A LIE ALGEBRA ACTION ON THE CHOW RING OF THE HILBERT SCHEME OF POINTS OF A K3 SURFACE

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Abstract. We show that for Hilbert scheme of points on K3 surfaces the action of the Neron-Severi part of the Verbitsky Lie algebra on cohomology lifts to an action on Chow groups. This yields a new argument for the representation-theoretic part of Maulik and Negut's proof of Beauville's conjecture that the cycle class map is injective on the subring generated by divisor classes. The key ingredients in the proof are Lehn's formula in Chow proven by Maulik and Negut, and an explicit formula for Lefschetz duals in terms of Nakajima operators. Our results also yield a formula for the monodromy action on Hilbert schemes in terms of Nakajima operators.

1. Introduction

1.1. **Chow.** Let X be a smooth complex projective variety of dimension m. Let $h \in \operatorname{End} H^*(X, \mathbb{Q})$ be the operator that acts on $H^i(X, \mathbb{Q})$ by multiplication with $i - m$. Let also $e_a \in \operatorname{End} H^*(X, \mathbb{Q})$ denote the operator of cup product with a given element $a \in H^2(X, \mathbb{Q})$. The element a is called *Lefschetz* if there exist an operator $f_a \in \operatorname{End} H^*(X, \mathbb{Q})$ such that e_a, f_a, h satisfy the $\mathfrak{sl}_2$-commutation relations

$$[e_a, f_a] = h, \quad [h, e_a] = 2e_a, \quad [h, f_a] = -2f_a.$$

In this case we say (e_a, f_a, h) is a Lefschetz triple. The operator f_a, if it exists, is unique and is called the Lefschetz dual to e_a. By the Hard Lefschetz theorem every ample class on X is Lefschetz. More generally, an element a is Lefschetz precisely if the morphism $e_a^s : H^{m-s}(X) \to H^{m+s}(X)$ is an isomorphism for every $s \geq 0$. In particular being Lefschetz is a Zariski open condition.

The total Lie algebra of X introduced by Looijenga and Lunts [7] and Verbitsky [14] is the Lie subalgebra

$$\mathfrak{g}(X) \subset \operatorname{End} H^*(X, \mathbb{Q})$$

generated by all Lefschetz triples (e_a, f_a, h). We also consider the Neron-Severi Lie algebra of X which is defined as the Lie subalgebra

$$\mathfrak{g}_{\mathrm{NS}}(X) \subset \mathfrak{g}(X)$$

generated by all Lefschetz triples such that a is algebraic, i.e. $a \in H^{1,1}(X, \mathbb{Q})$.

Assume now that X is irreducible holomorphic symplectic, that is it is simply connected and $H^0(X, \Omega_X)$ is generated by a holomorphic symplectic form σ. The prime example of such a variety is the Hilbert scheme of points of a K3 surface. By a result of Verbitsky [14] we have

$$\mathfrak{g}(X) \otimes \mathbb{R} = \mathfrak{so}_{\mathbb{R}}(4, b_2(X) - 2), \quad \mathfrak{g}_{\mathrm{NS}}(X) \otimes \mathbb{R} = \mathfrak{so}_{\mathbb{R}}(2, \rho(X))$$

where $b_i(X)$ are the Betti numbers and $\rho(X)$ is the Picard rank of X.

Date: August 26, 2019.

Let $A^*(X)$ denote the Chow ring of X taken here always with $\mathbb{Q}$-coefficients. The group of correspondences $A^*(X \times X)$ carries a natural ring structure given by composition. The cycle class map $\mathrm{cl} : A^*(X \times X) \to \operatorname{End} H^*(X)$ is a ring homomorphism. Our main result says that for Hilbert schemes of points of K3 surfaces the action of the Neron-Severi Lie algebra on cohomology lifts to an action on Chow groups by correspondences:

Theorem 1.1. *Let X be the Hilbert scheme of points of a smooth projective K3 surface. There exists a Lie algebra homomorphism $\rho : \mathfrak{g}_{\mathrm{NS}}(X) \to A^*(X \times X)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}_{\mathrm{NS}}(X) & \xrightarrow{\ \rho\ } & A^*(X \times X) \\ & \searrow & \downarrow \mathrm{cl} \\ & & \operatorname{End} H^*(X, \mathbb{Q}). \end{array}$$

The operators e_a here lift under ρ to the cup product with the divisor class a. The claim that the Lefschetz dual f_a lifts to Chow is precisely the Grothendieck standard conjecture of Lefschetz type [5]. The standard conjectures have been proven for Hilbert schemes of points on surfaces by Arapura [1]. The main improvement of Theorem 1.1 is that we give an explicit lift of the operator f_a and show that also all the relations between the e_a's and f_b's lift. For example the operator h lifts to an endomorphisms of Chow which should yield a splitting of the Bloch-Beilinson filtration on Chow groups, see Remark 3.3.

