\mathrm{ev}_1^*\gamma \cap [\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}). \tag{9}$$

We call (9) the *Gromov–Witten correspondence.*

We introduce a factorization of (9) as follows. Consider the diagram

$$\begin{array}{ccc} \overline{M}_{0,1}(X,\beta) & \xrightarrow{\mathrm{ev}} & X \\ \downarrow{\scriptstyle p} & & \\ \overline{M}_{0,0}(X,\beta) & & \end{array} \tag{10}$$

with p the forgetful map (which is flat). We define morphisms[7]

$$\Phi_1 : H^i(X,\mathbb{Q}) \to H_{4n-2-i}(\overline{M}_{0,0}(X,\beta),\mathbb{Q}), \quad \gamma \mapsto p_*(\mathrm{ev}^*\gamma \cap [\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}),$$
$$\Phi_2 = \mathrm{ev}_* p^* : H_{4n-2-i}(\overline{M}_{0,0}(X,\beta),\mathbb{Q}) \to H^i(X,\mathbb{Q}).$$

[6] We have suppressed an application of Poincaré duality here.

[7] We have suppressed an application of Poincaré duality in the definition of Φ_2.

Since β is irreducible, there is a Cartesian diagram of forgetful maps

$$\begin{array}{ccc} & \overline{M}_{0,2}(X,\beta) & \\ \swarrow & & \searrow \\ \overline{M}_{0,1}(X,\beta) & & \overline{M}_{0,1}(X,\beta) \\ \searrow & & \swarrow \\ & \overline{M}_{0,0}(X,\beta). & \end{array}$$

Hence the Gromov–Witten correspondence (9) factors as

$$\mathsf{GW}_\beta = \Phi_2 \circ \Phi_1 : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q}). \tag{11}$$

2.4. **Hodge classes.** Now let (X,H) be a very general polarized holomorphic symplectic 4-fold of $K3^{[2]}$ type. It is shown in [22, Section 3] that the Hodge classes in $H^4(X,\mathbb{Q})$ are spanned by H^2 and $c_2(X)$.

A surface $\Sigma \subset X$ is *Lagrangian* if the holomorphic 2-form σ on X restricts to zero on Σ. The class of any Lagrangian surface is a positive multiple of

$$v_X = 5H^2 - \frac{1}{6}(H,H)c_2(X) \in H^4(X,\mathbb{Q}), \tag{12}$$

where $(-,-)$ is the Beauville–Bogomolov form on $H^2(X,\mathbb{Z})$.[8]

Proposition 2.3. *If (X,H) is very general of $K3^{[2]}$ type and $\beta \in H_2(X,\mathbb{Z})$ is the primitive curve class, then for any Hodge class $\alpha \in H^4(X,\mathbb{Q})$, the class*

$$\mathsf{GW}_\beta(\alpha) \in H^4(X,\mathbb{Q})$$

is proportional to v_X.

Proof. We use the factorization (11). For any Hodge class $\alpha \in H^4(X,\mathbb{Q})$, the class

$$\Phi_1(\alpha) \in H_2(\overline{M}_{0,0}(X,\beta),\mathbb{Q})$$

is represented by curves. Hence $\mathsf{GW}_\beta(\alpha)$ can be expressed as a linear combination of classes of the form

$$[\mathrm{ev}(p^{-1}(C))] \in H^4(X,\mathbb{Q})$$

with $C \subset \overline{M}_{0,0}(X,\beta)$ a curve.

Moreover, we have

$$\mathrm{ev}^*\sigma = p^*\sigma'$$

for some holomorphic 2-form σ' on $\overline{M}_{0,0}(X,\beta)$. Hence any surface of the form $\mathrm{ev}(p^{-1}(C))$ is Lagrangian, and the proposition follows. $\square$

[8]This follows from a direct calculation of the constraint $[\Sigma]\cdot\sigma = 0 \in H^6(X,\mathbb{Q})$. The class v_X was first calculated by Markman.

Proposition 2.3 implies that the class v_X in (12) is an eigenvector of the Gromov–Witten correspondence

$$\mathsf{GW}_\beta : H^4(X, \mathbb{Q}) \to H^4(X, \mathbb{Q}).$$

An explicit formula for GW_β was calculated in [21] and is recalled in Appendix A.5.

3. Proof of the main theorem

In this section, we combine the ingredients in Sections 1 and 2 and prove Theorem 0.1. Let F be the Fano variety of lines in a very general cubic 4-fold Y, and let $\beta \in H_2(F, \mathbb{Z})$ be the primitive curve class.

3.1. **Divisorial contribution.** By Proposition 2.1, the moduli space of stable maps $\overline{M}_{0,1}(F, \beta)$ is pure of dimension 3. Recall the decomposition (4),

$$\overline{M}_{0,1}(F, \beta) = M^0 \cup M^1,$$

such that a general fiber of $\mathrm{ev} : M^i \to \mathrm{ev}(M^i) \subset F$ is of dimension i. We first analyze the component M^0.

