PROGRAM FOR THE SEMINAR ON "THOM SPECTRA AND UNITS" (AFTER ANDO-BLUMBERG-GEPNER-HOPKINS-REZK)

STEFFEN SAGAVE AND MARKUS SZYMIK

Introduction

So far the seminar has been structured into six sessions. Each session is planned to occupy one afternoon with two talks, and covers a rather localized topic. Needless to say, speakers therefore have the obligation to finish their topic by the end of the afternoon, especially if there is a soccer match scheduled later on, so that the next meeting can start fresh with a new topic. The semester leaves time for a seventh session; see below.

Session 1: Repetition and overview

Talk 1: Repetition. This will be a survey talk in which spectra are mentioned as late as possible. (As will certainly become clear during the talk, spectra – which are not just suspension spectra – naturally arise from *stable* bundles.) Maybe start with bundles in the form of maps $X \to \mathrm{B}G$ for suitable G to cover the case of vector bundles and spherical fibrations. Discuss orientations with respect to a (co)homology theory, and the Thom isomorphism. Examples of theories to bear in mind are ordinary theory with suitable coefficients, topological K-theories, and stable homotopy theory itself. Explain also the corresponding obstruction theories, for example those related to the first/second Stiefel-Whitney class. Finally, although twists will not play a rôle later on, this might be the best place to mention them, see [ABGHR, §2.5].

References: The classical theory is covered in many textbooks, see [Rud98] for example.

Comment: This is the ideal talk for someone with a distrust of categories and spectra, but who nevertheless wants to contribute to the seminar.

Talk 2: Overview of [ABGHR]**.** The aim of this talk is to give a survey of the approach to units and Thom spectra in [ABGHR]. A good way to start is to give the provisional definition of a Thom spectrum as in §1, see (1.3) and (1.8), and to explain how this suggests to treat orientations. The main adjunction in Theorem 2.1/Theorem 3.2 should be stated without proof, and the speaker is invited to be vague about what is meant by an E_∞ ring spectrum: both will be treated in detail in the following session. Assuming that, the "homotopy pushout" definition of the E_∞ Thom spectrum associated with a map $b \to bgl_1 R$ (Definition 2.6) can be

Date: September 9, 2009.

given, followed by a discussion of orientations, culminating in a proof of Theorem 2.10/Theorem 4.6.

References: Use §1, §2.1, and §4. It seems reasonable leave out the rest of §2 at this point.

Comment: If you intend to follow the seminar, but don't want to give a talk: consider volunteering for this one! You get an overview and learn how to use one of its main theorems, without much effort: with the axiomatic approach, everything becomes rather symbolic.

Session 2: E_∞ ring spectra and units

Talk 3: E_∞ ring spectra and E_∞ spaces. This talk should feature E_∞ ring spectra and their relation to E_∞ spaces. A good starting point is a brief review of the definition of an operad and an E_∞ operad, and of algebras over operads. The linear isometries operad and the Barratt-Eccles operad are important examples to have in mind. Then one can proceed with defining the model categories of E_∞ ring spectra and E_∞ spaces and the Quillen adjunction relating them (Proposition 3.27). It is also important to explain that the categories of E_∞ spaces over different E_∞ operads are related by a zig-zag of Quillen equivalences (Corollary 3.16). There is the option to take a slightly different approach to this talk which might be easier to digest for parts of the audience: the speaker could work with E_∞ symmetric spectra instead of C-spectra in the sense of Lewis-May-Steinberger. A discussion of these different accounts can be found in [May09, §13].

Reference: The summary at the beginning of §3 and §§3.1 - 3.3. For people without prior knowledge on operads, [MS04] is recommended. A good reference for the alternative approach are the last two sections of [EM06]. (When reading in [EM06], remember that an operad is a multicategory with one object!)

Comment: This talk is ideally suited for someone who wants to learn what an E_∞ ring spectrum is before learning more about Thom spectra.

