Summer School

Novikov Morse Theory for Closed 1-Forms

Kloster Steinfeld, 10. — 15. September 2006

Graduiertenkolleg 1150: Homotopy and Cohomology

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Program, Schedule and Application: see http://www.math.uni-bonn.de/people/GRK1150 → study program

1 Announcement

The Graduiertenkolleg GRK 1150 *Homotopy and Cohomology* will hold its next summer school on the topic of *Novikov Morse Theory* between September 10th and 15th at the monastery Steinfeld in the Eifel area, west of Bonn.

All participants must apply; the **deadline for applications** is June 26th, 2006; see details below. The summer school will be in the style of a seminar, i.e. all participants are expected to prepare a talk from the program below.

The places are limited, but external applications are well-come.

The prerequisites are a good knowledge in differential topology, in singular homology and cohomology, and in classical Morse theory.

For information about the location, see http://www.kloster-steinfeld.de

2 Topic

Morse theory is a very appropriate mean to study the topology of a manifold. One studies on a (compact) manifold X of dimension n a smooth function $f : X \to \mathbb{R}$ with only non-degenerate critical points. A non-critical point can be pushed along the (negative) gradient flow towards a minimum of f as long as no critical point comes in its way. If f has only one critical point, we expect X to be contractible; if f has only two critical points, we expect X to be a sphere. In general, we expect the topology (or rather the homotopy type, or the homology type) of X to be determined by the number, the indices, and the interconnections of critical points via flow lines.
See not only the textbooks [Milnor, Matsumoto] for this, but also the surveys [Bott-1982, Bott-1988].

In Morse theory it is not really the function f, but its gradient vector field (or gradient flow), or dually the 1-form $\omega = \mathrm{d}f$, which is used to define critical points, their index, a cell or handle decomposition and to derive the famous Morse inequalities. Thus it is natural to ask for a theory where one starts with a closed 1-form ω instead of a function f to arrive at similar or refined theorems; one assumes that ω has only Morse-Type zeroes. This is Novikov-Morse theory. Note that a Morse function gives the 1-form $\omega = \mathrm{d}f$; since this is an exact form, its cohomology class $\xi = [\omega]$ in the first DeRham cohomology $H^1_{dR}(X) = H^1(X;\mathbb{R})$ is trivial. We will see that most results depend only on a cohomology class $\xi \in H^1(X;\mathbb{R})$ and that classical Morse theory is the special case $\xi = 0$.

The main results of classical Morse theory are a relation between the number $c_j(f)$ of critical points of index j of f and the Betti numbers $b_j^{\mathbb{F}} := \dim_{\mathbb{F}} H_j(X;\mathbb{F})$, for $j = 0, 1, \ldots, n$. The Morse inequalities are

$$\begin{aligned} b_j^{\mathbb{F}} &\leq c_j(f) &(1)\\ \sum_{k=0}^{j} (-1)^{j-k} b_k^{\mathbb{F}} &\leq \sum_{k=0}^{j} (-1)^{j-k} c_k(f) &(2) \end{aligned}$$

for $j = 0, 1, \ldots, n$ and any field coefficients $\mathbb{F}$.
One can rephrase this by setting $b_j := b_j^{\mathbb{Q}}$, which is the rank (i.e. the minimal number of generators) of the free part of the integral homology $H_j(X;\mathbb{Z})$, and setting $q_j(X)$ to be the minimal number of generators of the torsion part of $H_j(X;\mathbb{Z})$. Then we have for $j = 0, 1, \ldots, n$:

$$\begin{aligned} b_j + q_j + q_{j-1} &\leq c_j(f) &(3)\\ q_j + \sum_{k=0}^{j} (-1)^{j-k} b_k &\leq \sum_{k=0}^{j} (-1)^{j-k} c_k(f) &(4) \end{aligned}$$

The main result of Novikov Morse theory will be an analogue result to the last two equations, where $c_j(f)$ is replaced by the number of critical points $c_j(\omega)$ of a closed 1-form of

