Global Analysis 2: Pseudodifferential Operators

Lecture course at the University of Bonn in the WS 2018/2019

Batu Güneysu

Batu Güneysu, Mathematisches Institut der Universität Bonn, Germany
E-mail address: gueneysu@math.uni-bonn.de

1. Basics of Locally Convex Spaces

In this section I have mainly followed the presentation from the manuscript by Matthias Lesch [5]. Classical references are [6, 4]. Proofs have been given in the lecture course.

We fix $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ and understand all vector spaces to be over $\mathbb{K}$. A *topological vector space* X is a vector space which is also a topological space such that the multiplication with scalars

$$\mathbb{K} \times X \longrightarrow X$$

and the addition

$$X \times X \longrightarrow X$$

is continuous. Given a linear space X a map $p : X \to [0, \infty)$ is called a *seminorm*, if $p(ax) = |a|x$, $p(x + y) \leq p(x) + p(y)$ for all $a \in \mathbb{K}$, $x, y \in X$. It follows that $p(0) = 0$, but $p(x) = 0$ need not imply $x = 0$.

Lemma 1.1. *Let X be a linear space and let $(p_i)_{i \in I}$ be a family of seminorms on X. Then there exists a unique topology τ on X with the following two properties:*

- *(X, τ) is a topological linear space,*
- *the family of sets $A \subset X$ of the form*

$$A = \bigcap_{j=1}^{n} \{p_{i_j} < \epsilon_j\}, \quad n \in \mathbb{N}, i_1, \ldots, i_n \in I, \epsilon_1, \ldots, \epsilon_n > 0,$$

forms a basis of neighbourhoods of $0 \in X$.

This leads to:

Definition 1.2. A topological linear space X is called a *locally convex space*, if its topology is induced by a family of seminorms in the sense of the above lemma. Any such family is called a *defining family of seminorms*.

A seminorm p on a LCS X is continuous, if and only if there exists a defining family $(p_i)_{i\in I}$ of seminorms with the following property: there exists $r \in \mathbb{N}$, $i_1, \ldots i_r \in I$, $c > 0$, such that for all $x \in X$ one has

$$p(x) \leq C \max_{j=1,\ldots,r} p_{i_j}(x).$$

Then the family of all continuous seminorms is also a defining family of seminorms for X.

The Hausdorffness of a LCS can be checked easily:

Lemma 1.3. *Assume X is a locally convex space. Then X is Hausdorff, if any only if there exists a defining family $(p_i)_{i\in I}$ of seminorms with the following property: for all $x \in X \setminus \{0\}$ there exists $i \in I$ with $p_i(x) \neq 0$.*

A net $(x_\alpha)_{\alpha\in A}$ in a LCS X is called a *Cauchy net*, if there exists a defining family $(p_i)_{i\in I}$ of seminorms with the following property: for all $i \in I$ there exists $\epsilon > 0$, $\alpha_0 \in A$ such that for all $\alpha, \beta \geq \alpha_0$ one has $p(x_\alpha - x_\beta) < \epsilon$. Convergence to $x \in X$ is defined likewise, and X is called *complete*, if every Cauchy net is convergent.

Apart from normed spaces, the main examples of LCS's in our course will be Frechet spaces (F) and limit Frechet (LF) spaces:

Definition 1.4. A LCS X is called an *F space*, if X is complete, Hausdorff and there exists a countable defining family of seminorms.

Every Banach space is an F space, and any F space carries a translation invariant metric which induces the original topology.

Assume on a linear space X we are given an increasing sequence $X_n \subset X$ of linear subspaces which are all F spaces such that

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

and such that the inclusion map $X_n \to X_{n+1}$ is continuous for all n. Call a seminorm p on X *admissible* w.r.t. (X_n), if $p|_{X_n}$ is continuous for all n. This induces a LCS topology on X.

Definition 1.5. A LCS X is called an *LF space*, if there exists a sequence (X_n) as above which induces the original topology on X. Such a sequence is called a *defining sequence* (for the LF topology on X).

LF spaces need not be Hausdorff, but G. Köthe has proved:

Theorem 1.6. *Let X be an LF space. Then:*
a) X is complete.
b) A sequence $(f_n)_{n\in\mathbb{N}}$ in X converges in the topology of X, if and only if there exists a defining sequence $(X_r)_{r\in\mathbb{N}}$ and $r_0 \in \mathbb{N}$ such that

- *$x_n \in X_{r_0}$ for all (large) $n \in \mathbb{N}$,*
- *$(x_n)_{n\in\mathbb{N}}$ converges in the topology of X_{r_0}.*

c) If Y is a LCS then for a linear map $T : X \to Y$ the following statements are equivalent:

- *T is continuous.*
- *T is sequentially continuous*[1].
- *$T|_{X_n} : X_n \to Y$ is continuous for all $n \in \mathbb{N}$.*

The natural maps of LCS are continuous linear maps (which turn LCS into a category).

Proposition 1.7. *Assume X, Y are LCS's and that $T : X \to Y$ is linear. Then the following statements are equivalent:*

- *T is continuous.*
- *T is continuous at 0.*
- *There exist defining families of seminorms $(p_i)_{i\in I}$, $(q_j)_{j\in J}$ for X and Y, respectively, with the following property:*

$$\forall j \in J \quad \exists r \in \mathbb{N} \quad \exists i_1, \dots, i_r \in I \quad \exists C > 0 \quad \forall x \in X \quad q_j(Tx) \leq C \max_{j=1,\dots,r} p_{i_l}(x).$$

For LCS's X, Y we denote the linear space of continuous linear maps $X \to Y$ with $\mathscr{L}(X, Y)$. Specifically, one has the *space of continuous linear functionals* $X' := \mathscr{L}(X, \mathbb{K})$ *on* X. This space becomes a LCS with the topology given by the family of seminorms $\phi \mapsto |\phi(x)|$, where x runs through X. This is called *weak-*- topology* on X. The Hahn-Banach Theorem states that for all $x \in X \setminus \{0\}$ there exists $\phi \in X'$ with $\phi(x) \neq 0$. The proof uses Zorn's Lemma.

Definition 1.8. Given $T \in \mathscr{L}(X, Y)$, the *transpose* $T' \in \mathscr{L}(Y', X')$ is defined by $T'y'(x) := y'(Tx)$.

A central result is:

Theorem 1.9 (Open mapping theorem)**.** *Let X, Y be LF spaces and let $T \in \mathscr{L}(X, Y)$ be surjective. Then T maps open subsets to open subsets.*

If X, Y are F spaces, then the proof is exactly the same as for Banach spaces (Baire's category Theorem!). The general case is technical and is usually attributed to Köthe.

Corollary 1.10. *Let X, Y be LF spaces and let $T \in \mathscr{L}(X, Y)$ be surjective. then T^{-1} is automatically continuous.*

A consequence of the latter corollary is:

Theorem 1.11 (Closed graph theorem)**.** *Let X, Y be LF spaces and let $T : X \to Y$ be linear. Then the following statements are equivalent:*

- *T is continuous.*
- *The graph $\{(x, y) : y = Tx\} \subset X \times Y$ is a closed subset.*

Another central result is:

---

[1] This is surprising, because X need not be Hausdorff, so one would expect that one has to check continuity on all nets!

Theorem 1.12 (Uniform boundedness principle)**.** *Let X be an LF space and let Y be a LCS. Suppose $(T_i)_{i\in I}$ is a family of continuous linear maps from X to Y which is* pointwise uniformly bounded*, that is, for all $x \in X$ and all continuous seminorms p on Y, there exists $C > 0$ such that for all $i \in I$ one has $p(T_i x) \leq C$. Then for all continuous seminorms p on Y there exists a continuous seminorm q on X such that for all $i \in I$ and all $x \in X$ one has $p(T_i x) \leq q(x)$.*

The following result is a straightforward Corollary to the uniform boundedness principle.

Corollary 1.13. *Let X,Y be F spaces, and let*

$$T : X \times Y \longrightarrow \mathbb{K}$$

be a bilinear and seperatly continuous map. Then T is already continuous.

2. Basic facts on linear operators in Hilbert spaces

We collect some standard facts for unbounded operators on Hilbert spaces. A classical reference is [3].

Let $\mathscr{H}_1, \mathscr{H}_2$ be infinite dimensional seperable complex Hilbert spaces. The underlying scalar products will simply be denoted with $\langle f, g\rangle$, and the induced norms with $\|f\| := \langle f, f\rangle^{1/2}$.

A continuous linear map $A : \mathscr{H}_1 \to \mathscr{H}_2$ is called *unitary*, if A is invertible with $A^{-1} = A^*$. Unitary maps preserve scalar products and thus norms.

We also recall that for a closed subspace M of $\mathscr{H}_1$ one has

$$\mathscr{H}_1 = M \oplus M^{\perp},$$

and that for every continuous linear functional $T : \mathscr{H}_1 \to \mathbb{K}$ there exists a unique $f_T \in \mathscr{H}_1$ such that $T(\psi) = \langle f_T, \psi\rangle$ for all $\psi \in \mathscr{H}_1$. One has

$$|T(\psi)| \leq \|f_T\| \, \|\psi\| \quad \text{ for all } \psi \in \mathscr{H}_1.$$

The assignment $T \mapsto f_T$ induces an antilinear isomorphism of LCS's between $\mathscr{H}_1'$ and $\mathscr{H}_1$ called the *Riesz-Fischer duality theorem*.

Let S be linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ which is defined on some dense subspace $\mathrm{Dom}(S) \subset \mathscr{H}_1$. Then S is called *bounded*, if

$$\inf\{\|S\psi\| : \psi \in \mathrm{Dom}(S), \|\psi\| \leq 1\} < \infty,$$

that is, if $S : \mathrm{Dom}(S) \to \mathscr{H}_2$ is continuous if $\mathrm{Dom}(S) \subset \mathscr{H}_1$ is equipped with subspace topology. Otherwise S is called *unbounded*.

S is called *closed*, if its graph $\{(f, g) : f \in \mathrm{Dom}(S), g = Sf\} \subset \mathscr{H}_1 \times \mathscr{H}_2$ is a closed subset (continuous linear operators are automatically closed), and S is called *closable*, if it has a closed extension.

Lemma 2.1. *If S is closable, then S has a smallest closed extension $\overline{S}$ which is given by*

$$\mathrm{Dom}(\overline{S}) = \{f \in \mathscr{H}_1 : \textit{there ex. sequ. } (f_n) \subset \mathrm{Dom}(S) \textit{ s.t. } f_n \to f \textit{ and s.t. } (Sf_n) \textit{ conv.}\},$$

and then $\overline{S}f = \lim_n Sf_n$.

The adjoint S^* of S is the operator from $\mathscr{H}_2$ to $\mathscr{H}_1$ defined by

$$\mathrm{Dom}(S^*) := \{f \in \mathscr{H}_2 : \text{there ex. } f^* \in \mathscr{H}_1 \text{ w. } \langle f^*, h\rangle = \langle f, Sh\rangle \text{ for all } h \in \mathrm{Dom}(H)\}$$

and then $S^*f := f^*$.