Talk 4: Units and the spectrum gl_1. This talk should feature a proof of Theorem 2.1 (= Theorem 3.2). It exhibits the spectrum of units gl_1 as a right adjoint in a sense to be made precise. The definition of the E_∞ space of units GL_1R of an E_∞ ring spectrum R is a good way to get started. This should be followed by a detour to §6, where the authors give an alternative definition GL_1R that becomes important later. Then one can proceed with the definition of the spectrum of units gl_1R and finish the talk with the proof of Theorem 2.1.

References: The summary at the beginning of §3, §§3.4 - 3.6, and §6.

Comment: This talk is a good choice if you want to learn something about an important notion in the algebra of structured ring spectra which is not only relevant for Thom spectra, but also for the study of algebraic K-theory. It is instructional to see how the classical construction of gl_1 is expressed in terms of the modern approaches to stable homotopy theory.

Session 3: A_∞ ring spectra and units

Talk 5: S-modules and $*$-modules. In order to set up a theory of Thom spectra in the A_∞ context, a rigidified model of A_∞ spaces is used in [ABGHR]. These $*$-*modules* are a space level analog of the S-modules of [EKMM97]. This talk should feature the definition of $*$-modules and their comparison to S-modules. For the preparation of this talk, it seems advisable to also consult [BCS08]. The axiomatic approach in [BCS08] might help in understand what one wants the $*$-modules to do, although we don't need all of those axioms for the present purpose.

Reference: §§5.1 - 5.3 and [BCS08]

Comment: If you were always wondering what an S-module in the sense of [EKMM97] actually is, then you should go for this talk. After all, understanding the definition of S-modules is likely to become easier by studying $*$-modules instead of just reading [EKMM97], because it is very advisable to see a second instance in which some of the main ideas behind S-modules get applied.

Talk 6: Thom spectra and orientations in the A_∞ context. This talk should deal with the definition of the Thom spectrum associated with a map into the space $B\mathrm{GL}_1 R$, where R is an A_∞ ring spectrum. To make this precise, one has to give a version of the bar construction for monoids in $*$-modules. Once the theory is set up, one can discuss orientations and the Thom isomorphism for the A_∞ case in a similar manner as before. A good ending for this would be comparison to the E_∞ case in Proposition 2.25.

Reference: §2.2, the motivation at the beginning of §5, §§5.4-5.5, and the relevant part of §2.4.

Comment: Giving this talk is a good way to learn some of the main points of [ABGHR] since it encompasses the formulation of the main results within the A_∞ world.

Session 4: Thom spectra and quasi-categories

Talk 7: An overview of ∞-categories. This talk has three objectives: (1) Introduce quasi-categories as a model for ∞-categories. Explain how ordinary categories and spaces, i.e. ∞-groupoids, give examples. Can you give an example of an ∞-category that is neither an ordinary category nor a space? (2) Explain the simplicial categories $\mathfrak{C}[\Delta^n]$ and the simplicial nerve construction, which turns simplicial categories into ∞-categories. (3) Define colimits in ∞-categories as initial objects in certain undercategories. Obviously, this requires a preliminary discussion of initial objects and undercategories in ∞-categories; the latter in turn requiring join categories.

Reference: All this is covered in [HTT, Chapter 1].

Comment: If you know what a simplicial set is, but don't know what a quasi-category is, this is the talk to go for. If you already know what a quasi-category is: be more ambitious!