Morse type, and the new Novikov Betti numbers $b_j(\xi)$ and Novikov torsion numbers $q_j(\xi)$ are defined as follows. The cohomology class $\xi = [\omega] \in H^1(X;\mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(\pi_1(x),\mathbb{R})$ defines a homomorphism $g \mapsto t^{\xi(g)}$ from the group ring $\mathbb{Z}[\pi_1(X)]$ to the Novikov ring $\mathcal{N} = \mathrm{Nov}(\mathbb{R})$ consisting of all formal power series $\sum_{\gamma\in\mathbb{R}} n_\gamma t^\gamma$ in one variable t, with integral coefficients n_γ such that for any $a \in \mathbb{R}$ there are only finitely many $\gamma > a$ with $n_\gamma \neq 0$. We therefore have a local coefficient $\mathcal{L}_\xi$ system associated to ξ; and we define the Novikov homology by $H_j(X;\mathcal{L}_\xi)$. It is important that the Novikov ring $\mathcal{N}$ is a principal ideal domain and that $H_j(X;\mathcal{L}_\xi)$ is a finitely generated module over $\mathcal{N}$. Now finally, $b_j(\xi)$ resp. $q_j(\xi)$ is defined to be the minimal number of generators of the free part resp. of the torsion part of $H_j(X;\mathcal{L}_\xi)$, regarded as modules over $\mathcal{N}$.

These numbers satisfy $b_j(\xi) \leq b_j(0) = b_j$ and $\sum_j^n (-1)^j b_j(\xi) = \sum_j^n (-1)^j b_j(0) = \chi(X)$, the Euler characteristic of X. The Novikov-Morse inequalities are now

$$
\begin{aligned}
b_j(\xi) + q_j(\xi) + q_{j-1}(\xi) &\leq c_j(\omega) &(5)\\
q_j(\xi) + \sum_{k=0}^{j} (-1)^{j-k} b_k(\xi) &\leq \sum_{k=0}^{j} (-1)^{j-k} c_k(\omega) &(6)
\end{aligned}
$$

for any closed 1-form ω of Morse-type and cohomology class $\xi = [\omega]$, and $j = 0, 1, \ldots, n$. See [Novikov-1981, Novikov-1982] for the origins of this theory.

Further applications of this theory include a signature theorem, symplectic circle actions, and the Lusternik-Schnirelmann category of the manifold.

List of talks

1. Classical Morse theory

Since Novikov Morse theory generalizes classical Morse theory, it might be useful to give first a review of Morse theory. Define the notion of a Morse function $f : X \to \mathbb{R}$ and its gradient vector field ω and gradient flow, its critical points, their non-degeneracy and their index. Construct the Morse complex $C_\bullet(f)$, with the critical points as basis, and define the boundary operator ∂ via the gradient flow connections of critical points. Construct the cell decomposition and handle decomposition associated of f. This leads to the Morse inequalities between the Betti numbers b_j of the manifold and the numbers of critical points $c_j(f)$ of the Morse function as mentioned above.
References: [Bott-1982], [Bott-1988], [Matsumoto], [Milnor].

2. Closed 1-forms

The geometry of a Morse function f depends only on its differential $\omega = \mathrm{d}f$. This may suggest to extend Morse theory from exact 1-forms to closed 1-forms which are not necessarily exact, — and this is what Novikov's generalisation is about. Explain how circle-valued Morse theory is a special case of this principle, proof [Farber, Lemma 2.1]. Give a quick

reminder of DeRham cohomology and its isomorphism to singular cohomology with real coefficients $H^*(X;\mathbb{R})$. Explain the period homomorphism $\mathrm{Per}_\xi : H_1(X;\mathbb{Z}) \to \mathbb{R}$ associated to $\xi = [\omega]$. Define its rank and image group $\Gamma \leq \mathbb{R}$. Consider the covering manifold associated to the kernel of the period homomorphism, where the pull-back of ω is exact.
Reference: [Farber, 2.1].

3. Novikov rings

Define the Novikov ring $\mathrm{Nov}(\Gamma)$ of an additive subgroup Γ of $\mathbb{R}$. Describe its group of units, explain that this is a principal ideal domain and its flatness properties. The Novikov ring contains the group ring $\mathbb{Z}[\Gamma]$ and the inverses of some of its elements. Therefore, it contains the corresponding localisation $\mathcal{R}(\Gamma)$. Explain that it is a principal ideal domain, and that the Novikov ring is flat over $\mathcal{R}(\Gamma)$.
The construction of the universal Novikov complex works over $\mathbb{Z}[\pi]$-algebras in which certain square matrices are invertible. Show that $\mathcal{R}(\Gamma)$ (and hence the Novikov ring) satisfies this condition. Explain that there is a universal ring for this situation, the Cohn localisation. Other examples are given by the ring $\mathbb{C}$, using transcendental numbers, and the Novikov-Sikorav completion of the group ring.
References: [Farber, 1.2, 1.3, and 3.1].

4. Novikov homology

Explain local coefficient systems $\mathcal{L}$, and homology with local coefficients $H_*(X;\mathcal{L})$, and the relation to the (co)homology of (universal) coverings. Introduce the local system $\mathcal{L}_\xi$ associated to a cohomology class $\xi \in H^1(X;\mathbb{R})$. Explain the notions of monodromy and flatness of bundles. Define the Novikov (co)homology as $H_*(X;\mathcal{L}_\xi)$.
Reference: [Farber, 1.4].