Lemma 2.2. *The operator S^* is always closed. One has*

$$\mathrm{Ran}(S)^\perp = \mathrm{Ker}(S^*).$$

S is called *symmetric*, if $S \subset S^*$ and *self-adjoint*, if $S = S^*$ (both implies $\mathscr{H}_1 = \mathscr{H}_2$). Symmetric operators are closable (as adjoints are closed) and self-adjoint ones are closed.

The *resolvent set* $\mathrm{Res}(S)$ is defined by all $\lambda \in \mathbb{K}$ s.t. $S - \lambda$ is an invertible linear map $\mathrm{Dom}(S) \to \mathscr{H}_2$, such that the *resolvent* $(S-\lambda)^{-1}$, which is a linear map $\mathscr{H}_2 \to \mathrm{Dom}(S)$, is continuous as a linear map $\mathscr{H}_2 \to \mathscr{H}_1$.

Lemma 2.3. *If S is closed, then* $\mathrm{Res}(S)$ *is an open set and if $S - \lambda$ is invertible for some $\lambda \in \mathbb{C}$, then λ is automatically in* $\mathrm{Res}(S)$*, that is, $(S-\lambda)^{-1}$ is continuous as a map from $\mathscr{H}_2 \to \mathscr{H}_1$.*

A point $\lambda \in \mathbb{K}$ is called an *eigenvalue* of S, if $\mathrm{Ker}(S-\lambda) \neq \{0\}$, and then the dimension of this space is called the *multiplicity* of λ. Vectors in $\mathrm{Ker}(S-\lambda)\setminus\{0\}$ are called *eigenvectors*.

The *spectrum* of S is defined by $\mathrm{Spec}(S) := \mathbb{C}\setminus\mathrm{Res}(S)$, a set which contains all eigenvalues of S and which by the above is closed, if S is so.

Lemma 2.4. *Symmetric operators have real eigenvalues, and self-adjoint operators have a real spectrum. A symmetric operator is self-adjoint, if and only if its spectrum is real (!). The closure of a symmetric operator S is self-adjoint, if* $\mathrm{Ker}(S^* \pm i) = \{0\}$*. If the closure of a symmetric operator is self-adjoint, then its closure is its unique self-adjoint extension and the operator is called* essentially self-adjoint*.*

Assume S is self-adjoint for the moment. Then S has a functional calculus, that is, one can define $f(S)$ in a 'consistent' way for every Borel function $f : \mathbb{R} \to \mathbb{C}$, and then $f(S)$ is self-adjoint if f is real-valued, and bounded if f is bounded on the spectrum of S. For example, given $\psi \in \mathscr{H}_1$, $\psi(t) := e^{-itS}\psi$ is the unique solution of the abstract Schrödinger operator $\dot{\psi}(t) = -iH\psi(t)$, $t \in \mathbb{R}$, $\psi(0) = \psi \in \mathrm{Dom}(S)$. If $S \geq c$ for a constant $c \in \mathbb{R}$, that is, $\langle Sf, f\rangle$ (which is always a real number as S is symmetric) is $\geq c\,\|f\|^2$ for all $f \in \mathrm{Dom}(S)$, then one can also define $\psi(t) := e^{-tS}\psi$ which solves the abstract heat equation $\dot{\psi}(t) = -H\psi(t)$, $t > 0$, $\psi(0) = \psi \in \mathscr{H}_1$ (in the Schrödinger case the initial vector has to be in the domain of S, in the heat case the exponential has a smoothing effect, as $S \geq c$!). Moreover, $S \geq c$ is equivalent to $\mathrm{Spec}(S) \subset [c, \infty)$.

A continuous linear map $K : \mathscr{H}_2 \to \mathscr{H}_1$ is called *compact*, if for all ONB $(e_n)_{n\in\mathbb{N}}$ of $\mathscr{H}_1$ and $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{H}_2$ one has $\langle Ke_n, f_n\rangle \to 0$. This is equivalent to requiring that for every bounded sequence ψ_n in $\mathscr{H}_1$ the sequence $K\psi_n$ in $\mathscr{H}_2$ has a convergent subsequence. Compositions of compact linear maps with continuous linear maps are again compact. For example, given a Riemann manifold M and complex metric vector bundles $E, F \to M$ we can canonically construct the complex metric vector bundle

$$E \boxtimes F \longrightarrow M \times M$$

whose fiber at $(x, y) \in M \times M$ is given by $\mathrm{Hom}(E_x, F_y)$, and then every 'integral kernel'

$$k \in \Gamma_{L^2}(M \times M, E \boxtimes F)$$

determines the continuous linear operator

$$K : \Gamma_{L^2}(M, E) \longrightarrow \Gamma_{L^2}(M, F), \quad Kf(x) = \int_M k(x, y) f(y) d\mu(y),$$

where μ denotes the Riemann volume measure. One can easily prove using Fubini that

$$\sum_n |\langle Ke_n, f_n \rangle|^2 < \infty,$$

the defining property of a Hilbert-Schmidt operator, showing that K is compact (so Hilbert-Schmit operators are compact).

Theorem 2.5. *Assume S is closed and that $(S - \lambda)^{-1}$ is compact for some (equivalently: all) $\lambda \in \mathrm{Res}(S)$. Then S has at most countably many eigenvalues all having a finite multiplicity, and the spectrum of S contains no other points. Moroever, there exists an orthonormal basis of $\mathscr{H}_1$ given by eigenvectors of S, and if (λ_j) denotes the eigenvalues, then either $\sharp(\lambda_j) < \infty$ or $|\lambda_j| \to \infty$ as $j \to \infty$.*

The following result is central for the Atiyah-Singer index theorem.

Theorem 2.6. *Assume S is closed and* invertible modulo compact operators*, that is, there exist continuous linear maps $A_1, A_2 : \mathscr{H}_2 \to \mathscr{H}_1$, and compact linear maps $K_j : \mathscr{H}_j \to \mathscr{H}_j$, such that*

$$A_1 S = 1 + K_1, \quad S A_2 = 1 + K_2.$$

Then S is Fredholm*, that is,* $\mathrm{Ker}(S)$ *is finite dimensional,* $\mathrm{Ran}(S)$ *is closed and its codimension in $\mathscr{H}_2$ is finite.*

Fredholm operators have a well-defined index $\mathrm{ind}(S) := \mathrm{dimKer}(S) - \mathrm{codimRan}(S)$.

In principle, the main goal of this course is to show that every elliptic pseudodifferential operator P acting on sections of metric vector bundles over a Riemannian manifold M is invertible modulo smoothing operators. If M is compact this automatically shows that P is Fredholm (as then every smoothing operator is Hilbert-Schmidt and so compact) and together with Rellich's compactness theorem (which we will prove later on) this also implies that P has a compact resolvent, so all implications of the above two Theorems hold true.

3. Function spaces, distributions and the Fourier transform

We collect some functions spaces that are important examples of LCS's and that will be the central objects of the lecture course. We understand all our function spaces to be over $\mathbb{C}$.
In the sequel let $\Omega \subset \mathbb{R}^m$ be an arbitrary open subset, and fix $l \in \mathbb{N}$. Both $\mathbb{R}^m$ and $\mathbb{C}^l$ are equipped with their standard Euclidean scalar products $(\cdot, \cdot)$ and the induced norms $|\cdot|$. Let $e_1, \dots, e_l \in \mathbb{C}^l$ denote the standard basis in $\mathbb{C}^l$.
The Lebesgue measure on Ω is considered to be defined on the Borel sigma algebra of Ω, rather than on the Lebesgue sigma algebra (although this makes no essential difference). The Lebesgue integral of an appropriate function $f : \Omega \to \mathbb{C}^l$ is denoted with symbols such as

$$\int_\Omega f, \quad \int_\Omega f(x)dx,$$

and 'a.e.' will stand for 'Lebesgue almost everywhere'. The Lebesgue measure of a Borel set $A \subset \Omega$ set will be denoted with

$$|A| = \int_A 1 \in [0, \infty]$$

We record the following regularity property of the Lebesgue measure:

Remark 3.1. For every Borel set $A \subset \Omega$ with a finite Lebesgue measure and every $\epsilon > 0$ there exists an open subset $U \subset \Omega$ and a compact set $K \subset \Omega$ such that

$$K \subset A \subset U, \quad |U \setminus K| < \epsilon.$$

An index 'c' in a function space such as $C_c^\infty(\Omega)$ will stand for 'compactly supported'.
Open Euclidean balls will be denoted with

$$B_r(x) = \{y \in \mathbb{R}^m : d(x, y) = |x - y| < r\}, \quad r > 0, \quad x \in \mathbb{R}^m.$$

The closed r-neighbourhood of a set $A \subset \mathbb{R}^m$ is defined by

$$(A)_r := \{x \in \mathbb{R}^m : \inf\{d(x, a) : a \in A\} \leq r\}, \quad r > 0.$$

Example 3.2. Assume $p \in [1, \infty]$. Then the linear space $L^p(\Omega, \mathbb{C}^l)$ of all Lebesgue equivalence classes of Borel maps $f : \Omega \to \mathbb{C}^l$ such that

$$\|f\|_{L^p}^\Omega := \begin{cases} \int_\Omega |f|^p, & \text{if } p < \infty \\ \inf\{C \geq 0 : |f| \leq C \quad \text{a.e. in } \Omega\}, & \text{if } p = \infty \end{cases} < \infty,$$

becomes a Banach space (thus an F space) with respect to $\|\cdot\|_{L^p}^\Omega$. The asserted completeness is a classical result by Riesz and Fischer, which is in fact true on every measure space (and which is proved in every course on Lebesgue integration theory). For $p = 2$ the space $L^p(\Omega, \mathbb{C}^l)$ is in fact a Hilbert space w.r.t.

$$\langle f_1, f_1 \rangle^\Omega := \int_\Omega (f_1, f_2).$$

Example 3.3. For $k \in \mathbb{N}_{\geq 0}$ the linear space

$$C_b^k(\Omega, \mathbb{C}^l) := \Big\{ f \in C^k(\Omega, \mathbb{C}^l) : \partial^\alpha f \text{ is bounded for all } \alpha \in \mathbb{N}^m \text{ with } |\alpha| := \sum_{i=1}^m \alpha_i \leq k \ \Big\}$$

becomes a Banach space with respect to the norm

$$\|f\|^\Omega_{\infty,k} := \sum_{|\alpha|\leq k} \sup_{x\in\Omega} |\partial^\alpha f(x)|.$$

Indeed, given a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$, $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq k$, it follows immediately that for all $x \in \Omega$ the sequence $(\partial^\alpha f_n(x))_n$ is Cauchy in $\mathbb{C}^l$ and thus has a limit which we denote by $F_\alpha(x)$, where we set $f(x) := F_{(0,\dots,0)}(x)$. Then using the triangle inequality (and an $\epsilon/2$ argument) one easily checks that the function $x \mapsto f(x)$ lies in $C_b^k(\Omega, \mathbb{C}^l)$ wit $F_\alpha = \partial^\alpha f$ and

$$\lim_n \|f_n - f\|^\Omega_{\infty,k} = 0.$$

Another important space is the Banach space $C_0^k(\Omega, \mathbb{C}^l)$, which is defined as the closure of $C_c^k(\Omega, \mathbb{C}^l)$ with respect to the norm $\|\cdot\|^\Omega_{\infty,k}$.