We expect Theorem 1.1 to hold for all irreducible holomorphic symplectic varieties X. The standard conjectures for irreducible holomorphic symplectic varieties deformation equivalent to the Hilbert schemes of points of K3 surfaces have been proven by Charles and Markman [3]. The difficulty in extending Theorem 1.1 beyond Hilbert schemes is to deform the relations between the operators e_a and f_b.

For the proof of the theorem we consider the action of the Nakajima operators $\mathfrak{q}_n$ on the direct sum of Chow groups

$$A^*(\operatorname{Hilb} S) = \bigoplus_{n=0}^{\infty} A^*(\operatorname{Hilb}^n(S)),$$

see Section 2.3 for details on Nakajima operators. The action of the operators e_a on cohomology was expressed in terms of Nakajima operators by Lehn [6]. By recent work of Maulik and Negut the formula of Lehn holds also on the level of Chow groups [10]. We prove Theorem 1.1 by explicitly writing the operators f_a in terms of Nakajima operators and show they satisfy the required commutation relations.

As application of Theorem 1.1 we obtain the following result which was conjectured by Beauville and first proven by Maulik and Negut.

Corollary 1.2 ([10])**.** *Let S be a K3 surface. The cycle map $A^*(\operatorname{Hilb}^n(S)) \to H^*(\operatorname{Hilb}^n(S))$ is injective on the subring generated by divisor classes.*

Proof. The subring of $A^*(\operatorname{Hilb}^n(S))$ generated by divisor classes is an irreducible representation of the simple Lie algebra $\mathfrak{g}_{\mathrm{NS}}(X)$ hence the cycle class map restricted to it is either injective or zero. □

Theorem 1.1 and hence the proof of Corollary 1.2 is not independent from [10] and should be rather viewed as replacing its representation-theoretic part. In [10]

Lehn's formula is used to construct an action of the product of the Heisenberg and Virasoro algebra on $A^*(\mathrm{Hilb}(S))$. Beauville's conjecture is then deduced from Schur's Lemma. Our approach here also relies on Schur's Lemma but has the advantange that the Lie algebra $\mathfrak{g}_{\mathrm{NS}}(X)$ involved is much smaller (for once it is finite-dimensional) and that the argument might generalize to other cases.[1]

An analog of Theorem 1.1 and Corollary 1.2 for abelian varieties was obtained by Moonen in [11].

1.2. **Application to monodromy.** Recall that the locus of Hilbert schemes of points of K3 surfaces form a divisor in the moduli space of irreducible holomorphic symplectic varieties. In particular, for $n \geq 2$ the monodromy group of $X = \mathrm{Hilb}^n(S)$ is strictly larger than the monodromy group of the underlying K3 surface S. On the other hand the Nakajima operators define a basis of the cohomology of $\mathrm{Hilb}^n(S)$ which strongly depends on the Hilbert scheme structure. A basic question is how the monodromy group acts on this basis, and whether its action on cohomology can be written in terms of Nakajima operators.

In Theorem 3.1 we describe the action of the total Lie algebra $\mathfrak{g}(X)$ on cohomology in terms of Nakajima operators. This leads to a formula for the monodromy action as follows. The degree zero part of the Lie algebra $\mathfrak{g}_0(X)$ is isomorphic to $\mathfrak{so}(H^2(X,\mathbb{Q})) \oplus \mathbb{Q}h$ where $H^2(X,\mathbb{Q})$ is endowed with the Beauville-Bogomolov quadratic form. Its action on $H^*(X)$ integrates to an action $\sigma : \mathrm{SO}(H^2(X,\mathbb{Z})) \to \operatorname{End} H^*(X,\mathbb{Q})$. By a result of Markman [9] the monodromy group of X is

$$\mathrm{Mon}(X) = \widetilde{O}^+(H^2(X,\mathbb{Z}))$$

where the right hand side stands for orthogonal transformations which preserve the orientation and act by ± 1 on the discriminant. By [8, Lemma 4.13] the monodromy action on cohomology agrees with σ on an index 2 subgroup of the intersection

$$\mathrm{SO}(H^2(X,\mathbb{Z})) \cap \widetilde{O}^+(H^2(X,\mathbb{Z})).$$

This leads to the desired formulas up to finite index.

The description of the monodromy in terms of Nakajima operators was the original motivation for considering the operators f_a in the Nakajima basis. It will also play an important role in holomorphic anomaly equations for Hilbert schemes of points of K3 surfaces in forthcoming work.