Talk 8: Thom spectra in ∞-categories of spectra. Let $\mathcal{M}$ be a model category of spectra which is both symmetric monoidal and simplicial, and let R be a fibrant cofibrant monoid in it which is associative but not necessarily commutative. Define the ∞-categories $(R-\mathrm{mod}_{\mathcal{M}})$, $(R-\mathrm{line}_{\mathcal{M}})$, and $(R-\mathrm{triv}_{\mathcal{M}})$, and that show that there is a map $(R-\mathrm{triv}_{\mathcal{M}}) \to (R-\mathrm{line}_{\mathcal{M}})$ which models $\mathrm{EGL}_1 R \to \mathrm{BGL}_1 R$. Explain that a map $f\colon X \to (R-\mathrm{line}_{\mathcal{M}}) \subset (R-\mathrm{mod}_{\mathcal{M}})$ has a colimit $\mathrm{M}f$, the Thom spectrum of f in this context. Explain its mapping property, define the space of orientations of f, and explain the obstruction theory in this context.

References: Follow §B, incorporate also §§2.3, 7.1 and 7.7 until and including Proposition 7.38.

Comment: If you know what a quasi-category is, but don't know what they can be good for, this is the talk to go for. If you already know what quasi-categories can be good for: be more ambitious!

Session 5: Thom spectra via ∞-categories

Talk 9: Stable ∞-categories. This talk has two objectives. (1) Expose Lurie's point of view on stable homotopy theory: Give a definition of a stable ∞-category, and of a morphism between such. Explain how to stabilize a given pointed ∞-category such as the ∞-category of ∞-groupoids. It will also be relevant to know about the $(\Sigma^\infty_+, \Omega^\infty_-)$-adjunction, and that the stabilization of the ∞-category of ∞-groupoids is a symmetric monoidal ∞-category. (2) Give a brief indication how this compares to symmetric spectra: The stabilization of the ∞-category of ∞-groupoids is equivalent to the simplicial nerve of the simplicial category of fibrant-cofibrant symmetric spectra, and the ∞-category of algebras in the stabilization of the ∞-category of ∞-groupoids is equivalent to the simplicial nerve of the simplicial category of fibrant-cofibrant symmetric ring spectra, in a way which induces equivalences between the corresponding module categories. As the material is spread all over [DAGI], [DAGII], and [DAGIII], this talk is probably best suited for someone who is already familiar with the foundations in [HTT]. Read at least §7.2 and the beginning of §7.4 to see where and how this enters the main text (and the following talk).

References: [DAGI, DAGII, DAGIII], [ABGHR, §7.2,7.4].

Comment: If you think you know what quasi-categories are good for, but have never had a look into Lurie's DAG, this is the talk to go for. If you have peeked into DAG at least once: why not go for talk 10 or 11?

Talk 10: Bundles and Thom spectra in the ∞-category setting. Ideally, the speaker explains the necessary bundle theories of ∞-groupoids, stable ∞-groupoids, and R-lines (§7.3, §7.4, §7.5), presents Thom R-modules and orientations in this setting (§7.6), and compares the result with the approach in §B, which uses ∞-categories of other symmetric monoidal models of spectra, as done in §7.7 after Prop. 7.38.

Reference: [ABGHR, §7.3-7.7].

Comment: You may be inclined to focus on the main ideas which lead to the definition of Thom spectra in this particular model, and blame the previous speaker for leaving out many technical result needed in the course. No remorse: This might be the only reasonable way through it.

Session 6: Comparison and perspective

Talk 11: The equivalence of the different approaches. The aim of this talk is to show that the definition of Thom spectra in Talk 8 (in terms of quasi-categories) is actually equivalent to the definitions in the context of E_∞ and A_∞ ring spectra. One of the main ingredients of the proof is Morita theory in the context of structured ring spectra. This should be reviewed first. Moving on to ∞-categories, ideas from Morita theory can be used to classify colimit preserving functors out of the stabilization of the ∞-groupoid valued presheaves on an ∞-category. The latter classification can be used to characterize the ∞-categorical Thom spectrum construction (Corollary 8.13). With this description at hand, it can be shown that it coincides with the the definition given in Talk 6.

Reference: The motivation at the beginning of §8 and §§8.1 - 8.4, and also the slightly different approach in §2.4. The arguments also rely on results from [HTT] and [DAGI] which ideally have been covered in Talks 7 and 9. A reference for Morita theory is [SS03].