By construction, the family of maps $p : D \to S$ in (2) has a factorization

$$\phi : D \to M^0 \xrightarrow{\mathrm{ev}} F.$$

We have seen in (5) that

$$\phi_*[D] = 60H \in H^2(F, \mathbb{Q}).$$

On the other hand, by Theorem A.2[9] together with property (b) of the virtual fundamental class, we find

$$\mathrm{ev}_*[M^0] = \mathrm{ev}_*[\overline{M}_{0,1}(F, \beta)] = \mathrm{ev}_*[\overline{M}_{0,1}(F, \beta)]^{\mathrm{vir}} = 60H \in H^2(F, \mathbb{Q}).$$

To conclude $M^0 = D$, it suffices to prove the following proposition.

Proposition 3.1. *For very general F, each $s \in S$ yields a distinct rational curve $\phi(p^{-1}(s)) \subset F$.*

Proof. Let $s_1, s_2 \in S$ be two distinct points and suppose

$$\phi(p^{-1}(s_1)) = \phi(p^{-1}(s_2)) \subset F.$$

For $i = 1, 2$, let $l_i \subset Y$ be the line corresponding to s_i, and let $P_i \subset \mathbb{P}^5$ be the 3-dimensional linear subspace spanned by the tangent planes along l_i. Then necessarily $P_1 = P_2$. Otherwise, the intersection $P_1 \cap P_2$ is a plane that contains all lines in Y corresponding to the points on $\phi(p^{-1}(s_i))$. The fact that Y contains a plane violates the very general assumption. We also know $l_1 \cap l_2 = \emptyset$. Otherwise, the plane spanned by l_1 and l_2 is tangent to Y along both l_1 and l_2, which is impossible.

[9]By [2], we have $(\beta, \beta) = \frac{3}{2}$ and $(\beta, -) = \frac{1}{2}H \in H^2(F, \mathbb{Q})$.

Consider the Gauss map[10] associated to the cubic Y,

$$\mathcal{D} : \mathbb{P}^5 \to \mathbb{P}^{5*}.$$

By definition, the image $\mathcal{D}(l_i) \subset \mathbb{P}^{5*}$ is a line which is dual to $P_i \subset \mathbb{P}^5$. Following the argument of Clemens and Griffiths [6, Section 6], we may assume that l_1, l_2 are given by the equations

$$X_2 = X_3 = X_4 = X_5 = 0,$$
$$X_0 = X_1 = X_4 = X_5 = 0.$$

Then the condition $P_1 = P_2$ forces $\mathcal{D}(l_1) = \mathcal{D}(l_2)$ to be given by the equations

$$X_0^* = X_1^* = X_2^* = X_3^* = 0.$$

As a result, the cubic polynomial of Y takes the form

$$X_4Q_4^1(X_0, X_1) + X_5Q_5^1(X_0, X_1) \\ + X_4Q_4^2(X_2, X_3) + X_5Q_5^2(X_2, X_3) + R_1 + R_2. \quad (13)$$

Here the Q_i^j are quadratic polynomials, R_1 consists of terms of degree at least 2 in $\{X_4, X_5\}$, and R_2 consists of terms of degree 1 in each of $\{X_0, X_1\}, \{X_2, X_3\}, \{X_4, X_5\}$. The total number of possibly nonzero coefficients in (13) is

$$4 \cdot 3 + (4 \cdot 3 + 4) + 2 \cdot 2 \cdot 2 = 36.$$

On the other hand, the subgroup of $\mathrm{GL}(\mathbb{C}^6)$ fixing two disjoint lines in $\mathbb{P}^5$ is of dimension

$$4 + 4 + 3 \cdot 4 = 20,$$

resulting in a locus of dimension $36 - 20 = 16$ in the moduli space of cubic 4-folds. This again contradicts the very general assumption of Y. □

3.2. **Non-contribution.** We use the Gromov–Witten correspondence introduced in (9) to eliminate the component M^1. Recall that by property (b) of the virtual fundamental class, the class $[\overline{M}_{0,2}(F, \beta)]^{\mathrm{vir}}$ in (9) equals the ordinary fundamental class.

We begin by calculating the contribution of $M^0 = D$ to the Gromov–Witten correspondence

$$\mathsf{GW}_\beta : H^4(F, \mathbb{Q}) \to H^4(F, \mathbb{Q}). \quad (14)$$

Recall the diagram (2) and consider morphisms

$$\Phi_1^D = p_*\phi^* : H^4(F, \mathbb{Q}) \to H^2(S, \mathbb{Q}),$$
$$\Phi_2^D = \phi_*p^* : H^2(S, \mathbb{Q}) \to H^4(F, \mathbb{Q}).$$

[10] It is called the *dual mapping* in [6].