Example 3.4. The space $C^\infty(\Omega, \mathbb{C}^l)$ becomes a locally convex space with respect to the family of seminorms

$$p^\Omega_{\alpha,K}(f) := \sup_{x\in K} |\partial^\beta f(x)|, \quad \alpha \in \mathbb{N}^m, \ \ K \subset \Omega \text{ compact.} \tag{1}$$

This locally convex space (that is, the pair given by $C^\infty(\Omega, \mathbb{C}^l)$ and the above topology) plays a distinguished role and is usually

$$\text{denoted by} \quad \mathscr{E}(\Omega, \mathbb{R}^l).$$

This is in fact an F space: Every sequence $(K_r)_{r\in\mathbb{N}}$ of compact subsets of Ω with

$$\bigcup_{r\in\mathbb{N}} K_r = \Omega, \quad K_1 \subset \mathring{K}_2 \subset K_2 \subset \mathring{K}_3 \subset \cdots \tag{2}$$

induces the countable family of (semi)norms p^Ω_{α,K_r}, $\alpha \in \mathbb{N}^m$, $r \in \mathbb{N}$, which is equivalent to (1). The Hausdorff property of $(p^\Omega_{\alpha,K_r})_{\alpha,r}$ is obvious in view of Lemma **??**, and the completeness can be seen much as in the previous example: If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $(p^\Omega_{\alpha,K_r})_{\alpha,r}$, then certainly for all $x \in \Omega$ and all $\alpha \in \mathbb{N}^m$ the sequence $(\partial^\alpha f_n(x))_n$ is a Cauchy sequence in $\mathbb{C}^l$ thus has a limit which we denote with $F_\alpha(x)$, where $f(x) := F_{(0,\dots,0)}(x)$. In fact, $x \mapsto f(x)$ is in $C_c^\infty(\Omega, \mathbb{C}^l)$ with $F_\alpha = \partial^\alpha f(x)$ and one has

$$\lim_n p^\Omega_{\alpha,K_r}(f_n - f) = 0 \quad \text{ for all } \alpha \in \mathbb{N}^m,\ r \in \mathbb{N}.$$

As above, this follows from the subadditivity

$$p^\Omega_{\alpha,K_r}(\psi + \phi) \leq p^\Omega_{\alpha,K_r}(\psi) + p^\Omega_{\alpha,K_r}(\phi) \text{ for all } \psi, \phi \in C_c^\infty(\Omega, \mathbb{C}^l)$$

and an $\epsilon/2$ argument.

In a complete analogy to the smooth case one can consider C^k-functions:

Example 3.5. Given $k \in \mathbb{N}_{k\geq 0}$ the space $C^k(\Omega, \mathbb{C}^l)$ becomes a locally convex space with respect to the family of seminorms $p^{\Omega}_{\alpha,K}$, where $\alpha \in \mathbb{N}^m$ satisfies $|\alpha| \leq k$ and $K \subset \Omega$ is compact. As above one sees that this is in fact an F space.

Example 3.6. Given a compact subset $K \subset \Omega$ set

$$\mathscr{D}_K(\Omega, \mathbb{C}^l) := \left\{ f \in C^\infty(\Omega, \mathbb{C}^l) : \mathrm{supp}(f) \subset K \right\}.$$

Then $\mathscr{D}_K(\Omega, \mathbb{C}^l)$ is a F space with respect to the above defined seminorms $(p^{\Omega}_{\alpha,K})_{\alpha\in\mathbb{N}^m}$, and if $K \subset K'$ are compact subsets of Ω, then the natural embedding[2]

$$\mathscr{D}_K(\Omega, \mathbb{C}^l) \hookrightarrow \mathscr{D}_{K'}(\Omega, \mathbb{C}^l)$$

is continuous. Note that

$$C^\infty_c(\Omega, \mathbb{C}^l) = \bigcup_{K \subset \Omega \text{ compact}} \mathscr{D}_K(\Omega, \mathbb{C}^l).$$

We equip $C^\infty_c(\Omega, \mathbb{C}^l)$ with the family of seminorms p on $C^\infty_c(\Omega, \mathbb{C}^l)$ such that $p|_{\mathscr{D}_K(\Omega,\mathbb{C}^l)}$ is continuous for all compact $K \subset \Omega$. Then $C^\infty_c(\Omega, \mathbb{C}^l)$ together with this family of seminorms is denoted by $\mathscr{D}(\Omega, \mathbb{C}^l)$ and called *the space of test functions on* Ω.

The latter space is in fact an LF space:

Proposition 3.7. *a)* $\mathscr{D}(\Omega, \mathbb{C}^l)$ *is an LF space. In fact, for every sequence* $(K_r)_{r\in\mathbb{N}}$ *of compact subsets of* Ω *with*

$$\bigcup_{r\in\mathbb{N}} K_r = \Omega, \quad K_1 \subset \mathring{K}_2 \subset K_2 \subset \mathring{K}_3 \subset \cdots, \tag{3}$$

the topology on $\mathscr{D}(\Omega, \mathbb{C}^l)$ *is induced by the sequence of F spaces* $(\mathscr{D}_{K_r}(\Omega, \mathbb{C}^l))_{r\in\mathbb{N}}$.
b) A sequence $(f_n)_{n\in\mathbb{N}}$ *in* $\mathscr{D}(\Omega, \mathbb{C}^l)$ *converges to* $f \in \mathscr{D}(\Omega, \mathbb{C}^l)$*, if and only if there exists a compact set* $K \subset \Omega$ *such that*

- $\mathrm{supp}(f_n) \subset K$ *for all* $n \in \mathbb{N}$,
- $\partial^\alpha f_n$ *converges to* $\partial^\alpha f$ *uniformly as* $n \to \infty$ *for all* $\alpha \in \mathbb{N}$.

Proof: a) It only remains to prove that a seminorm p on $\mathscr{D}(\Omega, \mathbb{C}^l)$ is continuous, if and only $p|_{\mathscr{D}_{K_r}(\Omega,\mathbb{C}^l)}$ is continuous for all $r \in \mathbb{N}$. Here '$\Rightarrow$' is trivial, so assume $p|_{\mathscr{D}_{K_r}(\Omega,\mathbb{C}^l)}$ is continuous for all $r \in \mathbb{N}$ and let $K \subset \Omega$ be compact. Pick $n_0 \in \mathbb{N}$ with $K \subset \mathring{K}_{n_0} \subset K_{n_0}$ (this is possible since $(\mathring{K}_r)_{r\in\mathbb{N}}$ is an open covering of K). As $p|_{\mathscr{D}_{K_{n_0}}(\Omega,\mathbb{C}^l)}$ is continuous, we have an estimate of the form

$$p|_{\mathscr{D}_{K_{n_0}}(\Omega,\mathbb{C}^l)} \leq C \max_{j=1,\ldots,s} p^{\Omega}_{\alpha_j,K_{n_0}}$$

for some $C > 0$, $s \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_s \in \mathbb{N}^m$, and so

$$p|_{\mathscr{D}_K(\Omega,\mathbb{C}^l)} \leq C \max_{j=1,\ldots,s} p^{\Omega}_{\alpha_j,K_0} = C \max_{j=1,\ldots,s} p^{\Omega}_{\alpha_j,K},$$

[2]In this course, 'embedding' simply means 'linear and injective'.

showing the continuity of p.
b) This is an immediate consequence of Theorem 1.6 b). ■

Again there is a C^k-variant:

Example 3.8. Given $k \in \mathbb{N}_{\geq 0}$ and a compact subset $K \subset \Omega$ set

$$C^k_K(\Omega, \mathbb{C}^l) := \left\{ f \in C^k(\Omega, \mathbb{C}^l) : \mathrm{supp}(f) \subset K \right\}.$$

Then $C^k_K(\Omega, \mathbb{C}^l)$ is a F space with respect to the seminorms $(p^{\Omega}_{\alpha,K})_{\alpha\in\mathbb{N}^m,|\alpha|\leq k}$, and if $K \subset K'$ are compact subsets of Ω, then the natural embedding

$$C^k_K(\Omega, \mathbb{C}^l) \hookrightarrow C^k_{K'}(\Omega, \mathbb{C}^l)$$

is continuous. Note that as above

$$C^k_c(\Omega, \mathbb{C}^l) = \bigcup_{K \subset \Omega \text{ compact}} C^k_K(\Omega, \mathbb{C}^l).$$

One equips $C^k_c(\Omega, \mathbb{C}^l)$ with the family of seminorms p on $C^k_c(\Omega, \mathbb{C}^l)$ such that $p|_{C^k_K(\Omega,\mathbb{C}^l)}$ is continuous for all compact $K \subset \Omega$. Then $C^k_c(\Omega, \mathbb{C}^l)$ together with this family of seminorms becomes an LF space, which checked with the same arguments as in the smooth case.

Example 3.9. The space of *Schwartz test functions* or *tempered test functions* $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ is the linear space defined by

$$\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) := \left\{ f \in C^\infty(\mathbb{R}^m, \mathbb{C}^l) : \sup_{x\in\mathbb{R}^m} |x^\alpha \partial^\beta f(x)| < \infty \text{ for all } \alpha, beta \in \mathbb{N}^m \right\}.$$

It is an F space with respect to the family of seminorms given by

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^m} |x^\alpha \partial^\beta f(x)|, \quad \alpha, \beta \in \mathbb{N}^m.$$

The Hausdorff property and the completeness will be proved as an exercise. The Schwartz space is the natural space on which the Fourier transform is a priori defined (later!).

Example 3.10. Assume $p \in [1, \infty]$. Then the linear space $L^p_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$ of all Lebesgue equivalence classes of Borel maps $f : \Omega \to \mathbb{C}^l$ with[3]

$$\|1_K f\|^{\Omega}_{L^p} < \infty \quad \text{for all compact } K \subset \Omega,$$

becomes a locally convex space with respect to the family of seminorms

$$\|1_K \cdot\|^{\Omega}_{L^p}, \quad K \subset \Omega \text{ compact},$$

called the *space of locally L^p-integrable 'functions' on* Ω. It is in fact a F space, which is checked by picking a compct exhaustion on Ω as above.

[3] that with 1_A the indicator function of a subset A

Example 3.11. Assume $p \in [1, \infty]$. Define the linear space

$$L^p_c(\Omega, \mathbb{C}^l) := \left\{ f \in L^p(\Omega, \mathbb{C}^l) : f \text{ has a compactly supported Lebesgue representative} \right\}$$

of *compactly supported L^p-'functions' on* Ω. Given $K \subset \Omega$ compact define

$$L^p_K(\Omega, \mathbb{C}^l) :=$$
$$\left\{ f \in L^p(\Omega, \mathbb{C}^l) : f \text{ has a Lebesgue representative which is compactly supported in } K \right\}.$$

Then $L^p_K(\Omega, \mathbb{C}^l)$ is a Banach space with respect to $\|1_K \cdot\|^{\Omega}_{L^p}$ and we turn $L^p_c(\Omega, \mathbb{C}^l)$ into a locally convex space with the family of seminorms p on $L^p_c(\Omega, \mathbb{C}^l)$ such that

$$p|_K := p|_{L^p_K(\Omega, \mathbb{C}^l)}$$

is continuous for all compact $K \subset \Omega$. It is then easily checked (with arguments as above) that $L^p_c(\Omega, \mathbb{C}^l)$ is in fact an LF space.