1.3. **Plan.** In Section 2 we give preliminaries on the Lie algebra, Nakajima operators, and the Chow ring of K3 surfaces. In Section 3 we state the formulas for the Lefschetz duals f_a and give the proof of Theorem 1.1.

1.4. **Acknowledgements.** I would like thank Hsueh-Yung Lin and Andrei Negut for interesting discussions on how one may deform the algebra action, and the latter for giving an inspiring talk in Bonn in June 2019. I'm also grateful to Junliang Shen and Qizheng Yin for useful discussions.

[1] On ther other hand, Maulik and Negut's argument yields the stronger statement that the cycle class is injective on the subring generated by all small tautological classes [10].

2. Preliminaries

2.1. **The Lie algebra made explicit.** Let V be a vector space with a non-degenerate symmetric bilinear form $(-,-)$ on it. The wedge prouct $\wedge^2 V$ carries naturally the structure of a Lie algebra. The Lie bracket is defined by

$$[a\wedge b, c\wedge d] = (a,d) b\wedge c - (a,c) b \wedge d - (b,d) a\wedge c + (b,c) a \wedge d$$

for all $a,b,c,d \in V$. There exist a natural Lie algebra isomorphism $\wedge^2 V \to \mathfrak{so}(V)$ by letting $a\wedge b$ act on V via the endomorphism $(a\wedge b) v = (b,v)a - (a,v)b$.

Let X be an irreducible holomorphic symplectic variety. The Beauville-Bogomolov form is a non-degenerate quadratic form on $H^2(X,\mathbb{Z})$. Let U be the hyperbolic lattice $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with standard basis e, f. By [7, 14] we have

$$\mathfrak{g}(X) = \mathfrak{so}(H^2(X,\mathbb{Q}) \oplus U_{\mathbb{Q}}).$$

After identifying the right hand side with the second wedge product of $H^2(X,\mathbb{Q}) \oplus U_{\mathbb{Q}}$ as before, this isomorphism is given explicitly by

$$e_a = e\wedge a, \quad f_a = \frac{-2}{(a,a)} f \wedge a, \quad h = 2\cdot e\wedge f$$

for all $a \in H^2(X)$ with $(a,a) \neq 0$. We will also use

$$\widetilde{f}_a = -2 \cdot f \wedge a$$

which is defined for all a, is linear in a and satisfies $\widetilde{f}_a = (a,a) f_a$ whenever $(a,a)\neq 0$.

2.2. **The Chow ring of a K3 surface.** Let S be a smooth projective K3 surface and let

$$c \in A^2(S)$$

be the class of any point on any rational curve of S. Beauville and Voisin [2] prove the following basic relations:

$$c_2(T_S) = 24c, \quad \ell \cdot \ell' = (\ell, \ell')c$$

for all $\ell, \ell' \in A^1(S)$. They also establish the following decomposition of the class of the small diagonal Δ_{123} in the Chow ring of $S\times S\times S$:

$$[\Delta_{123}] = \Delta_{12} c_3 + \Delta_{13} c_2 + \Delta_{23} c_1 - c_1 c_2 - c_1 c_3 - c_2 c_3, \tag{1}$$

where following [10] we write c_i for the pullback of c along the projection to the i-th factor and Δ_{ij} for the pullback of the class of the diagonal in S^2 along the projection to the (i,j)-factor, etc. Parallel conventions will be followed throughout. We will also use the following relation from [2]:

$$\Delta \cdot c_1 = \Delta \cdot c_2 = c_1 \cdot c_2$$
$$\Delta \cdot \ell_1 = \Delta \cdot \ell_2 = c_1 \ell_2 + \ell_1 c_2.$$

2.3. **Nakajima operators.** Let S be a smooth projective surface. We recall the definition of Nakajima operators [12, 4].

For $n \geq 0$ and $i > 0$ consider the closed subscheme

$$Z_{n,n+i} = \{ (\xi, x, \eta) \in \mathrm{Hilb}^n(S) \times S \times \mathrm{Hilb}^{n+i}(S) | \xi \subset \eta, \mathrm{Supp}(\eta/\xi) = \{x\} \}$$

and let $p_1 : Z_{n,n+i} \to \mathrm{Hilb}^n(S)$, $p_2 : Z_{n,n+i} \to S$ and $p_3 : Z_{n,n+i} \to \mathrm{Hilb}^{n+i}(S)$ be the projection to the factors. The Nakajima operators act on $A^*(\mathrm{Hilb}(S))$ and are defined by

$$\mathfrak{q}_i = (p_2 \times p_3)_* p_1^*$$
$$\mathfrak{q}_{-i} = (-1)^i \cdot (p_1 \times p_2)_* p_3^*.$$