Comparing with (10) and (11), we see that $\Phi_2^D \circ \Phi_1^D = \phi_* p^* p_* \phi^*$ gives the contribution of D to the Gromov–Witten correspondence (14).

Let $c = c_2(\mathcal{U}_F^*) \in H^4(F, \mathbb{Q})$. Using the short exact sequence

$$0 \to T_F \to T_{\mathrm{Gr}(2,6)}|F \to \mathrm{Sym}^3(\mathcal{U}_F^*) \to 0,$$

we find

$$8c = 5H^2 - c_2(F) = v_F \in H^4(F, \mathbb{Q})$$

where v_F is the class defined in (12).[11] There is the following explicit calculation.

Proposition 3.2. *We have*

$$\phi_* p^* p_* \phi^* c = 945c \in H^4(F, \mathbb{Q}).$$

Proof. The argument in Proposition 2.3 shows that c is an eigenvector of $\phi_* p^* p_* \phi^*$. To determine the eigenvalue, it suffices to compute the intersection number

$$\int_F \phi_* p^* p_* \phi^* c \cdot H^2. \tag{15}$$

By the projection formula, we have

$$\begin{aligned}\int_F \phi_* p^* p_* \phi^* c \cdot H^2 &= \int_D p^* p_* \phi^* c \cdot \phi^* H^2 \\ &= \int_S p_* \phi^* c \cdot p_* \phi^* H^2 = \int_F \phi_* p^* p_* \phi^* H^2 \cdot c.\end{aligned}$$

Again by the argument in Proposition 2.3, we know that $\phi_* p^* p_* \phi^* H^2$ is proportional to c. Hence we can deduce the intersection number (15) by calculating instead

$$\int_F \phi_* p^* p_* \phi^* H^2 \cdot H^2 = \int_S (p_* \phi^* H^2)^2.$$

Let ξ be the relative hyperplane class of the projective bundle

$$p : D = \mathbb{P}(\mathcal{N}_{S/F}) \to S.$$

By [1, Proposition 6] and the projective bundle formula, we find

$$p_* \phi^* H^2 = p_*(7p^* H_S + 3\xi)^2 = 42 H_S - 9c_1(\mathcal{N}_{S/F}) \in H^2(S, \mathbb{Q}),$$

where H_S is the restriction of H to S. Moreover, Corollary 1.3 yields

$$c_1(\mathcal{N}_{S/F}) = -c_1(S) = 3H_S \in H^2(S, \mathbb{Q}).$$

Hence we obtain

$$p_* \phi^* H^2 = 15 H_S \in H^2(S, \mathbb{Q}).$$

[11] The proportionality of c and v_F also follows from the fact that c is represented by a rational (and hence Lagrangian) surface.

Applying Lemma 1.1, we find the intersection number

$$\int_F \phi_* p^* p_* \phi^* H^2 \cdot H^2 = \int_S (p_* \phi^* H^2)^2 = 15^2 \cdot 315 = 70875.$$

Finally, by the intersection numbers calculated in [1, Lemma 4], we have

$$\int_F \phi_* p^* p_* \phi^* c \cdot H^2 = \int_F \phi_* p^* p_* \phi^* H^2 \cdot c = 70875 \cdot \frac{27}{45} = 42525$$

and hence

$$\phi_* p^* p_* \phi^* c = \frac{42525}{45} c = 945c \in H^4(F, \mathbb{Q}). \qquad \square$$

The eigenvalue in Proposition 3.2 coincides with the one in Theorem A.4,

$$\mathsf{GW}_\beta(c) = 945c \in H^4(F, \mathbb{Q}).$$

Hence the final step is to show that if the component M^1 is nonempty, then it has to contribute nontrivially to the Gromov–Witten correspondence (14).

If $M' \subset M^1$ is a nonempty irreducible component, consider the restriction of (10)

$$\begin{array}{ccc} M' & \xrightarrow{\mathrm{ev}} & F \\ \downarrow{\scriptstyle p} & & \\ T' & & \end{array}$$

where $T' \subset p(M^1) \subset \overline{M}_{0,0}(F, \beta)$ is the base of M'. We define morphisms

$$\Phi_1^{M'} : H^4(F, \mathbb{Q}) \to H_2(T', \mathbb{Q}), \quad \gamma \mapsto p_*(\mathrm{ev}^* \gamma \cap [M']),$$
$$\Phi_2^{M'} = \mathrm{ev}_* p^* : H_2(T', \mathbb{Q}) \to H^4(F, \mathbb{Q}).$$

By definition, the composition $\Phi_2^{M'} \circ \Phi_1^{M'}$ gives the contribution of M' to the Gromov–Witten correspondence (14).