Remark 3.12. In the sequel, whenever there is no danger of confusion, we will omit 'Ω' in the notation for the various (semi-)norms, so $\|\cdot\|_{L^p} := \|\cdot\|^{\mathbb{R}^m}_{L^p}$ etc. Furthermore, if $l = 1$ we are going to omit $\mathbb{C}^l$ in the notation for the function spaces, so $L^p(\Omega) := L^p(\Omega, \mathbb{C})$ etc.

Our next aim is to determine nice dense subspaces of these spaces. The key tool in this context is a construction that goes back to K. Friedrichs (1940!), usually referred to as Friedrichs mollifiers. The essential idea is to convolute a 'bad function' with a nice function (smooth and compactly supported). To this end, we are going to need precise L^p-estimates for convolutions.

Given Borel functions $f : \mathbb{R}^m \to \mathbb{C}$, $g : \mathbb{R}^m \to \mathbb{C}^l$ with

$$\int_{\mathbb{R}^m} |f(y) g(x - y)| dy < \infty \qquad \text{for every or (a.e.) } x \in \mathbb{R}^m,$$

we define their convolution to be the function

$$f * g : \mathbb{R}^m \longrightarrow \mathbb{C}^l, \quad f * g(x) := \int_{\mathbb{R}^m} f(y) g(x - y) dy.$$

Then from the transformation formula for Lebesgue integrals we have

$$f * g(x) = \int_{\mathbb{R}^m} f(x - y) g(y) dy.$$

Proposition 3.13. *Let $p \in [1, \infty]$, let $f \in L^1(\mathbb{R}^m)$, $g \in L^p(\mathbb{R}^m, \mathbb{C}^l)$. Then $f * g(x)$ is well-defined for a.e. $x \in \mathbb{R}^m$, and one has the estimate*

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

Proof: Case $p = \infty$: Obviously

$$|f * g(x)| \leq \int_{\mathbb{R}^m} |f(y)||g(x - y)| dy \leq \|f\|_{L^1} \|g\|_{L^\infty}.$$

Case $p < \infty$: We are going to prove that for all $x \in \mathbb{R}^m$ one has

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} |f(y)||g(x - y)| dy \right)^p dx \leq \|f\|^p_{L^1} \|g\|^p_{L^p},$$

which also shows a posteriori that $f * g$ is well-defined a.e. in $\mathbb{R}^m$ (if the integral of nonnegative function taking a priori values in $[0, \infty]$ is finite, then the function must be finite a.e.).

Case $p = 1$: One has

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |f(y)||g(x-y)| dy dx = \|f\|^p_{L^1} \|g\|^p_{L^1}$$

by Fubini-Tonelli.

Case $1 < p < \infty$: Set $q := p/(p-1)$, so $1/q + 1/p = 1$. Then by Hoelder's inequality

$$\begin{aligned}\left(\int_{\mathbb{R}^m} |f(y)||g(x-y)| dy \right)^p &= \left(\int_{\mathbb{R}^m} |f(x-y)|^{1/p} |g(y)| |f(x-y)|^{1/q} dy \right)^p \\ &\leq \left(\int_{\mathbb{R}^m} |f(x-y)||g(y)|^p dy \right) \left(\int_{\mathbb{R}^m} |f(y)| dy \right)^{p/q},\end{aligned}$$

so that

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} |f(y)||g(x-y)| dy \right)^p dx \leq \|f\|^{1+p/q}_{L^1} \|g\|^p_{L^p},$$

completing the proof in view of $1 + p/q = p$.

■

More generally, one can prove *Hausdorff-Young's inequality*, which states that for all $p, q, r \in [1, \infty]$ with $1/p + 1/q = 1/r + 1$ one has

$$\|f * g\|_{L^r} \leq \|f\|_{L^q} \|g\|_{L^p},$$

whenever the right hand side is finite. We will not need the latter generalization, which also holds on general unimodular groups, if one replaces $x - y$ with $y^{-1}x$ and the Lebesgue measure with the (up to a constant uniquely determined) Haar measure. If $p, q, r \in (1, \infty)$, then one can even prove an inequality of the form

$$\|f * g\|_{L^r} \leq c_{p,q} \|f\|_{L^q} \|g\|_{L^p},$$

where $c_{p,q} < 1$. This is a highly nontrivial result by Brascamp and Lieb from 1976.

Definition 3.14. Let $0 \leq \varrho \in C^\infty_c(\mathbb{R}^m)$ be such that

- $\mathrm{supp}(\varrho) \subset B_1(0)$.
- $\int_{\mathbb{R}^m} \varrho = 1$.

For all $\lambda > 0$ define $0 \leq \varrho_\lambda \in C^\infty_c(\mathbb{R}^m)$ by the scaling $\varrho_\lambda(x) := \lambda^{-m} \varrho(x/\lambda)$. Then for all $p \in [1, \infty)$, all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{C}^l)$, the net of functions

$$(\varrho_\lambda * f)_{\lambda > 0}$$

is called the *Friedrichs mollification of f* (with respect[4] to ϱ).

Note that the definition of ϱ_λ is justified by the fact that the support of this function is controlled according to $\mathrm{supp}(\varrho_\lambda) \subset B_\lambda(0)$, while still $\int_{\mathbb{R}^m} \varrho_\lambda = 1$.

[4]The particular choice of ϱ with the above properties will play no role in the sequel.

Remark 3.15. In the above sitution, the following assertions hold true for all $\lambda > 0$:
i) One has $\varrho_\lambda * f \in C^\infty(\mathbb{R}^m, \mathbb{C},)$. This follows from

$$\varrho_\lambda * f(x) = \int_{\mathbb{R}^m} \varrho_\lambda(x-y) f(y) dy$$

and differentiating under the integral (make yourself clear that the latter is possible!). In fact, for all $\alpha \in \mathbb{N}^m$ one has $\partial^\alpha(\varrho_\lambda * f) = (\partial^\alpha \varrho_\lambda) * f$, and if $f \in C^{|\alpha|}(\mathbb{R}^m, \mathbb{C}^l)$ then the latter is also equal to $\varrho_\lambda * (\partial^\alpha f)$, which follows from integrating by parts.
iii) One has $\mathrm{supp}(\varrho_\lambda * f) \subset \big(\mathrm{supp}(f)\big)_\lambda$.

A first application of Friedrichs mollifiers is provided by the construction of nice cut-off functions:

Proposition 3.16. *Let $K \subset \Omega$ be compact. Then there exists a function $\chi \in C_c^\infty(\Omega)$ with $0 \leq \chi \leq 1$ and $\chi = 1$ in an open neighbourhood of K.*

Proof: Pick open and relatively compact subsets $V_1, V_2 \subset \Omega$ (meaning that the closure of V_j is a compact subset of Ω) which satisfy $K \subset V_1 \subset \overline{V_1} \subset V_2$. Then for all $\lambda > 0$ one has $\varrho_\lambda * 1_{V_2} \in C^\infty(\mathbb{R}^m)$, and clearly

$$0 \leq \varrho_\lambda * 1_{V_2}(x) = \int_{V_2} \varrho_\lambda(x-y) dy \leq \int_{\mathbb{R}^m} \varrho_\lambda(y) dy = 1 \text{ for all } x \in \mathbb{R}^m.$$

If λ is sufficiently small (depending on V_2), the function $\varrho_\lambda * 1_{V_2}$ is compactly supported in Ω. Moreover, if λ is sufficiently small (depending on V_j and K) we have $B_\lambda(x) \subset V_2$ for all $x \in V_1$ and so $1_{V_2}|_{B_\lambda(x)} = 1$ for all $x \in V_1$. Thus for such x's we have

$$\varrho_\lambda * 1_{V_2}(z) = \int_{\mathbb{R}^m} \varrho_\lambda(z-y) 1_{V_2}(y) dy = \int_{B_\lambda(x)} \varrho_\lambda(y) dy = 1.$$

Thus we can pick some $\lambda_{K,V_j} > 0$ such that $\chi := (\varrho_\lambda * 1_{V_2})|_\Omega$ does the job for all $0 < \lambda < \lambda_{K,V_j}$. ■

All density results can be derived from the following two auxiliary results:

Proposition 3.17. *For all $f \in C(\mathbb{R}^m, \mathbb{C}^l)$ one has $\varrho_\lambda * f \to f$ as $\lambda \to 0+$ in the topology of $C(\mathbb{R}^m, \mathbb{C}^l)$.*

Proof: Given an arbitrary compact subset $K \subset$ we have to show that $\varrho_\lambda * f \to f$ uniformly on K. For all $\lambda < 1$ and all $x \in \mathbb{R}^m$ we have

$$|\varrho_\lambda * f(x) - f(x)| = \left| \int_{B_\lambda(0)} \varrho_\lambda(y)(f(x-y) - f(x)) dy \right| \leq \sup_{y \in B_\lambda(0)} |f(x-y) - f(y)|,$$

where we have used

$$\int_{B_{\lambda(0)}} \varrho_\lambda(y) dy = 1.$$

Thus,

$$\sup_{x \in K} |\varrho_\lambda * f(x) - f(x)| \leq \sup_{x \in K, y \in B_\lambda(0)} |f(x-y) - f(x)| \quad \text{ for all } x \in K, \lambda > 0,$$

which goes to zero as $\lambda \to 0+$, as $f|_K$ is uniformly continuous. ■

Proposition 3.18. *For all $p \in [1,\infty)$, $f \in L^p(\mathbb{R}^m, \mathbb{C}^l)$ one has $\varrho_\lambda * f \to f$ as $\lambda \to 0+$ in the topology of $L^p(\mathbb{R}^m, \mathbb{C}^l)$.*

Proof: We can assume $l = 1$. Note first that indeed $\varrho_\lambda * f \in L^p(\mathbb{R}^m)$ for all $\lambda > 0$, which follows e.g. from Lemma 3.13.
We first show that $C_c^\infty(\mathbb{R}^m)$ is dense. To this end, as simple functions of the form $\sum_{j=1}^r c_j 1_{A_j}$ with $c_j \in \mathbb{C}$ and A_j Borel with finite Lebesgue measure are dense, given a Borel set $A \subset \mathbb{R}^m$ with finite Lebesgue measure and $\epsilon > 0$ we need to find $h \in C_c^\infty(\mathbb{R}^m)$ with

$$\|h - 1_A\|_{L^p}^p < \epsilon.$$

By the regularity of the Lebesgue measure we can pick $U \subset \mathbb{R}^m$ open and $K \subset \mathbb{R}^m$ compact with

$$K \subset A \subset U, \quad |U \setminus K| < \epsilon.$$

By Proposition 3.16 we can find $h \in C_c(U) \subset C_c(\mathbb{R}^m)$ with $0 \leq h \leq 1$ in U and $h = 1$ in K. Thus one has $h = 1 = 1_A$ in K, $h = 0 = 1_A$ in $\mathbb{R}^m \setminus U$, and $|h - 1_A| \leq 1$ in $U \setminus K$, so that

$$\|h - 1_A\|_{L^p}^p = \int_{U\setminus K} |h - 1_A|^p \leq |U \setminus K| < \epsilon.$$

Now given $f \in L^p(\mathbb{R}^m)$, $\epsilon > 0$, pick $h \in C_c(\mathbb{R}^m)$ with $\|f - h\|_{L^p}^p < \epsilon/3$. Then we can write

$$\|f - \varrho_\lambda * f\|_{L^p}^p \leq \|f - h\|_{L^p}^p + \|h - \varrho_\lambda * h\|_{L^p}^p + \|\varrho_\lambda * (h - f)\|_{L^p}^p \tag{4}$$

$$\leq \|f - h\|_{L^p}^p + \|h - \varrho_\lambda * h\|_{L^p}^p + \|h - f\|_{L^p}^p \tag{5}$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3, \tag{6}$$

where $\|h - \varrho_\lambda * h\|_{L^p}^p$ can be made $< \epsilon/3$ for small λ by Proposition 3.17. ■

Proposition 3.18 fails for $p = \infty$ (exercise).
The latter two results remain true on open subsets of $\mathbb{R}^m$: To this end,
given a function $f : \Omega \to \mathbb{C}^l$ we denote by $\underline{f} : \mathbb{R}^m \to \mathbb{C}^l$ its extension by 0 to $\mathbb{R}^m$.