We also set $\mathfrak{q}_0 = 0$. Following [10] the $\mathfrak{q}_i$ here are viewed as operators

$$\mathfrak{q}_i : A^*(\mathrm{Hilb}^n(S)) \to A^*(\mathrm{Hilb}^{n+i}(S) \times S).$$

The composition $\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k}$ of Nakajima operators is understood as an operator

$$\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k} : A^*(\mathrm{Hilb}^n(S)) \to A^*(\mathrm{Hilb}^{n+i_1+\ldots+i_k}(S) \times S^k)$$

where the operator $\mathfrak{q}_{i_j}$ acts by its definition on the Hilbert scheme and by the identity on all remaining S-factors. We have the Heisenberg commutation relations

$$[\mathfrak{q}_m, \mathfrak{q}_n] = m\delta_{m+n,0} \mathrm{id} \times \Delta. \tag{2}$$

For $\alpha \in A^*(S)$ we also write

$$\mathfrak{q}_i(\alpha) = p_{3*}(p_1^*(\,\cdot\,) \cup p_2^*(\alpha))$$

and similarly for negative i. The commutation relations read

$$[\mathfrak{q}_m(\alpha), \mathfrak{q}_n(\beta)] = m\delta_{m+n,0}\langle \alpha, \beta \rangle \mathrm{id}.$$

More general given $\Gamma \in A^*(S^k)$ we let

$$\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k}(\Gamma) : A^*(\mathrm{Hilb}^n(S)) \to A^*(\mathrm{Hilb}^{n+i_1+\ldots+i_k}(S))$$

be the map obtained by viewing $\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k}$ as a correspondence from S^k to $\mathrm{Hilb}^{n+\sum_j i_j}(S)$ and applying it to Γ.

3. Formulas and proofs

3.1. **Formulas.** Let S be a smooth projective K3 surface. Let $\Delta_{\mathrm{Hilb}^n(S)} \subset \mathrm{Hilb}^n(S)$ be the divisor parametrizing non-reduced subschemes and let

$$\delta = -\frac{1}{2}[\Delta_{\mathrm{Hilb}^n(S)}].$$

By definition $\delta = 0$ if $n \leq 1$. For all $n \geq 1$ we have the orthogonal decomposition

$$H^2(\mathrm{Hilb}^n(S), \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$$

The restriction of the Beauville-Bogomolov form to the first factor is the intersection pairing on S. Moreover, $(\delta, \delta) = 2 - 2n$. Similarly, for algebraic classes we have

$$A^1(\mathrm{Hilb}^n(S)) \cong A^1(S) \oplus \mathbb{Z}\delta. \tag{3}$$

We will identify classes in $A^1(S) \oplus \mathbb{Z}\delta$ with their image in $A^1(\mathrm{Hilb}^n(S))$ under the isomorphism (3) and similarly for cohomology.

Let e_a be the operator on $A^*(\mathrm{Hilb}(S))$ which acts on $A^*(\mathrm{Hilb}^n(S))$ by cup product with the class $a \in A^1(S) \oplus \mathbb{Z}\delta$. By the results of Lehn [6] and Maulik-Negut [10] we have for all $\alpha \in A^1(S)$

$$\begin{aligned} e_\alpha &= -\sum_{n>0} \mathfrak{q}_n \mathfrak{q}_{-n}(\Delta_* \alpha) \\ e_\delta &= -\frac{1}{6} \sum_{i+j+k=0} :\mathfrak{q}_i \mathfrak{q}_j \mathfrak{q}_k(\Delta_{123}): \end{aligned} \tag{4}$$

where $\Delta : S \to S^2$ is the inclusion of the diagonal, and $: - :$ is the normal ordered product defined by

$$: \mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k} : = \mathfrak{q}_{i_{\sigma(1)}} \cdots \mathfrak{q}_{i_{\sigma(k)}}$$

where σ is a permutation such that $i_{\sigma(1)} \geq \ldots \geq i_{\sigma(k)}$.

The formulas (4) hold also in cohomology for all $\alpha \in H^2(S, \mathbb{Q})$.