Comment: This talk requires some understanding of the previous models and techniques; the approach taken here makes this ideally suited for someone who is interested in Morita theory.

Talk 12: Thom spectra that are symmetric spectra. This talk should present Schlichtkrull's approach to Thom spectra in [Sch08]. It turns out that in symmetric spectra, it is actually possible to construct strictly commutative symmetric ring spectra which model the Thom spectra associated with appropriate input data. The speaker should explain this construction and the preliminaries about symmetric spectra and $\mathcal{I}$-spaces needed for this. It would be good to include some of the examples from [Sch08]. If time permits, it could also be explained how these techniques lead to a symmetric spectrum version of some of the results in Talk 5 (compare [BCS08]).

Reference: [Sch08] and [BCS08].

Comment: As a final chord, this talk presents another model for Thom spectra, which has a rather different flavor: it is very explicit. As it only assumes knowledge of symmetric spectra, but none of the results of the previous talks. Therefore, this talk is fairly independent of the rest – unless you intend to explain the connection to the results in [ABGHR, § 5].

Session 7: Topic to be determined

The last session is left for talks whose necessity perhaps becomes clear only during the seminar. If you think that more time should be devoted to certain aspects of the theory or certain applications, feel free to suggest so to the organizers. One such topic might be the comparison with the Thom spectra of Lewis, May and Sigurdsson. Another talk might be about results of Mahowald and others which recognize that certain spectra are Thom spectra or cannot be Thom spectra. And yet another topic might be Waldhausen's theorem that the units of the sphere spectrum do not split off its K-theory. Last but not least, the discussion of the next topic has to fit into this session, too.

References

[ABGHR] M. Ando, A. J. Blumberg, D. J. Gepner, M. J. Hopkins, and C. Rezk. Units of ring spectra and Thom spectra, 2008. `arXiv:0810.4535`.

[BCS08] A. J. Blumberg, R. L. Cohen, and C. Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space, 2008. `arXiv:0811.0553`.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[EM06] A. D. Elmendorf and M. A. Mandell. Rings, modules, and algebras in infinite loop space theory. *Adv. Math.*, 205(1):163–228, 2006. `DOI:10.1016/j.aim.2005.07.007`.

[LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure, `DOI:10.1007/BFb0075778`.

[HTT] J. Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. `http://www-math.mit.edu/~lurie/papers/highertopoi.pdf`.

[DAGI] J. Lurie. Stable infinity categories, 2009. Preprint, available at `http://www-math.mit.edu/~lurie/papers/DAG-I.pdf`.

[DAGII] J. Lurie. Noncommutative algebra, 2009. Preprint, available at `http://www-math.mit.edu/~lurie/papers/DAG-II.pdf`.

[DAGIII] J. Lurie. Commutative algebra, 2009. Preprint, available at `http://www-math.mit.edu/~lurie/papers/DAG-III.pdf`.

[May09] J. P. May. What precisely are E_∞ ring spaces and E_∞ ring spectra? In *New Topological Contexts for Galois Theory and Algebraic Geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 215–282. Geom. Topol. Publ., Coventry, 2009. `DOI:10.2140/gtm.2009.16.215`.

[MS04] J. E. McClure and J. H. Smith. Operads and cosimplicial objects: an introduction. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 133–171. Kluwer Acad. Publ., Dordrecht, 2004. `arXiv:math/0402117`.

[Rud98] Y. B. Rudyak. *On Thom spectra, orientability, and cobordism*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller, `DOI:10.1007/978-3-540-77751-9`.

[Sch08] C. Schlichtkrull. Thom spectra that are symmetric spectra, 2008. `arXiv:0811.0592`.

[SS03] S. Schwede and B. E. Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003. `DOI:10.1016/S0040-9383(02)00006-X`.

E-mail address: `sagave@math.uni-bonn.de`

E-mail address: `markus.szymik@ruhr-uni-bochum.de`