Proposition 3.19. *For all $h \in C(\Omega, \mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{h})|_\Omega \to h$ as $\lambda \to 0+$ in the topology of $C(\Omega, \mathbb{C}^l)$.*

Proof: Given a compact set $K \subset \Omega$ pick a continuous function $\psi : \mathbb{R}^m \to \mathbb{C}^l$ which coincides with h in an open neighboorhood $V \subset \Omega$ of K. For example you can use Proposition 3.16 to do that, or simply use Urysohn's lemma. Then for all sufficiently small $\lambda > 0$ (depending on K and V) we have $B_\lambda(x) \subset V$ for all $x \in K$, thus

$$\varrho_\lambda * \underline{h}(x) = \int_{B_\lambda(x)} \varrho_\lambda(y)\underline{h}(y)y = \varrho_\lambda * \psi(x)$$

which tends to h uniformly in K as $\lambda \to 0+$ by Proposition 3.17.

■

Proposition 3.20. *For all $p \in [1,\infty)$, $h \in L^p(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{h})|_\Omega \to h$ as $\lambda \to 0+$ in the topology of $L^p(\Omega,\mathbb{C}^l)$.*

Proof: Apply Proposition 3.18 with $f = \underline{h}$. ■

We collect all possible convergence results for Friedrichs mollifiers in the following Theorem:

Theorem 3.21. *i) For all $p \in [1,\infty)$, $f \in L^p(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $L^p(\Omega,\mathbb{C}^l)$.*
ii) For all $p \in [1,\infty)$, $f \in L^p_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $L^p_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$.
iii) For all $p \in [1,\infty)$, $f \in L^p_c(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $L^p_c(\Omega,\mathbb{C}^l)$.
iv) For all $k \in \mathbb{N}_{\geq 0}$, $f \in C^k_c(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $C^k_c(\Omega,\mathbb{C}^l)$.
v) For all $f \in \mathscr{D}(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $\mathscr{D}(\Omega,\mathbb{C}^l)$.
vi) For all $k \in \mathbb{N}_{\geq 0}$, $f \in C^k(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $C^k(\Omega,\mathbb{C}^l)$.
vii) For all $f \in \mathscr{E}(\Omega,\mathbb{C}^l)$ one has $(\varrho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of $\mathscr{E}(\Omega,\mathbb{C}^l)$.
viii) For all $f \in \mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$ one has $(\varrho_\lambda * f) \to f$ as $\lambda \to 0+$ in the topology of $\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$.

Proof: Exercise. ■

As a consequence we get that $\mathscr{D}$ is dense in any of the above space:

Theorem 3.22. *$\mathscr{D}(\Omega,\mathbb{C}^l)$ is dense in...*
i) $L^p(\Omega,\mathbb{C}^l)$ for all $p \in [1,\infty)$,
ii) $L^p_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$ for all $p \in [1,\infty)$,
iii) $L^p_c(\Omega,\mathbb{C}^l)$ for all $p \in [1,\infty)$,
iv) $C^k_c(\Omega,\mathbb{C}^l)$ for all $k \in \mathbb{N}_{\geq 0}$,
v) $C^k(\Omega,\mathbb{C}^l)$ for all $k \in \mathbb{N}_{\geq 0}$,
vi) $\mathscr{E}(\Omega,\mathbb{C}^l)$,
vii) $\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$, if $\Omega = \mathbb{R}^m$.

Proof: i) Clearly $L^p_c(\Omega,\mathbb{C}^l)$ is dense in $L^p(\Omega,\mathbb{C}^l)$ (approximate $f \in L^p(\Omega,\mathbb{C}^l)$ with $1_{K_n} f$ using dominated convergence, where (K_n) is a compact exhaustion of Ω), but the elements of $L^p_c(\Omega,\mathbb{C}^l)$ can be approximated with Friedrichs mollifiers, which now are compactly supported.
ii) By the same argument as in i) one finds that $L^p_c(\Omega,\mathbb{C}^l)$ is dense in $L^p_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$, but the latter functions can be approximated by Friedrichs mollifiers. Noting that the support of the Friedrichs mollification of a compactly supported function gets smaller with λ getting smaller, this also proves iii).
iv) This follows immediately using Friedrichs mollifiers, using again that the support of the Friedrichs mollification of a compactly supported function gets smaller with λ getting smaller.
v) Note first that $C^k_c(\Omega,\mathbb{C}^l)$ is dense in $C^k(\Omega,\mathbb{C}^l)$ (approximate $f \in C^k(\Omega,\mathbb{C}^l)$ with $f_n(x) =$

$\phi(x/n)f(x)$ where $\phi \in \mathscr{D}(\Omega)$). Then one can use Friedrichs mollifiers.
vi) This follows from applying v) order by order.
vii) Approximate $f \in \mathscr{S}(\Omega, \mathbb{C}^l)$ with $f_n(x) = \phi(x/n)f(x)$ where $\phi \in \mathscr{D}(\mathbb{R}^m)$. ■

The following result is usually referred to as the *fundamental lemma of distribution theory*:

Lemma 3.23. *Assume $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$ and that*

$$\int_\Omega (f, \psi) = 0 \quad \textit{for all } \Psi \in \mathscr{D}(\Omega, \mathbb{C}^l).$$

Then one has $f = 0$ (a.e. in Ω).

Proof: We are going to prove

$$\phi f^{(i)} = 0 \quad \text{a.e. in } \Omega \text{ for all } \phi \in \mathscr{D}(\Omega),\ j = 1, \dots, l. \tag{7}$$

To see this, note that for all $\psi \in \mathscr{D}(\mathbb{R}^m)$ we have $\phi\psi|_\Omega \in \mathscr{D}(\Omega)$, and so by using the assumption with $\Psi = \phi\psi|_\Omega e_j$ we get the second equality in

$$\int_{\mathbb{R}^m} \psi \cdot \underline{\phi f^{(j)}} = \int_\Omega \big(f, \phi\psi|_\Omega e_j\big) = 0,$$

where for the first equality we have used the support properties and assumed that ψ is real-valued. Applying this with $\psi = \varrho_{1/n}(x - \cdot)$ gives

$$\varrho_{1/n} * (\underline{\phi f^{(j)}})(x) = \int_{\mathbb{R}^m} \varrho_{1/n}(x - y)\underline{\phi f^{(j)}}(y)dy = 0 \quad \text{for all } n \in \mathbb{N},$$

which implies $\underline{\phi f^{(i)}} = 0$ a.e. in $\mathbb{R}^m$ and so (7), as

$$\underline{\phi f^{(i)}} = \lim_n \varrho_{1/n} * (\underline{\phi f^{(j)}}) \quad \text{in } L^1(\mathbb{R}^m, \mathbb{C}^l),$$

which completes the proof. ■

Let us now come to spaces of distributions. In the sequel, the space of continuous linear functionals X' on a locally convex space is always equipped with its weak topology, unless otherwise stated.

Definition 3.24. a) $\mathscr{D}'(\Omega, \mathbb{C}^l) := \mathscr{D}(\Omega, \mathbb{C}^l)'$ is called *the space of distributions on* Ω.
b) $\mathscr{E}'(\Omega, \mathbb{C}^l) := \mathscr{E}(\Omega, \mathbb{C}^l)$ is called *the space of distributions with compact support*[5].
c) $\mathscr{S}'(\mathbb{R}^m, \mathbb{C}) := \mathscr{S}(\mathbb{R}^m, \mathbb{C})'$ is called *the space of tempered distributions* or *the space of Schwartz distributions.*

Lemma 3.25. *The inclusion maps*

$$\mathscr{D}(\Omega, \mathbb{C}^l) \hookrightarrow \mathscr{E}(\Omega, \mathbb{C}^l), \quad \mathscr{D}(\mathbb{R}^m, \mathbb{C}^l) \hookrightarrow \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \hookrightarrow \mathscr{E}(\mathbb{R}^m, \mathbb{C}^l)$$

are continuous.
b) The restriction maps

$$\mathscr{E}'(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{D}'(\Omega, \mathbb{C}^l), \quad \mathscr{E}'(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{D}'(\mathbb{R}^m, \mathbb{C}^l)$$

[5]This notion will become clear later.

are continuous embeddings.