Define the following operators on $A^*(\mathrm{Hilb}^n(S))$:

$$h = 2 \sum_{n>0} \frac{1}{n} \mathfrak{q}_n \mathfrak{q}_{-n}(c_2 - c_1) \tag{5}$$

$$\widetilde{f}_\alpha = -2 \sum_{n>0} \frac{1}{n^2} \mathfrak{q}_n \mathfrak{q}_{-n}(\alpha_1 + \alpha_2) \tag{6}$$

$$\widetilde{f}_\delta = -\frac{1}{3} \sum_{i+j+k=0} : \mathfrak{q}_i \mathfrak{q}_j \mathfrak{q}_k \left(\frac{1}{k^2} \Delta_{12} + \frac{1}{j^2} \Delta_{13} + \frac{1}{i^2} \Delta_{23} + \frac{2}{j \cdot k} c_1 + \frac{2}{i \cdot k} c_2 + \frac{2}{i \cdot j} c_3 \right) :$$

We define $\widetilde{f}_a$ for all $a \in A^1(S) \oplus \mathbb{Q}\delta$ by linearity in a. If $(a, a) \neq 0$ we also set

$$f_a = \frac{1}{(a, a)} \widetilde{f}_a.$$

The following implies Theorem 1.1.

Theorem 3.1. *Let S be a smooth projective K3 surface and let $n \geq 1$ be an integer.*

(a) *For every $a \in A^1(S) \oplus \mathbb{Q}\delta$ we have*

$$[h, e_a] = 2e_a, \quad [h, \widetilde{f}_a] = -2\widetilde{f}_a, \quad [e_a, \widetilde{f}_a] = (a, a)h$$

as operators on $A^*(\mathrm{Hilb}^n(S))$.

(b) *If $(a, a) \neq 0$ then (e_a, f_a, h) specializes to a Lefschetz triple in cohomology.*

(c) *The Lie subalgebra of $A^*(\mathrm{Hilb}^n(S) \times \mathrm{Hilb}^n(S))$ generated by $e_a, \widetilde{f}_a, h$ for all $a \in A^1(S) \oplus \mathbb{Q}\delta$ is isomorphic to $\mathfrak{so}(A^1(\mathrm{Hilb}^n(S)) \oplus U_{\mathbb{Q}})$.*

We make several remarks.

Remark 3.2. Consider Lefschetz duals on Hilbert schemes of points of arbitrary smooth projective surfaces S. By the discussion in Section 3.2, for any $\alpha \in H^2(S, \mathbb{Q})$ of non-zero square the operator e_α admits the Lefschetz dual $\widetilde{f}_\alpha / (\alpha \cdot \alpha)$ where $\widetilde{f}_\alpha$ is defined as in (6). However, the Lefschetz dual of more general elements $a \in H^2(\mathrm{Hilb}^n(S))$ do not seem to admit a nice expression in terms of Nakajima operators. For example on $\mathrm{Hilb}^n(\mathbb{P}^2)$ we have $[f_a, f_b] \neq 0$ in general and computer calculations suggest that the expression for f_a involves expressions in Nakajima operators $\mathfrak{q}_n$ of arbitrarily high degree. That the Lefschetz duals on $\mathrm{Hilb}(K3)$ can be expressed as quadratic and cubics in Nakajima operators is remarkable. It requires both $K = 0$ and $e(S) = 24$, see the proof below.

Remark 3.3. Let $X = \mathrm{Hilb}^n(S)$ where S is a smooth projective K3 surface and recall from [13] the Beauville-Voisin filtration on the Chow group of zero cycles

$$S^0 A_0(X) \subset S^1 A_0(X) \subset \ldots \subset S^n A_0(X) = A_0(X) \tag{7}$$

where $S^i A_0(X)$ is spanned by the classes $[z]$ for all $z \in \mathrm{Hilb}^n(S)$ such that $c_2(\mathcal{O}_Z) = y + (n - i)c$ where y is an effective zero cycle on S of degree i.

We consider how the operator h interacts with this filtration. For simplicity let $x_1, \dots, x_n \in S$ be distinct points and consider the subscheme

$$z = \{x_1, \dots, x_n\} \in X.$$

Then

$$h([z]) = 2\sum_{j=1}^{n}[\{x_1, \dots, \hat{x_j}, \dots, x_n, c_0\}]$$

where $c_0 \in S$ is a representative of the Beauville-Voisin class $c \in A^2(S)$. This leads to $h(S^i A_0(X)) \subset S^{i-1}A_0(X)$ for $i > 0$ and $h|_{S^0 A_0(X))} = 2n \cdot \mathrm{id}$.

More generally, we expect $h^n(A^*(X)) = h^{n+1}(A^*(X))$ and that the sequence of subspaces

$$h^n(A^*(X)) \subset h^{n-1}(A^*(X)) \subset \ldots \subset h(A^*(X)) \subset A^*(X)$$

is equal to the filtration induced by the motivic Chow-Künneth decomposition of X defined in [16, Section 1].