5. Novikov numbers

Define the Novikov Betti numbers $b_j(\xi)$ and the Novikov torsion numbers $q_j(\xi)$ of a closed 1-form ω resp. its cohomology class $\xi = [\omega]$ in three equivalent ways [Farber, 1.5.1–1.5.3]. Prove the main properties, in particular show that the Novikov Betti numbers are bounded by the usual Betti numbers and are equal to the usual Betti numbers in case $\xi = 0$, and that the Novikov Euler characteristic equals the usual Euler characteristic. Discuss some complexes with two 1-cells and one 2-cell.
References: [Farber, 1.5 and 1.6].

6. The geometry of Novikov theory

A Riemannian metric on a manifold X associates to a closed 1-form ω a vector field and thus a flow. (Do not mention that a symplectic form does the same. This will be important later, but might be confusing at this time.) The flow lines of these vector fields may differ from those of an exact 1-form in two ways: there may be closed orbits, and there may be

homoclinic orbits. Give the intuitive idea behind the Novikov complex, which uses the lift of the vector field and its flow to the covering determined by the 1-form, and explain how one may try to use equivariant intersection numbers to define the Novikov complex.
Reference: [Farber, 2.2].

7. The universal Novikov complex I

Construct the universal complex in the rank 1 case. While not logically necessary, it might be helpful to see this before the general case. It uses the chain collapse operation. Make sure to emphasize the appearance of the localisation.
Reference: [Farber, 4.1 and 4.2].

8. The universal Novikov complex II

The purpose of this and the following talk is to construct the universal complex in the general case. For motivation, refer to the rank 1 case as often as possible.
Reference: [Farber, first part of 4.3].

9. The universal Novikov complex III

See previous talk.
Reference: [Farber, second part of 4.3].

10. The Novikov inequalities

The existence of a geometric model for the chain complex of a manifold leads to a connection between the geometry and the homotopy type of the underlying manifold. Mention the analogous situation for the cellular complex of a CW-decomposition. In the case at hand, one obtains a relationship between the number of singularities of the closed 1-form ω and the Novikov-Betti and Novikov torsion numbers $b_j(\xi)$ resp. $q_j(\xi)$ of its cohomology class $\xi = [\omega]$, as stated above.
Reference: [Farber, 1.1, 2.3, perhaps 3.2 and 3.3].

11. Examples

At this time of the program, everybody will be curious whether the theory can be applied to some non-trivial examples. Give three of these to illustrate the techniques learned so far: mapping tori (again), 3-manifolds obtained by surgery along connected sums of trefoil knots, and 3-manifolds obtained by adding a handle to a lens space.
Reference: [Farber, 3.4].

12. Symmetries I: equivariant Novikov theory

Discuss basic 1-forms for actions of a compact Lie group on X. Define equivariant Novikov numbers as the Novikov numbers of the Borel quotient. Prove the equivariant Novikov inequalities, asserting that the equivariant Morse data dominate the equivariant Novikov data.
References: [Farber, 7.1, 7.2].

13. Symmetries II: symplectic circle actions

On a symplectic manifold X, the symplectic form induces a correspondence between closed 1-form and symplectic vector fields. In particular, a symplectic circle action determines a symplectic vector field and hence a closed 1-form. This is to be thought of as a generalised moment map, being the differential of the moment map in the case of a Hamilton action. Prove that in this situation, the equivariant Novikov Betti numbers of the generalised moment map are determined by the Betti numbers of the fixed point set.
Reference: [Farber, 7.3].

14. Symmetries III: a signature theorem

Prove that the signature of a symplectic manifold with a symplectic circle action which has only isolated singularities is determined by the equivariant Novikov numbers of its generalised moment map.
Reference: [Farber, 7.4].

15. Exactness of the Novikov inequalities

If $\pi_1(X)$ is infinite cyclic, and the dimension n of the manifold X is at least 6, there is in every cohomology class ξ a form ω such that the first of the Novikov-Morse inequalities is indeed an equality.
This generalises Smale's theorems which lead to the proof of the Poincare conjecture in high dimensions. And it re-proves the Brouwder-Levine result which characterises mapping tori of simply-connected manifolds.
The proof of the main theorem would take too long, so emphasize the interrelationship of the results.
Reference: [Farber, 8.1].