Proof: a) Using Proposition 3.7 b) one finds that convergence in $\mathscr{D}$ (support in a fixed compact set and local uniform convergence of all derivatives thereon) implies convergence in $\mathscr{E}$ (local uniform convergence of all derivatives) and in $\mathscr{S}$ (global uniform convergence of all derivatives multiplied with polynomials). Likewise, convergence in $\mathscr{S}$ implies convergence in $\mathscr{E}$.
b) Note first that $\mathscr{D}$ in dense in any of these spaces (cf. Theorem 3.22). Now the claim follows from the following simple observation: If X and Y are LCS's such that there is an continuous embedding $\iota : X \hookrightarrow Y$ with a dense image, then the map

$$r : Y' \longrightarrow X', \quad r(T)(\phi) := T(\iota(\phi))$$

is a continuous embedding. ■

Our abstract continuity results for linear maps between locally convex spaces (lecture course; to be added to the manuscript later) immediately imply:

Lemma 3.26. *a) A linear functional $T : \mathscr{D}(\Omega, \mathbb{C}^l) \to \mathbb{C}$ is a distribution, if and only if for every compact set $K \subset \Omega$ there exist $N \in \mathbb{N}_{\geq 0}$, $C > 0$, such that*

$$|T(\phi)| \leq C \max_{|\alpha| \leq N} p^{\Omega}_{\alpha,K}(\phi) \quad \textit{for all } \phi \in \mathscr{D}_K(\Omega, \mathbb{C}^l).$$

b) A linear functional $T : \mathscr{E}(\Omega, \mathbb{C}^l) \to \mathbb{C}$ is a distribution with compact support, if and only if there exist a compact set $K \subset \Omega$, $N \in \mathbb{N}_{\geq 0}$, $C > 0$, such that

$$|T(\phi)| \leq C \max_{|\alpha| \leq N} p^{\Omega}_{\alpha,K}(\phi) \quad \textit{for all } \phi \in \mathscr{E}(\Omega, \mathbb{C}^l).$$

c) A linear functional $T : \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \to \mathbb{C}$ is a tempered distribution, if and only if there exist $\alpha, \beta \in \mathbb{N}^m$, $C > 0$, such that

$$|T(\phi)| \leq C p_{\alpha,\beta}(\phi) \quad \textit{for all } \phi \in \mathscr{S}(\Omega, \mathbb{C}^l).$$

We continue with some important examples:

Example 3.27. Every $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$ induces the distribution $T_f \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ given by

$$T_f(\phi) := \int_\Omega (f, \phi), \quad \phi \in \mathscr{D}(\Omega, \mathbb{C}^l).$$

Indeed, for all $K \subset \Omega$ compact one has

$$|T_f(\phi)| \leq \Big(\int_K |f| \Big) \sup_K |\phi| =: C_{K,f} p^{\Omega}_{(0,\dots,0),K}(\phi) \tag{8}$$

The assignment $f \mapsto T_f$ induces a continuous embedding

$$L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^l) \hookrightarrow \mathscr{D}'(\Omega, \mathbb{C}^l).$$

Indeed, the injectivity follows from the fundamental lemma of distribution theory, and the continuity (in the weak topology) follows from

$$|T_f(\phi)| \leq C_{K,\phi} \left\| 1_K f \right\|^{\Omega}_{L^1},$$

which is implied by (8). Distributions that arise from locally integrable functions in the above way are called *regular distributions*.
In a complete analogy, one gets continuous embeddings

$$L^1_c(\Omega,\mathbb{C}^l)\hookrightarrow \mathscr{E}'(\Omega,\mathbb{C}^l),\quad f\mapsto T_f:=\int_\Omega (f,\cdot)$$

and

$$L^p(\mathbb{R}^m,\mathbb{C}^l)\hookrightarrow \mathscr{S}'(\mathbb{R}^m,\mathbb{C}^l),\quad f\mapsto T_f:=\int_{\mathbb{R}^m} (f,\cdot)\quad \text{for all } p\in[1,\infty].$$

The prototype of a distribution which is not regular is given by the δ-distribution:

Example 3.28. For every fixed $a\in\Omega$ put

$$\delta_a:\mathscr{E}(\Omega)\longrightarrow\mathbb{C},\quad \phi\longmapsto\phi(a).$$

Then δ_a is a distribution with compact support, called the *δ-distribution with mass in a*. Indeed, this follows from the inequality

$$|\delta_a(\phi)|=|\phi(a)|\leq p^{\Omega}_{(0\dots,o),\{a\}}(\phi).$$

In particular (by restriction) one has $\delta_a\in\mathscr{D}'(\Omega)$ and $\delta_a\in\mathscr{S}'(\mathbb{R}^m)$. Let us show that δ_a is not a regular distribution: for if $\delta_a=T_f\in\in\mathscr{D}'(\Omega)$ for some $f\in L^1_{\mathrm{loc}}(\Omega)$, then for all $\phi\in\mathscr{D}(\Omega,\mathbb{C}^l)$ with support in $\Omega\setminus\{a\}$ one has

$$0=\delta_a(\phi)=T_f(\phi)=\int_\Omega \overline{f}\phi=\int_{\Omega\setminus\{a\}}\overline{f}\phi,$$

which implies $f=0$ a.e. in $\Omega\setminus\{a\}$ by the fundamental lemma of distribution theory and thus $f=0$ a.e in Ω. But this shows $\delta_a(\phi)=T_f(\phi)=0$ for all $\phi\in\mathscr{D}(\Omega)$, which obviously cannot be true.

Althoug the δ-distribution is not regular, it can be approximated by regular distributions:

Example 3.29. One has

$$\lim_{\lambda\to 0+} T_{\varrho_\lambda(a-\cdot)}=\delta_a\quad \text{in } \mathscr{D}'(\Omega).$$

Indeed, by definition one has to show that

$$\lim_{\lambda\to 0+} T_{\varrho_\lambda(a-\cdot)}(\phi)=\lim_{\lambda\to 0+}\int_\Omega \varrho_\lambda(a-y)\phi(y)dy=\phi(a)\quad \text{for all } \phi\in\mathscr{D}(\Omega),$$

which follows from Theorem 3.21 v).

Although $x\mapsto e^{-i(x,b)}$ is not in $L^1(\mathbb{R}^m)$, one can use distribution theory to give meaning to $\int_{\mathbb{R}^m} e^{-i(x,b)}dx$ by considering b as a variable. More precisely:

Proposition 3.30. *For the net of bounded functions*

$$\mathbb{R}^m\ni y\longmapsto \int_{B_R(0)} e^{-i(x,a-y)}dx\in\mathbb{C},\quad R>0,$$

the limit

$$\lim_{R\to\infty} T_{\int_{B_R(0)} e^{-i(x,a-\bullet)dx}} \quad \textit{exists in } \mathscr{S}'(\mathbb{R}^m) \textit{ (and thus in } \mathscr{D}'(\mathbb{R}^m)).$$

In fact, the limit is proportional to δ_a.

Proof: Exercise. ■

Distribution theory also allows to differentiate nondifferentiable functions in a consistent 'weak' sense. To this end, we add:

Lemma 3.31. *For all* $k \in \mathbb{N}_{\geq 0}$ *and all differential operators*

$$P : \mathscr{E}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'}) \quad \textit{of order} \leq k$$

there exists a unique differential operator

$$P^\dagger : \mathscr{E}(\Omega, \mathbb{C}^{l'}) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^l) \quad \textit{of order} \leq k,$$

the formal adjoint of P, *such that*

$$\int_\Omega (P\phi, \psi) = \int_\Omega (\phi, P^\dagger \psi) \quad \textit{for all } \phi \in \mathscr{E}(\Omega, \mathbb{C}^l),\ \psi \in \mathscr{E}(\Omega, \mathbb{C}^l), \tag{9}$$

at least one of which having a compact support.

Proof: Uniqueness follows from the fundamental lemma of distribution theory. For the existence, if $P = \sum_{|\alpha|\leq k} P_\alpha \partial^\alpha$, one may set

$$P^\dagger \psi := \sum_{|\alpha|\leq k} (-1)^{|\alpha|} \partial^\alpha (P_\alpha^\dagger \psi),$$

where $(P_\alpha^\dagger)_{ij}(x) = \overline{(P_\alpha)_{ji}}(x)$ denotes the adjoint matrix for $x \in \Omega$. By the Leibnitz rule, $P^\dagger$ indeed is a partial differential operator of order $\leq k$. The equality (9) then follows from integrating by parts. ■

In particular, the formal adjoint of ∂^α is $(-1)^{|\alpha|}\partial^\alpha$. Using the formal adjoint, we can give:

Definition 3.32. Let

$$P : \mathscr{E}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'})$$

be a differential operator.

a) Given $T \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ we set

$$PT(\phi) := T(P^\dagger \phi) \quad \text{for all } \phi \in \mathscr{D}(\Omega, \mathbb{C}^l).$$

b) Given $T \in \mathscr{E}'(\Omega, \mathbb{C}^l)$ we set

$$PT(\phi) := T(P^\dagger \phi) \quad \text{for all } \phi \in \mathscr{E}(\Omega, \mathbb{C}^l).$$

c) Given $T \in \mathscr{D}'(\mathscr{S}, \mathbb{C}^l)$ and if $P = \sum_{|\alpha|\leq k} P_\alpha \partial^\alpha$ is such that $(P_{\alpha ij}) \in \mathscr{S}(\mathbb{R}^m)$ for all α with $|\alpha| \leq k$ and all $i = 1, \ldots, l, j = 1, \ldots, l'$, we again set

$$PT(\phi) := T(P^\dagger \phi) \quad \text{for all } \phi \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l).$$

Note that this definition is consistent with the embeddings

$$\mathscr{E}(\Omega,\mathbb{C}^l)\hookrightarrow\mathscr{D}'(\Omega,\mathbb{C}^l) \tag{10}$$

$$\mathscr{D}(\Omega,\mathbb{C}^l)\hookrightarrow\mathscr{E}'(\Omega,\mathbb{C}^l), \tag{11}$$

$$\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)\hookrightarrow\mathscr{S}'(\mathbb{R}^m,\mathbb{C}^l), \tag{12}$$

In the sense that in each case one has $PT_f = T_{Pf}$. This follows from integrating by parts. Moreover, the linear maps

$$\begin{aligned}
&P:\mathscr{D}(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{D}(\Omega,\mathbb{C}^{l'}),\\
&P:\mathscr{E}(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{E}(\Omega,\mathbb{C}^{l'}),\\
&P:\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)\longrightarrow\mathscr{S}(\mathbb{R}^m,\mathbb{C}^{l'})
\end{aligned}$$

are easily checked to be continuous (where in the Schwartz case we assume Schwartz coefficients), which implies that so are the induced maps

$$\begin{aligned}
&P:\mathscr{D}'(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{D}'(\Omega,\mathbb{C}^{l'}),\\
&P:\mathscr{E}'(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{E}'(\Omega,\mathbb{C}^{l'}),\\
&P:\mathscr{S}'(\mathbb{R}^m,\mathbb{C}^l)\longrightarrow\mathscr{S}'(\mathbb{R}^m,\mathbb{C}^{l'}),
\end{aligned}$$

since these maps are just the transposed maps in the sense of LCS's of continuous maps.

Example 3.33. 1. The derivative of the distribution $T_H\in\mathscr{D}'(\mathbb{R})$ which is induced by the locally integrable function $H := 1_{[0,\infty)}\in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, the so called Heaviside function, is given by $\partial T_H = \delta_0$. Indeed, for all $\phi\in\mathscr{D}'(\mathbb{R})$ one has

$$\partial T_H(\phi) = -T_H(\partial\phi) = -\int_{[0,\infty)}\phi' = \phi(0).$$

2. The derivative of the distribution $T_f\in\mathscr{D}'(-1,1)$ which is induced by the locally integrable function $f(x) := \log(x)$ for $x>0$ and $f(x)=0$ for $x\leq 0$ is given by

$$\partial T_f(\phi) = \int_0^1\frac{\phi(x)-\phi(0)}{x}dx\quad\text{ for all }\phi\in\mathscr{D}(-1,1).$$

3. The partial derivatives of $\delta_a\in\mathscr{E}'(\Omega)$ are given by

$$\partial^\alpha\delta_a(\phi) = \delta_a((-1)^{|\alpha|}\partial^\alpha\phi) = (-1)^{|\alpha|}\partial^\alpha\phi(a)\quad\text{ for all }\phi\in\mathscr{E}(\Omega).$$

The first two examples shows that the derivatives of regular distributions that are induced by nondifferentiable functions need not be regular.