Proposition 3.3. *If $M' \subset M^1$ is a nonempty irreducible component, then we have*

$$\Phi_2^{M'} \circ \Phi_1^{M'}(c) = Nc \in H^4(F, \mathbb{Q})$$

for some $N > 0$.

Proof. Let $Z' = \mathrm{ev}(M') \subset F$. By Remark 2.2, the normalization of Z' is isomorphic to the projective plane $\mathbb{P}^2$, and the rational curves in Z' parametrized by T' correspond to lines in $\mathbb{P}^2$. Hence the normalization of T' is isomorphic to $\mathbb{P}^{2*}$, that is, the space of lines in $\mathbb{P}^2$. There is the following diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\tilde{\tau}} & M' & \xrightarrow{\mathrm{ev}} & F \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} & & \\ \mathbb{P}^{2*} & \xrightarrow{\tau} & T' & & \end{array} \qquad (\widetilde{\mathrm{ev}} : \mathcal{L} \to F)$$

where $\mathcal{L}$ is the universal line in $\mathbb{P}^2$.

We calculate $\Phi_1^{M'}(c) \in H_2(T', \mathbb{Q})$. By the projection formula, we have[12]

$$\Phi_1^{M'}(c) = p_*(\mathrm{ev}^* c \cap [M']) = p_* \tilde{\tau}_* \widetilde{\mathrm{ev}}^* c = \tau_* \tilde{p}_* \widetilde{\mathrm{ev}}^* c \in H_2(T', \mathbb{Q}).$$

On the other hand, the morphism $\widetilde{\mathrm{ev}}$ factors as

$$\widetilde{\mathrm{ev}} : \mathcal{L} \xrightarrow{q} \mathbb{P}^2 \xrightarrow{\iota} F$$

where q is the projection and ι is the normalization map. Hence we find

$$\Phi_1^{M'}(c) = \tau_* \tilde{p}_* q^* \iota^* c \in H_2(T', \mathbb{Q}).$$

Now since Z' is rational (and hence Lagrangian), we have

$$[Z'] = \iota_*[\mathbb{P}^2] = N'c \in H^4(F, \mathbb{Q})$$

for some $N' > 0$. The intersection numbers calculated in [1, Lemma 4] imply

$$\iota^* c = 27N'[x] \in H^4(\mathbb{P}^2, \mathbb{Q})$$

for any point $x \in \mathbb{P}^2$. This yields

$$\Phi_1^{M'}(c) = 27N' \tau_* \tilde{p}_* q^*[x] = 27N' \tau_*[l_x] \in H_2(T', \mathbb{Q}),$$

where $l_x \subset \mathbb{P}^{2*}$ is the line corresponding to lines in $\mathbb{P}^2$ passing through x. In particular, we see that $\Phi_1^{M'}(c) \in H_2(T', \mathbb{Q})$ is an effective curve class.

As a result, the class

$$\Phi_2^{M'} \circ \Phi_1^{M'}(c) = \mathrm{ev}_* p^* \Phi_1^{M'}(c) \in H^4(F, \mathbb{Q})$$

is an effective sum of classes of Lagrangian surfaces, and hence a positive multiple of c. $\square$

We conclude $M^1 = \emptyset$, and the proof of Theorem 0.1 is complete.

Appendix A. Gromov–Witten calculations

Based on [21], we present formulas for the 1-pointed Gromov–Witten class in the $K3^{[n]}$ case and the Gromov–Witten correspondence in the $K3^{[2]}$ case.

A.1. **Quasi-Jacobi forms.** Jacobi forms are holomorphic functions in variables[13] $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ with modular properties; see [8] for an introduction. Here we will consider Jacobi forms as formal power series in the variables

$$q = e^{2\pi i \tau}, \quad y = -e^{2\pi i z}$$

expanded in the region $|q| < |y| < 1$.

Recall the Jacobi theta function

$$\Theta(q, y) = (y^{1/2} + y^{-1/2}) \prod_{m \geq 1} \frac{(1 + yq^m)(1 + y^{-1}q^m)}{(1 - q^m)^2}$$

---

[12]Since $\mathcal{L}$ is nonsingular, we have suppressed an application of Poincaré duality here.