3.2. **The surface part.** We begin with some general remarks that hold for every smooth projective surface S. For a correspondence $\Gamma \in A^*(S \times S)$ we let Γ' be its transpose which is defined as $\tau_*(\Gamma)$ where τ is the automorphism of S^2 that swaps the factors. The correspondence Γ acts on $A^*(S)$ via

$$\Gamma(\gamma) = \pi_{2*}(\pi_1^*(\gamma) \cdot \Gamma).$$

Given two correspondences Γ and $\widetilde{\Gamma}$ their composition as operators on $A^*(S)$ is

$$\Gamma \circ \widetilde{\Gamma} = \pi_{13*}(\widetilde{\Gamma}_{12} \cdot \Gamma_{23}),$$

where $\pi_{13} : S^3 \to S \times S$ is the projection to the outer factors.

Let $\deg(\Gamma)$ denote the degree of the homogeneous correspondence Γ, that is $\Gamma \in A^{\deg(\Gamma)}(S \times S)$. Define the following operator on $A^*(\mathrm{Hilb}(S))$:

$$T_\Gamma = -\sum_{n>0} n^{\deg(\Gamma)-3}\mathfrak{q}_n\mathfrak{q}_{-n}(\Gamma'). \tag{8}$$

Lemma 3.4. *For any $\mathsf{C} \in A^*(S^k)$ and homogeneous correspondence Γ,*

$$\begin{aligned}[T_\Gamma, \mathfrak{q}_{n_1}\cdots\mathfrak{q}_{n_k}(C)] = &\sum_{i:n_i>0} n_i^{\deg(\Gamma)-2}\mathfrak{q}_{n_1}\cdots\mathfrak{q}_{n_k}(\mathrm{id}_{S^{i-1}} \times \Gamma \times \mathrm{id}_{S^{k-i}}(\mathsf{C}))\\ &+ (-1)^{\deg(\Gamma)-3}\sum_{i:n_i<0} n_i^{\deg(\Gamma)-2}\mathfrak{q}_{n_1}\cdots\mathfrak{q}_{n_k}(\mathrm{id}_{S^{i-1}} \times \Gamma' \times \mathrm{id}_{S^{k-i}}(\mathsf{C}))\end{aligned}$$

Proof. We commute T_Γ through the Nakajima operators. If $n_i > 0$ then the i-th term contributes

$$\begin{aligned}&- n_i^{\deg(\Gamma)-3}\mathfrak{q}_{n_1}\cdots\underbrace{\mathfrak{q}_{n_i}[\mathfrak{q}_{-n_i}}_{\text{from } T_\Gamma}, \mathfrak{q}_{n_i}]\mathfrak{q}_{n_{i+1}}\cdots\mathfrak{q}_{n_k}(\mathsf{C}_{\{i,i+1\}^c} \cdot \Gamma'_{i,i+1})\\ =&n_i^{\deg(\Gamma)-2}\mathfrak{q}_{n_1}\cdots\mathfrak{q}_{n_k}(\pi_{\{i+1,i+2\}^c*}(\Delta_{i+1,i+2} \cdot \mathsf{C}_{\{i,i+1\}^c} \cdot \Gamma'_{i,i+1}))\\ =&n_i^{\deg(\Gamma)-2}\mathfrak{q}_{n_1}\cdots\mathfrak{q}_{n_k}(\mathrm{id}_{S^{i-1}} \times \Gamma \times \mathrm{id}_{S^{k-i}}(\mathsf{C}))\end{aligned}$$

where we write $\mathsf{C}_{\{i,i+1\}^c}$ for the pullback of C to S^{k+2} along the projection which forgets the factors i and $i+1$, etc. The case $n_i < 0$ is similar. □

Corollary 3.5. $[T_\Gamma, T_{\widetilde{\Gamma}}] = T_{[\Gamma, \widetilde{\Gamma}]}$ *for any homogeneous correspondences* $\Gamma, \widetilde{\Gamma}$.

By Corollary 3.5 for every $n \geq 1$ we have an embedding of Lie algebras

$$T : A^*(S \times S) \to A^*(\mathrm{Hilb}^n(S) \times \mathrm{Hilb}^n(S)),\, \Gamma \mapsto T_\Gamma.$$

We now specialize to the case of K3 surfaces. For every $\alpha \in A^1(S)$ consider the correspondences

$$e_\alpha = \Delta_*(\alpha) = c_1\alpha_2 + \alpha_1 c_2, \quad \widetilde{f}_\alpha = 2(\alpha_1 + \alpha_2), \quad h = 2(c_2 - c_1). \tag{9}$$

Either by a direct check or because $\mathbb{Q}1 \oplus A^1(S) \oplus \mathbb{Q}c$ is an invariant subring of $A^*(S)$ which injects into cohomology, the correspondences (9) satisfy the relations of part (a) of Theorem 3.1. Applying T to these correpondences precisely yields the operators (4), (5), (6). Using Corollary 3.5 we conclude that Theorem 3.1(a) holds for all $n \geq 1$ and $\alpha \in A^1(S)$.