Let us now come to the Fourier transform:

Definition 3.34. Given $f\in\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$ its *Fourier transform* $\widehat{f}:\mathbb{R}^m\to\mathbb{C}^l$ is the function defined by

$$\widehat{f}(\zeta) := \widehat{f}(\zeta) := (2\pi)^{-m/2}\int_{\mathbb{R}^m}e^{-i(x,\zeta)}f(x)dx.$$

The Fourier transform of f is well-defined since $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \subset L^1(\mathbb{R}^m, \mathbb{C}^l)$, and in fact a smooth function (differentiate under the integral. Moreover, this operation switches multiplication by monomials to differentiation and vice versa. More precisely:

Proposition 3.35. *Let $f \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$, $\zeta \in \mathbb{R}^m$.*
a) One has $D^\alpha \widehat{f}(\zeta) = -\widehat{x^\beta f}(\zeta)$, where $D^\alpha := (-i)^{|\alpha|}\partial^\alpha$.
b) One has $\zeta^\alpha \widehat{f}(\zeta) = \widehat{D^\alpha f}(\zeta)$.
c) The linear map

$$F : \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l), \quad f \longmapsto \widehat{f}$$

is well-defined and continuous.
d) For all $\lambda \in \mathbb{R} \setminus \{0\}$, $v \in \mathbb{R}^m$, $f \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ one has the following scaling/translation behaviour:

$$\widehat{f(\lambda \cdot)}(\zeta) = \lambda^{-m} \widehat{f}\left(\frac{\cdot}{\lambda}\right)(\zeta), \quad \widehat{f(\cdot - v)}(\zeta) = e^{-i(v,\zeta)}\widehat{f}(\zeta).$$

Proof: a) Differentiation under the integral gives

$$(2\pi)^{-m/2}\int_{\mathbb{R}^m} \partial_\zeta^\alpha e^{-i(x,\zeta)} f(x)dx = (2\pi)^{-m/2}\int_{\mathbb{R}^m} (-i)^{|\alpha|} e^{-i(x,\zeta)} f(x)dx,$$

which is the claimed formula.
b) One has

$$(2\pi)^{-m/2}\int_{\mathbb{R}^m} \zeta^\alpha e^{-i(x,\zeta)} f(x)dx = (-1)^{|\alpha|}(2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{-i(x,\zeta)}\partial_x^\alpha f(x)dx,$$

where we have integrated by parts.
c) By parts a), b) we can estimate as follows

$$\begin{aligned}
|\zeta^\alpha D^\beta \widehat{f}(\zeta)| &= \left|D^\alpha \widehat{(x^\beta f)}(\zeta)\right| \\
&\leq \int_{\mathbb{R}^m} |D^\alpha (x^\beta f)(x)| dx \\
&= \int_{\mathbb{R}^m} |D^\alpha (x^\beta f)(x)|(1 + |x|^2)^{(m+1)/2}(1 + |x|^2)^{-(m+1)/2} dx \\
&\leq C \sup_{x \in \mathbb{R}^m} |D^\alpha (x^\beta f)(x)|(1 + |x|^2)^{(m+1)/2},
\end{aligned}$$

where

$$C := \int_{\mathbb{R}^m} (1 + |x|^2)^{-(m+1)/2} dx < \infty.$$

Using the Leibniz formula to calculate $D^\alpha(x^\beta f)$, we can pick multi-indices α', β' and $C' > 0$ such that

$$C \sup_{x \in \mathbb{R}^m} |D^\alpha (x^\beta f)(x)|(1 + |x|^2)^{(m+1)/2} \leq C' p_{\alpha',\beta'}(f).$$

d) Transformation formula for integrals. ■

The following corollary shows that one can solve PDE's with constant coefficients in the Schwartz space, if the polynomial underlying the differential operator is invertible:

Corollary 3.36. *Assume*

$$P(X) = \sum_{\alpha} P_\alpha X^\alpha \in \mathrm{Mat}_{\ell\times\ell}(\mathbb{C})[X^1, \dots, X^m]$$

is a polynomial with constant matrix coefficients, and let $P(\partial) = \sum_\alpha P_\alpha \partial^\alpha$ *be the associated differential operator with constant matrix coefficients. Then one has*

$$\widehat{P(\partial)f} = P(x)\widehat{f} \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \ .$$

Our next aim will be to prove that the Fourier transform is continuously invertible. To this end, the Fourier transform of the Gaussian function will be useful (for it is a fixed point of F):

Example 3.37. For the Schwartz function $\exp(-|\cdot|^2) \in \mathscr{S}(\mathbb{R}^m)$ one has

$$\widehat{e^{-|\cdot|^2/2}} = e^{-|\cdot|^2/2}.$$

There are many proofs of this formula (complex analysis, ...). We use ODE theory: firstly, in view of

$$e^{-|x|^2} = \prod_{j=1}^{m} e^{-|x_j|^2}$$

and Fubini, it is enough to consider the case $m = 1$. Set

$$h(t) := \widehat{e^{-|\cdot|^2}}(t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-ixt} e^{-x^2/2} dx.$$

Then one has

$$\begin{aligned}(d/dt)h(t) &= i(2\pi)^{-1} \int_{\mathbb{R}} e^{-ixt}(-x)e^{-x^2/2}dx = i(2\pi)^{-1} \int_{\mathbb{R}} e^{-ixt}(-x)e^{-x^2/2}dx \\ &= i(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ixt}(d/dx)e^{-x^2/2}dx = -i(2\pi)^{-1/2} \int_{\mathbb{R}} (-it)e^{-ixt}e^{-x^2/2}dx \\ &= -th(t).\end{aligned}$$

Thus

$$h(t) = h(0)e^{-t^2/2} = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-x^2/2}dx \cdot e^{-t^2/2} = e^{-t^2/2}.$$

The following lemma is a small variation of the Friedrichs construction:

Lemma 3.38. *Assume* $f_0 \in L^1(\mathbb{R}^m)$ $f \in L^\infty(\mathbb{R}^m, \mathbb{C}^l)$ *are such that*

- $\int_{\mathbb{R}^m} f_0 = 1$
- f *is continuous at* 0.

Then one has

$$\lim_{\lambda\to 0+} \int_{\mathbb{R}^m} \lambda^{-m} f_0(x/\lambda) f(x) dx = f(0).$$

Proof: The proof is very similar to the proof of Proposition 3.17 and can be left to the reader. ■

Now we can prove the following result:

Theorem 3.39. *F is continuously invertible and its inverse is given by*

$$F^{-1} : \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l), \quad F^{-1}f(x) = \check{f}(x) = Ff(-x),$$

the so called inverse Fourier transform*.*

Proof: Let G be defined by

$$G : \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l), \quad Gf(x) := Ff(-x).$$

The fact that G is continuous is checked similarly to the continuity of F. Let $f \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$.
To see that $GFf = f$ it suffices to prove

$$GFf(0) = f(0).$$

Indeed, together with the translation behaviour from Proposition 3.35 d), the latter equality allows the following calculation:

$$f(x) = f(0+x) = GFf(\cdot + x)(0) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} Ff(\cdot + x)(\zeta) e^{i(0,\zeta)} d\zeta$$
$$= (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i(x,\zeta)} Ff(\zeta) d\zeta = GF(x).$$

To show $GFf(0) = f(0)$, let $h := e^{-|\cdot|^2/2}$, $f_0 := (2\pi)^{-m/2} h$. Using the previous lemma, $Fh = h$, and the scaling bevaviour from Proposition 3.35 d), we can calculate[6]

$$\begin{aligned}
&(2\pi)^{m/2} f(0) \\
&= \lim_{\lambda \to 0+} \int_{\mathbb{R}^m} \lambda^{-m} h(x/\lambda) f(x) dx \\
&= \lim_{\lambda \to 0+} \int_{\mathbb{R}^m} \lambda^{-m} Fh(\zeta/\lambda) f(\zeta) d\zeta \\
&= \lim_{\lambda \to 0+} \int_{\mathbb{R}^m} Fh(\lambda \cdot)(\zeta) f(\zeta) d\zeta \\
&= \lim_{\lambda \to 0+} \int_{\mathbb{R}^m} h(\lambda\zeta) Ff(\zeta) d\zeta \\
&= \int_{\mathbb{R}^m} Ff(\zeta) d\zeta,
\end{aligned}$$

where we have used Fubini for the fourth equality and dominated convergence for the last equality. What I had not realized in the lecture was the following magic trick: $1 = e^{i(0,\zeta)}$, so the latter expression is

$$= \int_{\mathbb{R}^m} e^{i(0,\zeta)} Ff(\zeta) d\zeta.$$

[6] the trick I haven't realized in the lecture course was $1 = e^{i(0,\zeta)}$

Atogether we have shown

$$f(0) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i(0,\zeta)} Ff(\zeta) d\zeta = GFf(0).$$

The proof of $FGf(x) = f(x)$ is very similar, showing $G = F^{-1}$. ∎

We extend the Fourier transform by duality:

Definition 3.40. The continuous linear map

$$F : \mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l), \quad FT(f) := T(Ff)$$

is called the *distributional Fourier transform.*

The distributional Fourier transform is continuously invertible (this follows immediately from Theorem 3.39).

The invertibility of F gives a simple recipe for solveing PDE's with constant coefficients: assume that in the situation of Corollary 3.36 we want to find a solution f of $P(\partial)f = g$, where g is a Schwartz function (or a Schwartz distribution). Then we have

$$FP(\partial)f = PFf = Fg,$$

so if $P(x)$ is an invertible matrix for all $x \in \mathbb{R}^m$, then $f = F^{-1}P^{-1}Fg$ is the unique solution in the space of Schwartz functions (or Schwartz distributions) of $P(\partial)f = g$.
If g is is compactly supported and suffuciently regular, then f can be chosen compactly supported, for one has the following result which we record without proof (and which will not be used later on):

Theorem 3.41 (Hoermander)**.** *Assume*

$$P(X) = \sum_{\alpha} P_\alpha X^\alpha \in \mathrm{Mat}_{\ell\times\ell}(\mathbb{C})[X^1, \ldots, X^m]$$

is a polynomial with constant matrix coefficients, and let $g \in \mathscr{E}(\mathbb{R}^m, \mathbb{C}^l)$. Then $P(\partial)f = g$ has a solution $f \in \mathscr{E}(\mathbb{R}^m, \mathbb{C}^l)$, if and only if $P(x) \in \mathrm{GL}_{\ell\times\ell}(\mathbb{C})$ for all $x \in \mathbb{R}^m$ and $P^{-1}Fg$ stems as a regular distribution from an entire function. This solution is unique.

Remark 3.42. 1. A continuous linear operator $A : X \to Y$ between Hilbert spaces is called *unitary*, if A is invertible with $A^{-1} = A^*$. Unitary maps preserve scalar products and thus norms.
3. If X is a normed space and Y is a Banach spaces, $D \subset X$ is a dense subspace, and $T : D \to Y$ is a continuous linear map, then there exists a unique continuous linear extension $X \to Y$ of T.