[13]Let $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ denote the upper half-plane.

and the Weierstraß elliptic function

$$\wp(q,y) = \frac{1}{12} - \frac{y}{(1+y)^2} + \sum_{m\geq 1}\sum_{d|m} d((-y)^d - 2 + (-y)^{-d})q^m.$$

Define Jacobi forms $\phi_{k,1}$ of *weight* k and *index* 1 by

$$\phi_{-2,1}(q,y) = \Theta(q,y)^2, \quad \phi_{0,1}(q,y) = 12\Theta(q,y)^2\wp(q,y).$$

We also require the weight k and index 0 Eisenstein series

$$E_k(q) = 1 - \frac{2k}{B_k}\sum_{m\geq 1}\sum_{d|m} d^{k-1}q^m, \quad k = 2,4,6,$$

where the B_k are the Bernoulli numbers, and the modular discriminant

$$\Delta(q) = \frac{E_4^3 - E_6^2}{1728} = q\prod_{m\geq 1}(1-q^m)^{24}.$$

We define the ring of quasi-Jacobi forms of even weight as the free polynomial algebra

$$\mathcal{J} = \mathbb{Q}[E_2, E_4, E_6, \phi_{-2,1}, \phi_{0,1}].$$

The weight/index assignments to the generators induce a bigrading

$$\mathcal{J} = \bigoplus_{k\in\mathbb{Z}}\bigoplus_{m\geq 0}\mathcal{J}_{k,m}$$

by weight k and index m.

Lemma A.1 ([8, Theorem 2.2])**.** *Let $\phi \in \mathcal{J}_{*,m}$ be a quasi-Jacobi form of index $m \geq 1$. For all $d, r \in \mathbb{Z}$, the coefficient $[\phi]_{q^dy^r}$ only depends on $2d - \frac{r^2}{2m}$ and the set $\{\pm[r]\}$, where $[r] \in \mathbb{Z}/2m\mathbb{Z}$ is the residue of r.*

By Lemma A.1, we may denote the q^dy^r-coefficient of ϕ by

$$\phi\left[2d - \frac{r^2}{2m}, \pm[r]\right] = [\phi]_{q^dy^r}. \tag{16}$$

If ϕ is of index 0, we set $\phi[2d, 0] = [\phi]_{q^d}$. Lemma A.1 and (16) remain valid if we replace ϕ by $f(q)\phi$ for any power series $f(q)$.

A.2. **Beauville–Bogomolov form.** Let X be a holomorphic symplectic variety of dimension $2n$. The Beauville–Bogomolov form on $H^2(X,\mathbb{Z})$ induces an embedding

$$H^2(X,\mathbb{Z}) \hookrightarrow H_2(X,\mathbb{Z}), \quad \alpha \mapsto (\alpha, -),$$

which is an isomorphism after tensoring with $\mathbb{Q}$. Let

$$(-,-) : H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Q} \tag{17}$$

denote the unique $\mathbb{Q}$-valued extension of the Beauville–Bogomolov form.

If X is of $K3^{[n]}$ type with $n \geq 2$, there is an isomorphism of abelian groups

$$r : H_2(X,\mathbb{Z})/H^2(X,\mathbb{Z}) \to \mathbb{Z}/(2n-2)\mathbb{Z}$$

such that $r(\alpha) = 1$ for some $\alpha \in H_2(X,\mathbb{Z})$ with $(\alpha,\alpha) = \frac{1}{2-2n}$. The morphism r is unique up to multiplication by ± 1.

A.3. **Curve classes.** Consider a pair (X,β) where X is a holomorphic symplectic variety of $K3^{[n]}$ type, and $\beta \in H_2(X,\mathbb{Z})$ is a primitive curve class. The curve class β has the following invariants:

(i) the Beauville–Bogomolov norm $(\beta,\beta) \in \mathbb{Q}$, and
(ii) the residue $[\beta] \in H_2(X,\mathbb{Z})/H^2(X,\mathbb{Z})$.

The *residue set* of β is the subset

$$\pm[\beta] = \{\pm r([\beta])\} \subset \mathbb{Z}/(2n-2)\mathbb{Z}.$$

It is independent of the choice of map r. If $n = 1$, we set $\pm[\beta] = 0$.

Given a (quasi-)Jacobi form ϕ of index $m = n - 1$, we define

$$\phi_\beta = \phi[(\beta,\beta), \pm[\beta]].$$

By Markman [17] (see also [20, Lemma 23]), two pairs (X,β) and (X',β') are deformation equivalent through a family of holomorphic symplectic manifolds which keeps the curve class of Hodge type if and only if the norms and the residue sets of β and β' agree. Hence, by identifying $H^*(X)$ with $H^*(X')$ via parallel transport and by property (c) of the virtual fundamental class, the Gromov–Witten invariants of the pairs (X,β) and (X',β') are equal.[14]

A.4. **Uniruled divisors.** Define the quasi-Jacobi form

$$\phi = \left(-\wp + \frac{1}{12}E_2\right)\Theta^2.$$

Theorem A.2 ([21])**.** *Let X be a holomorphic symplectic variety of $K3^{[n]}$ type, and let $\beta \in H_2(X,\mathbb{Z})$ be a primitive curve class. Then we have*

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} = \left(\frac{\phi^{n-1}}{\Delta}\right)_\beta h \in H^2(X,\mathbb{Q})$$

where $h = (\beta,-) \in H^2(X,\mathbb{Q})$ is the dual of β with respect to (17)*.*

Theorem A.2 together with the positivity of the Fourier coefficients of ϕ imply the following corollary.