Further, by the cohomological version of Lemma 3.4 for every $u \in H^i(S)$ and for all $n \in \mathbb{Z}$ one has

$$[h, \mathfrak{q}_n(u)] = (i-2)\mathfrak{q}_n(u).$$

Since the Nakajima operators generate the cohomology of Hilbert schemes and $\mathfrak{q}_n(u)$ is of degree i, we conclude that h acts on $H^j(\mathrm{Hilb}^n(S))$ by multiplication by $j - 2n$. This shows part (b) of Theorem 3.1 for all $\alpha \in A^1(S)$. On S the Lie algebra generated by the correspondences (9) for all $\alpha \in A^1(S)$ is $\mathfrak{so}(A^1(S) \oplus U)$ (e.g. use again the argument with the invariant subring and that in cohomology we know the result from Verbitsky). Applying T proves the same on $\mathrm{Hilb}^n(S)$.

3.3. **The general case.** For all $a, b \in A^1(S) \oplus \mathbb{Z}\delta$ let $\kappa_{ab} = [e_a, \widetilde{f}_b]$. To prove the remainder of Theorem 3.1 we need to establish the following commutation relations:

$$[h, e_a] = 2e_a, \quad [h, \widetilde{f}_a] = -2\widetilde{f}_a, \quad [h, \kappa_{ab}] = 0$$

$$[e_a, e_b] = 0, \quad [\widetilde{f}_a, \widetilde{f}_b] = 0, \quad [e_a, \widetilde{f}_a] = (a,a)h$$

$$\kappa_{ab} + \kappa_{ba} = 2(a,b)h,$$

$$[\kappa_{ab}, e_c] = 2(a,b)e_c + 2(b,c)e_a - 2(a,c)e_b$$

$$[\kappa_{ab}, \widetilde{f}_c] = -2(a,b)\widetilde{f}_c + 2(b,c)\widetilde{f}_a - 2(a,c)\widetilde{f}_b$$

$$\frac{1}{2}[\kappa_{ab}, \kappa_{cd}] = (a,d)\kappa_{bc} - (a,c)\kappa_{bd} - (b,d)\kappa_{ac} + (b,c)\kappa_{ad} + ((a,c)(b,d) - (a,d)(b,c))h$$

By the discussion in Section 3.2 we know these relations when all classes involved are from $A^1(S)$. Moreover it suffices to check the relations on a basis, and we only need to check those relations that do not follow from the Jacobi identity and previously established relations. Hence it is enough to check for all $\alpha, \beta \in A^1(S)$ the following.

(a) $[h, e_\delta] = 2e_\delta$ and $[h, \widetilde{f}_\delta] = -2\widetilde{f}_\delta$
(b) $[\widetilde{f}_\alpha, \widetilde{f}_\delta] = 0$.
(c) $[e_\delta, \widetilde{f}_\delta] = (2-2n)h$ on $A^*(\mathrm{Hilb}^n(S))$.
(d) $\kappa_{\alpha\delta} = -\kappa_{\delta\alpha}$
(e) $[h, \kappa_{\alpha\delta}] = 0$
(f) $[\kappa_{\alpha\beta}, e_\delta] = 2(\alpha,\beta)e_\delta$.
(g) $[\kappa_{\alpha\beta}, \widetilde{f}_\delta] = -2(\alpha,\beta)\widetilde{f}_\delta$.

Except for (c) this is all a straightforward application of Lemma 3.4 and we skip the details. It can be also seen as follows: Each of the above is a relation between Nakajima operators that, after applying the commutation relations (2), reduces to a relation in S^k between classes which are polynomials in Δ_{ij}, c_i and α_j for $\alpha \in A^1(S)$. By Verbitsky [14] we know these hold in cohomology. Hence for $k \leq 5$ we know from Yin [15] that they also hold in Chow. Case (c) involves a relation on S^6 which in Picard rank 20 does not seem to follow immediately from this, hence we sketch a short proof below:

Proof of Relation (c). By [10, Thm.1.6] the operator

$$L_0 = \sum_{k>0} \mathfrak{q}_k \mathfrak{q}_{-k}(\Delta)$$

acts by multiplication by $-n$ on $A^*(\mathrm{Hilb}^n(S))$. Hence we need to show

$$[e_\delta, \widetilde{f}_\delta] = 2h + 2L_0(1)h.$$

We do this by expanding both sides in Nakajima operators ordered in normal product ordering. For the right hand side we obtain

$$2h + 2L_0(1)h = 4\sum_{k>0} \frac{1-k}{k} \mathfrak{q}_k \mathfrak{q}_{-k}(c_2 - c_1) + 4\sum_{k,\ell>0} \frac{1}{k} : \mathfrak{q}_k \mathfrak{q}_{-k}(c_2 - c_1)\mathfrak{q}_\ell \mathfrak{q}_{-\ell}(\Delta) :$$

For the left hand side we first consider the quartic terms, that is those of degree 4 in a normal ordering. These involve precisely one interaction of the Nakajima operators. Let $\mathcal{E}$ be the argument of $\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k$ in the definition of $\widetilde{f}_\delta$. Since the argument of the cubic term in e_δ and $\widetilde{f}_\delta$ is S_3-symmetric, the quartic term reads

$$\begin{aligned}
&9 \cdot (-\frac{1}{6}) \cdot (-\frac{1}{3}) \sum_{\substack{j_1+k_1=-i_1 \\ j_2+k_2=i_1}} : [\mathfrak{q}_{i_1}, \mathfrak{q}_{-i_1}]\mathfrak{q}_{j_1}\mathfrak{q}_{k_1}\mathfrak{q}_{j_2}\mathfrak{q}_{k_2}(\Delta_{134} \cdot \mathcal{E}_{256}) : \\
&= \frac{1}{2} \sum_{\substack{j_1+k_1=-i_1 \\ j_2+k_2=i_1}} (-i_1) : \mathfrak{q}_{j_1}\mathfrak{q}_{k_1}\mathfrak{q}_{j_2}\mathfrak{q}_{k_2}(\pi_{3456*}(\Delta_{134} \cdot \mathcal{E}_{256} \cdot \Delta_{12})) : \\
&= -\frac{1}{2} \sum_{\substack{a+b+c+d=0 \\ a+b\neq 0}} : \mathfrak{q}_a\mathfrak{q}_b\mathfrak{q}_c\mathfrak{q}_d \left(2\frac{c+d}{d^2}\Delta_{123} - \frac{4}{c}\Delta_{12}c_4 + 2\frac{(c+d)}{c \cdot d}c_1c_2 + \frac{1}{(c+d)}\Delta_{12}\Delta_{34} \right) :
\end{aligned}$$

The term with $\Delta_{12}\Delta_{34}$ cancels by symmetrizing. For the remaining terms we insert the decomposition (1) of the small diagonal Δ_{123} and observe that the sum vanishes when it istaken over all a, b, c, d such that $a + b + c + d = 0$. The sum of the terms we overcounted (those with $a + b = 0$) precisely yields (the negative of) the quartic term in $2h + 2L_0(1)h$.

For the quadric term we have two Nakajima interactions. We get

$$\begin{aligned}
\text{Quadric terms in } [e_\delta, \widetilde{f}_\delta] = &\sum_{\substack{i_1+j_1+k_1=0 \\ i_1,j_1<0}} [\mathfrak{q}_{i_1}, \mathfrak{q}_{-i_1}][\mathfrak{q}_{j_1}, \mathfrak{q}_{-j_1}]\mathfrak{q}_{k_1}\mathfrak{q}_{-k_1}(\Delta_{135} \cdot \mathcal{E}_{246}) \\
&+ \sum_{\substack{i_1+j_1+k_1=0 \\ i_1,j_1>0}} [\mathfrak{q}_{i_1}, \mathfrak{q}_{-i_1}][\mathfrak{q}_{-j_1}, \mathfrak{q}_{j_1}]\mathfrak{q}_{-k_1}\mathfrak{q}_{k_1}(\Delta_{146} \cdot \mathcal{E}_{235})
\end{aligned}$$

Using the Nakajima commutation relations and $\Delta \cdot \Delta = e(S) \cdot c_1 c_2$ this simplifies to

$$\sum_{k>0} \Big(4(k-1) - \sum_{\substack{i+j=k \\ i,j>0}} e(S) \frac{i \cdot j}{k^2} \Big) \mathfrak{q}_k \mathfrak{q}_{-k}(c_2 - c_1) = 4 \sum_{k>0} \frac{1-k}{k} \mathfrak{q}_k \mathfrak{q}_{-k}(c_2 - c_1)$$

as desired. Here we have used $\sum_{i+j=k} i \cdot j = \frac{1}{6}k(k^2-1)$ and $e(S) = 24$. $\square$

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