16. Lusternik-Schnirelmann category for 1-forms

The Lusterik-Schnirelmann category $\mathrm{cat}(X)$ of a space X is the minimal number l such that there are $l+1$ subsets $F_0, F_1, \ldots, F_l$ covering X with null-homotopic inclusions $h_k : F_k \to X$. The connection to Morse theory is that any Morse function $f : X \to \mathbb{R}$ on a manifold X must have at least $\mathrm{cat}(X) + 1$ critical points.
Given a 1-form ω on X one can replace the condition of h_0 being null-homotopic by the weaker condition that for the path integrals $\int_{\gamma_x} \omega \leq -N$ holds for any $x \in F_0$ and any positive number N; here $\gamma_x(t) = h_0(x,t)$ is the track curve of the point x under the homotopy h_0 with $h_0(x,0) = x$. This new number $\mathrm{cat}(X,\xi)$ depends on the homology class $\xi = [\omega]$, and one has $\mathrm{cat}(X,\xi) \leq \mathrm{cat}(X)$. Explain and prove [Farber, Theorem 10.14].
References: [Farber, 10.1–10.4].

References

[Bott-1982] R. Bott: *Lectures on Morse theory, old and new.* Bull. Amer. Math. Soc. (N.S.) 7 (1982) 331–358.

[Bott-1988] R. Bott: *Morse theory indomitable.* Publ. Math. Inst. Hautes Études Sci. 68 (1988) 99–114.

[Farber] M. Farber: *Topology of Closed One - Forms.* Mathematical Surveys vol. 108, American Mathematical Society (2004).

[Matsumoto] Y. Matsumoto: *An Introduction to Morse Theory.* Translations of Mathematical Monographs vol. 208, American Math. Soc. (2002).

[Milnor] J. Milnor: *Morse Theory.* Annals of Mathematics Studies vol. 51, Princeton University Press (1963).

[Novikov-1981] S.P. Novikov: *Multi-valued functions and functionals: An analogue of Morse theory.* Soviet Math. Doklady 24 (1981), 222-226.

[Novikov-1982] S.P. Novikov: *The Hamiltonian formalism and a multi-valued analogue of Morse theory.* Russian Mathematical Surveys 35:5 (1982), 1-56.

3 Schedule

Arrival: Sunday, September 10th, 2006, before dinner at 18:00,
bus from Bonn to Steinfeld departs at 16:00 from Beringstrasse.

Daily schedule:			Monday	Tuesday	Wednesday	Thursday	Friday
Breakfast	:	08:00 - 09:00					
Talk	:	09:00 - 10:00	1	5	9	11	15
Talk	:	10:30 - 11:30	2	6	10	12	16
Lunch	:	12:00 - 13:30					
Coffee	:	14:30 - 15:00					
Talk	:	15:00 - 16:00	3	7	-	13	
Talk	:	16:30 - 17:30	4	8	-	14	
Dinner	:	18:00 - 19:00					

Wednesday afternoon : free or excursion.

Departure: Friday, September 15th, 2006 after lunch,
bus from Steinfeld to Bonn departs at 14:00.

4 Application

Accommodation

Kloster Steinfeld is an active monastery, founded around the year 920 A.D., located in a very beautiful region to the west of Bonn. It is known for its organ and its Sunday afternoon concerts. It has a boarding school and offers seminar rooms and accommodation for external seminars as ours. Most rooms are double rooms; some rooms have their own bathroom and shower. We will be offered breakfast, lunch and dinner.

Transportation

There will be a bus from Bonn to Kloster Steinfeld (leaving on Sunday, September 10th, 2006, at 16:00 from Beringstrasse 1), and back from Kloster Steinfeld to Bonn (leaving Kloster Steinfeld on Friday, September 15th, 2006, after lunch at 14:00). The bus ride takes about 90 minutes.

Costs

The costs of 220,– Euro for the week include accommodation in a double room, the meals and the transportation from Bonn to Steinfeld (Sunday) and back (Friday). If you will occupy a single room, please add 50,– Euro.
For fellows of the GRK 1150 the costs, except for the single room surcharge of 50,– Euro, are covered.

Application

To apply, please send an email to the office of the GRK 1150 (grk1150@math.uni-bonn.de),

containing the following:
- your name,
- your university,
- your supervisor,
- two talks from the program above you are willing to prepare,
- whether you need a single room,
- whether you will use the bus.

The **deadline** is June 26th, 2006.

Questions
For questions about the organisation contact Frau S. George at the office of the GRK 1150 (email: grk1150@math.uni-bonn.de).
For questions concerning the talks contact Prof. Bödigheimer (email: boedigheimer@math.uni-bonn.de) or Dr. Szymik (email: markus.szymik@ruhr-uni-bochum.de).