[14]The (reduced) virtual fundamental class can also be defined via symplectic geometry and the twistor space of X; see [4]. Hence, the Gromov–Witten invariants are invariant also under (nonnecessarily algebraic) symplectic deformations of (X,β) which keep β of Hodge type. The invariance under nonalgebraic deformations is not needed for our application to the Fano variety of lines in a cubic 4-fold.

Corollary A.3. *Let (X, β) be as in Theorem A.2. Then there exists a uniruled divisor on X swept out by rational curves in curve class β if*

$$(\beta, \beta) = -2 + \sum_{i=1}^{n-1} 2d_i - \frac{1}{2n-2}\left(\sum_{i=1}^{n-1} r_i\right)^2,$$
$$\pm[\beta] = \pm\left[\sum_{i=1}^{n-1} r_i\right]$$

for some $d_i, r_i \in \mathbb{Z}$ satisfying $4d_i - r_i^2 \geq 0$, $i = 1, \ldots, n-1$. The converse holds if β is irreducible.

Proof. We have $[\phi]_{q^d y^r} > 0$ if and only if $2d - \frac{r^2}{2} \geq 0$. This implies the first claim. The second claim follows from Proposition 2.1 and property (b) of the virtual fundamental class. □

As a consequence, a very general holomorphic symplectic varieties of $K3^{[n]}$ type may contain *no* uniruled divisor swept out by rational curves in the primitive curve class. The first instance is given by a very general pair (X, β) of $K3^{[8]}$ type with $(\beta, \beta) = \frac{3}{14}$ and $\pm[\beta] = \pm[5]$.

In the $K3^{[2]}$ case, we write

$$f = \frac{\phi}{\Delta} = \left(-\wp + \frac{1}{12}E_2\right)\frac{\Theta^2}{\Delta}.$$

The first few values of f_β are listed in the following table.[15]

(β,β)	$-\frac{5}{2}$	-2	$-\frac{1}{2}$	0	$\frac{3}{2}$	2	$\frac{7}{2}$	4	$\frac{11}{2}$	6
f_β	0	1	4	30	120	504	1980	6160	23576	60720

Table 1. The first few multiplicities of uniruled divisors for $K3^{[2]}$.

A.5. **Gromov–Witten correspondence.** We specialize to the $K3^{[2]}$ case. Recall the Gromov–Witten correspondence GW_β in (9). We also define

$$g = \left(-\frac{12}{5}\wp - E_2\right)\frac{\Theta^2}{\Delta}.$$

Theorem A.4 ([21])**.** *Let X be a holomorphic symplectic 4-fold of $K3^{[2]}$ type, and let $\beta \in H_2(X, \mathbb{Z})$ be a primitive curve class. If $(\beta, \beta) \neq 0$, then GW_β is diagonalizable with eigenvalues*

$$\lambda_0 = 0, \quad \lambda_1 = (\beta,\beta) f_\beta, \quad \lambda_2 = (\beta,\beta) g_\beta,$$

[15]When $n = 2$, the value $(\beta, \beta) \in \mathbb{Q}$ uniquely determines $\pm[\beta] \subset \mathbb{Z}/2\mathbb{Z}$.

and eigenspaces

$$V_{\lambda_1} = \mathbb{Q}\langle h, h^3, (he_i)_{i=1,\dots,22}\rangle, \quad V_{\lambda_2} = \mathbb{Q}v.$$

Here $h = (\beta, -) \in H^2(X, \mathbb{Q})$ *is the dual of* β *with respect to* (17), $\{e_i\}_{i=1,\dots,22}$ *is a basis of the orthogonal of* h *in* $H^2(X,\mathbb{Q})$, *and*

$$v = 5h^2 - \frac{1}{6}(\beta,\beta)c_2(X) \in H^4(X,\mathbb{Q}).$$

One can show that the eigenvalues λ_1, λ_2 are integral, and if $(\beta,\beta) > 0$ then $\lambda_2 > \lambda_1 > 0$. The first few eigenvalues are listed in Table 2.

(β,β)	$-\frac{5}{2}$	-2	$-\frac{1}{2}$	0	$\frac{3}{2}$	2	$\frac{7}{2}$	4	$\frac{11}{2}$	6
λ_1	0	-2	-2	0	180	1008	6930	24640	129668	364320
λ_2	3	0	0	0	945	3840	53760	138240	1237005	2661120

TABLE 2. The first eigenvalues of GW_β for $K3^{[2]}$.

A.6. **Proof of Theorem A.2.** A very general pair (X, β) is of Picard rank 1.[16] Hence there exists $N_\beta \in \mathbb{Q}$ such that

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} = N_\beta h \in H^2(X,\mathbb{Q}).$$

We will evaluate N_β on the Hilbert scheme of n points on an elliptic $K3$ surface S with a section. By Section A.3, we may assume

$$\beta = \mathsf{B} + (d+1)\mathsf{F} + r\mathsf{A} \in H_2(\mathrm{Hilb}^n(\mathsf{S}), \mathbb{Z}), \quad d \geq -1, \; r \in \mathbb{Z}, \tag{18}$$

where $\mathsf{B}, \mathsf{F} \in H_2(\mathsf{S}, \mathbb{Z})$ are the classes of the section and fiber of the elliptic fibration, and $\mathsf{A} \in H_2(\mathrm{Hilb}^n(\mathsf{S}), \mathbb{Z})$ is the class of an exceptional curve (for $n \geq 2$). Here we apply the natural identification

$$H_2(\mathrm{Hilb}^n(\mathsf{S}), \mathbb{Z}) \simeq H_2(\mathsf{S}, \mathbb{Z}) \oplus \mathbb{Z}\mathsf{A}.$$

Let $\mathsf{F}_0 \subset \mathsf{S}$ be a nonsingular fiber, and let $x_1, \dots, x_{n-1} \in \mathsf{S} \setminus \mathsf{F}_0$ be distinct points. Consider the curve

$$\mathsf{C} = \{x_1 + \dots + x_{n-1} + x' : x' \in \mathsf{F}_0\} \subset \mathrm{Hilb}^n(\mathsf{S}).$$

Then $\int_{[\mathsf{C}]} h = 1$ and hence by the first equation in [21, Theorem 2], we find

$$N_\beta = \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*[\mathsf{C}] = \left[\frac{\phi^{n-1}}{\Delta}\right]_{q^d y^r} = \left(\frac{\phi^{n-1}}{\Delta}\right)_\beta. \qquad \square$$

[16] In this statement, we allow X to be a holomorphic symplectic manifold.

A.7. **Proof of Theorem A.4.** Consider the 2-pointed class

$$\mathsf{Z}_\beta = \mathrm{ev}_{12*}[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}} \in H^8(X \times X, \mathbb{Q}).$$

By the divisor equation [9] and Theorem A.2, we have

$$\int_{\mathsf{Z}_\beta} \gamma \otimes \delta = \left(\int_\beta \delta \int_\gamma h \right) f_\beta$$

for all $\delta \in H^2(X,\mathbb{Q})$ and $\gamma \in H^6(X,\mathbb{Q})$.[17] Hence

$$\mathsf{GW}_\beta(\delta) = \left(\int_\beta \delta \right) f_\beta h \in H^2(X,\mathbb{Q}),$$

$$\mathsf{GW}_\beta(\gamma) = \left(\int_\gamma h \right) f_\beta \beta \in H^6(X,\mathbb{Q}).$$

Now consider the $(4,4)$-Künneth factor of Z_β,

$$\mathsf{Z}_\beta^{4,4} \in H^4(X) \otimes H^4(X).$$

By monodromy invariance under the group $\mathrm{SO}(H^2(X,\mathbb{C}),h)$, we have

$$\mathsf{Z}_\beta^{4,4} = a_\beta h^2 \otimes h^2 + b_\beta (h^2 \otimes c_2(X) + c_2(X) \otimes h^2) + c_\beta c_2(X) \otimes c_2(X) + d_\beta (h \otimes h) c_{BB} + e_\beta [\Delta_X]^{4,4}$$

for some $a_\beta, b_\beta, c_\beta, d_\beta, e_\beta \in \mathbb{Q}$; see [12, Section 4]. Here

$$c_{BB} \in \mathrm{Sym}^2(H^2(X,\mathbb{Q})) \subset H^2(X,\mathbb{Q}) \otimes H^2(X,\mathbb{Q})$$

is the inverse of the Beauville–Bogomolov class.

Since $\int_{\mathsf{Z}_\beta} \sigma^2 \otimes \bar{\sigma}^2 = 0$, we have $e_\beta = 0$. Also, since the Gromov–Witten correspondence is equivariant with respect to multiplication by σ, we find

$$\mathsf{GW}_\beta(h\sigma) = \mathsf{GW}_\beta(h)\sigma = (\beta,\beta) f_\beta h \sigma.$$

Hence $d_\beta = f_\beta$. Together with Proposition 2.3 and $\int_X v^2 = 48(\beta,\beta)^2 \neq 0$, this implies

$$\mathsf{Z}_\beta^{4,4} = \psi_\beta \frac{v \otimes v}{48(\beta,\beta)^2} + f_\beta (h \otimes h) \left(c_{BB} - \frac{h \otimes h}{(\beta,\beta)} \right) \tag{19}$$

for some $\psi_\beta \in \mathbb{Q}$. It remains to determine ψ_β.

As in the proof of Theorem A.2, let S be an elliptic $K3$ surface with a section, and let β be as in (18). Consider the fiber class of the Lagrangian fibration $\mathrm{Hilb}^2(\mathsf{S}) \to \mathbb{P}^2$ induced by the elliptic fibration $\mathsf{S} \to \mathbb{P}^1$,

$$\mathsf{L} \in H^4(\mathrm{Hilb}^2(\mathsf{S}),\mathbb{Q}).$$

We have

$$\int_{\mathrm{Hilb}^2(\mathsf{S})} h^2 \mathsf{L} = 2, \quad \int_{\mathrm{Hilb}^2(\mathsf{S})} v \mathsf{L} = 10, \quad \int_{\mathrm{Hilb}^2(\mathsf{S}) \times \mathrm{Hilb}^2(\mathsf{S})} (h\mathsf{L} \otimes h\mathsf{L}) c_{BB} = 0.$$

---

[17] We have suppressed an application of Poincaré duality here.

Then [21, Theorem 1] and (19) imply the relation

$$\left(\frac{\Theta^2}{\Delta}\right)_\beta = \int_{\mathsf{Z}_\beta} \mathsf{L}\otimes\mathsf{L} = \frac{10^2}{48(\beta,\beta)^2}\psi_\beta - \frac{2^2}{(\beta,\beta)} f_\beta.$$

Hence

$$\psi_\beta = \frac{12(\beta,\beta)}{25}\left(4f + \mathcal{H}_1\left(\frac{\Theta^2}{\Delta}\right)\right)_\beta$$

where

$$\mathcal{H}_m = 2q\frac{d}{dq} - \frac{1}{2m}\left(y\frac{d}{dy}\right)^2, \quad m \geq 1$$

is the *heat operator*. Explicit formulas for the derivatives of Jacobi forms can be found in [21, Appendix B], and this yields $\psi_\beta = (\beta,\beta)g_\beta$ as desired. □

Appendix B. Sketch of a classical proof

We sketch a proof of Theorem 0.1 via the classical geometry of cubic hypersurfaces. Let $Y \subset \mathbb{P}^5$ be a very general cubic 4-fold, and let F be the Fano variety of lines in Y.

Consider the correspondence given by the universal family

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{q_Y} & Y \\ \downarrow{\scriptstyle q_F} & & \\ F. & & \end{array}$$

A rational curve $R \subset F$ corresponds to a surface $Z = q_Y(q_F^{-1}(R)) \subset Y$. If R lies in the primitive curve class of F, then we have

$$[Z] = H_Y^2 \in H^4(Y,\mathbb{Z})$$

with H_Y the hyperplane class on Y.

Step 1. Let $j : Y \hookrightarrow \mathbb{P}^5$ be the embedding. Since the surface $j(Z) \subset \mathbb{P}^5$ is of degree 3, we know from [11, Page 173] that $j(Z)$ lies in a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. Hence Z is contained in the hyperplane section

$$Y' = Y \cap \mathbb{P}^4 \subset \mathbb{P}^4.$$

Step 2. By [11, Page 525, Proposition], the surface $Z \subset Y'$ belongs to one of the following classes:

(i) a cubic rational normal scroll;
(ii) a cone over a twisted cubic curve;
(iii) a cubic surface given by a hyperplane section of $Y' \subset \mathbb{P}^4$.

Since (i) and (ii) cannot hold for a very general[18] cubic 4-fold, we find that Z is a cubic surface of the form

$$Z = Y \cap \mathbb{P}^3.$$

Step 3. The singularities of cubic surfaces were classified long ago; see [7, Chapter 9] and [15, Section 2]. Since Z is integral, it satisfies one of the following conditions:

(i) Z has rational double point singularities;
(ii) Z has a simple-elliptic singularity;
(iii) Z is integral but not normal.

By definition, the surface Z is swept out by a 1-dimensional family of lines parameterized by a rational curve. Hence we may narrow down to case (iii).

Step 4. By further classification results (see [15, Section 2.3]), the surface Z is projectively equivalent to one of the four surfaces with explicit equations:

$$X_0^2X_1 + X_2^2X_3 = 0,$$
$$X_1X_2X_3 + X_0^2X_3 + X_1^3 = 0,$$
$$X_1^3 + X_2^3 + X_1X_2X_3 = 0,$$
$$X_1^3 + X_2^2X_3 = 0.$$

In each of the four cases, the singular locus of Z is a line $l \subset Z$, and the 1-dimensional family of lines covering Z is given by the residual lines of the planes containing l. Hence we conclude that all rational curves in the primitive curve class of F are given by the uniruled divisor (2). The uniqueness part of Theorem 0.1 follows from Proposition 3.1.

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