Theorem 3.43 (Plancherel's Theorem)**.** *The linear map*

$$\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow L^2(\mathbb{R}^m, \mathbb{C}^l), \quad f \longmapsto Ff, \tag{13}$$

where the Schwartz space is equipped with the L^2-topology, is continuous. Thus this map has a unique continuous linear extension $L^2(\mathbb{R}^m, \mathbb{C}^l) \to L^2(\mathbb{R}^m, \mathbb{C}^l)$, which will be denoted again by F. This extension is unitary.

Proof: For all $f_1, f_2 \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ one has

$$\begin{aligned}\langle Ff_1, Ff_2\rangle &= \int_{\mathbb{R}^m} (Ff_1, Ff_2)\\ &\int_{\mathbb{R}^m} (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{+i(x,\zeta)}(f_1(x), Ff_2(\zeta))dxd\zeta\\ &\int_{\mathbb{R}^m} (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{+i(x,\zeta)}(f_1(x), Ff_2(\zeta))dxd\zeta\\ &\int_{\mathbb{R}^m} (f_1(x), F^{-1}Ff_2(x))dx\\ &= \langle f_1, f_2\rangle\,,\end{aligned}$$

showing that (13) is continuous in the asserted way. Let us denote for a moment with

$$F_{L^2} : L^2(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow L^2(\mathbb{R}^m, \mathbb{C}^l)$$

its continuous linear extension. As above, one shows that the map

$$G : \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow L^2(\mathbb{R}^m, \mathbb{C}^l), f \longmapsto F^{-1}f,$$

satisfies

$$\langle Gf_1, Gf_2\rangle = \langle f_1, f_2\rangle\,,$$

for all $f_1, f_2 \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$, and so is continuous, too (again with the Schwartz space is equipped with the L^2-topology). Denoting with

$$G_{L^2} : L^2(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow L^2(\mathbb{R}^m, \mathbb{C}^l)$$

its continuous linear extension, it follows now easily from a density argument and $F_{L^2}G_{L^2} = F_{L^2}G_{L^2}$ in $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ that $G_{L^2} = F_{L^2}^{-1}$. Finally, for all $f, g \in L^2(\mathbb{R}^m, \mathbb{C}^l)$ one has

$$\begin{aligned}\langle f, F_{L^2}g\rangle &= \lim_n \langle f_n, Fg_n\rangle\\ &= \lim_n \left\langle F^{-1}f_n, F^{-1}Fg_n\right\rangle_2 = \langle G_{L^2}, g\rangle\,,\end{aligned}$$

for some sequences f_n, g_n in $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ with $f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R}^m, \mathbb{C}^l)$. Thus we have shown $F_{L^2}^{-1} = G_{L^2} = F_{L^2}^*$, which completes the proof. ■

While the Fourier transform in L^2 is abstractly defined, it turns out that the Fourier transform on $L^1 \cap L^2$ can be evaluated explicitly. This results in the Riemann-Lebesgue 'Lemma', which states that the Fourier transform of an L^1-function is continuous and vanishes at ∞:

Theorem 3.44. *The linear map*

$$L^1(\mathbb{R}^m, \mathbb{C}^l) \cap L^2(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow C_0(\mathbb{R}^m, \mathbb{C}^l), \quad f \longmapsto Ff, \tag{14}$$

where the LHS is equipped with the L^1-topology, is well-defined and continuous. Its unique continuous linear extension

$$L^1(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow C_\infty(\mathbb{R}^m, \mathbb{C}^l)$$

is denoted with F again, and this map is injective. One has

$$Ff(\zeta) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,\zeta)} f(x)dx \quad \textit{for all } f \in L^1(\mathbb{R}^m, \mathbb{C}^l),\ \zeta \in \mathbb{R}^m. \tag{15}$$

Proof: Let us first check that for all $f \in L^1(\mathbb{R}^m, \mathbb{C}^l) \cap L^2(\mathbb{R}^m, \mathbb{C}^l)$ one has (15) for a.e. $\zeta \in \mathbb{R}^m$, where the Fourier transform Ff has already been defined above, as $f \in L^2(\mathbb{R}^m, \mathbb{C}^l)$. Indeed, pick a sequence f_n in $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ with $f_n \to f$ in $L^1(\mathbb{R}^m, \mathbb{C}^l)$ *and* in $L^2(\mathbb{R}^m, \mathbb{C}^l)$. Then from the L^1-convergence one gets

$$\lim_n (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,\zeta)} f_n(\zeta)d\zeta = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,\zeta)} f(x)dx \quad \text{for } \textit{all } \zeta \in \mathbb{R}^m,$$

while the L^2-convergence and the continuity of F in $L^2(\mathbb{R}^m, \mathbb{C}^l)$ implies $Ff_n \to Ff$ in $L^2(\mathbb{R}^m, \mathbb{C}^l)$. Thus, after possibly picking a subsequence one gets

$$Ff(\zeta) = \lim_n Ff_n(\zeta) = \lim_n (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,\zeta)} f_n(x)dx \quad \text{for a.e. } \zeta \in \mathbb{R}^m,$$

ultimately proving (15).

Formula (15) shows that for all $f \in L^1(\mathbb{R}^m, \mathbb{C}^l) \cap L^2(\mathbb{R}^m, \mathbb{C}^l)$, the RHS of (15) defines a continuous representative of Ff, which is bounded by $(2\pi)^{-m/2} \|f\|_{L^1(\mathbb{R}^m, \mathbb{C}^l)}$. Thus we have a continuous linear map

$$L^1(\mathbb{R}^m, \mathbb{C}^l) \cap L^2(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow C_b(\mathbb{R}^m, \mathbb{C}^l), \quad f \longmapsto Ff, \tag{16}$$

where the LHS is equipped with the $L^1(\mathbb{R}^m, \mathbb{C}^l)$-topology. It remains to show that Ff vanishes at ∞ for all $f \in L^1(\mathbb{R}^m, \mathbb{C}^l) \cap L^2(\mathbb{R}^m, \mathbb{C}^l)$. Pick a sequence f_n in $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ with $f_n \to f$ in $L^1(\mathbb{R}^m, \mathbb{C}^l)$. Then

$$Ff_n \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \subset C_0(\mathbb{R}^m, \mathbb{C}^l) \quad \text{for all } n$$

and by the continuity of (16) one has $Ff_n \to Ff$ in $C_b(\mathbb{R}^m, \mathbb{C}^l)$. It follows that $Ff \in C_0(\mathbb{R}^m, \mathbb{C}^l)$, as $C_0(\mathbb{R}^m, \mathbb{C}^l)$ is a closed subspace of $C_b(\mathbb{R}^m, \mathbb{C}^l)$.

In order to show that

$$F : L^1(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow C_0(\mathbb{R}^m, \mathbb{C}^l) \tag{17}$$

is injective, assume $h, g \in L^1(\mathbb{R}^m, \mathbb{C}^l)$ are given with $Fh = Fg$. As for all $f \in L^1(\mathbb{R}^m, \mathbb{C}^l)$ the formula (15) remains true (pick a sequence

$$(f_n) \subset \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \subset L^1(\mathbb{R}^m, \mathbb{C}^l) \cap L^2(\mathbb{R}^m, \mathbb{C}^l)$$

with $f_n \to f$ in $L^1(\mathbb{R}^m, \mathbb{C}^l)$ and use the continuity of (17)), it follows using Fubini that for all $\phi \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ one has

$$\int_{\mathbb{R}^m} (h, F\phi) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (2\pi)^{-m/2} e^{-i(x,\zeta)} (h(\zeta), \phi(x)) dx d\zeta$$
$$= \int_{\mathbb{R}^m} (Fh, \phi) = \int_{\mathbb{R}^m} (Fg, \phi) = \int_{\mathbb{R}^m} (g, F\phi).$$

Thus we have shown $FT_h = FT_g$ in $\mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l)$ and so $T_h = T_g$ in in $\mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l)$, as the Fourier transform is bijective on Schwartz distributions. Finally, we arrive at $h = g$ a.e. in $\mathbb{R}^m$ by the fundamental lemma of distribution theory. ■

Thus, putting all mapping properties together, the Fourier transform induces consistently defined continuously invertible maps

$$\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l),$$
$$\mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l),$$
$$L^2(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow L^2(\mathbb{R}^m, \mathbb{C}^l),$$

which are furthermore consistent with the continuous injective (and in fact not surjective) map

$$L^1(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow C_0(\mathbb{R}^m, \mathbb{C}^l),$$

and we have agreed on denoting all these maps by F. If $f \in L^1(\mathbb{R}^m, \mathbb{C}^l)$, then one has the explicit formula

$$Ff(\zeta) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,\zeta)} f(x) dx, \zeta \in \mathbb{R}^m,$$

and in all other cases Ff is defined implictly (either by duality in the case of Schwartz distributions, or by density in the L^2-case). Using the Hausdorff-Young inequality, one can show that the Fourier transform also defines continuous linear maps

$$L^p(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow L^p(\mathbb{R}^m, \mathbb{C}^l)$$

for all $p \in [1, 2]$ and $q \in [1, \infty]$ with $1/p + 1/q = 1$.

We continue with the Fourier convolution theorem:

Theorem 3.45. *Let $f, g \in L^2(\mathbb{R}^m)$. Then one has $Ff \cdot Fg \in L^2(\mathbb{R}^m)$, if and only if $f * g \in L^2(\mathbb{R}^m)$, and then*

$$Ff \cdot Fg = F(f * g). \tag{18}$$

Likewise, one has $fg \in L^2(\mathbb{R}^m)$, if and only if $Ff * Fg \in L^2(\mathbb{R}^m)$, and then

$$Ff * Fg = F(fg). \tag{19}$$

Proof: One of the students of the course has found a slicker proof: Set $C := 1/(2\pi)^{m/2}$. For all f, g in $\mathscr{S}(\mathbb{R}^m)$ one has

$$F(f * g)(\zeta) = (2\pi)^{-m/2} \int e^{-i(\zeta,x)} f * g(x) dx = (2\pi)^{-m/2} \int \int e^{-i(\zeta,x)} f(x-y) dx g(y) dy$$
$$= (2\pi)^{-m/2} \int \int e^{-i(\zeta,z+y)} f(z) dz g(y) dy = Ff(\zeta) Fg(\zeta).$$

Assume now $f, g \in \mathscr{S}(\mathbb{R}^m)$ and pick sequences f_n, g_n in $\mathscr{S}(\mathbb{R}^m)$ with $f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R}^m)$. This implies $Ff_n \cdot Fg_n \to Ff \cdot Fg$ in $L^1(\mathbb{R}^m)$: to see this, write

$$Ff_n \cdot Fg_n - Ff \cdot Fg = Ff_n \cdot Fg_n + Ff \cdot Fg_n - Ff \cdot Fg_n - Ff \cdot Fg$$

and use Cauchy-Schwarz. On the other hand, one has $f_n * g_n \to f * g$ in $L^\infty(\mathbb{R}^m)$, which is seen likewise, writing

$$f_n * g_n - f * g = f_n * g_n + f * g_n - f * g_n - f * g$$

and using Cauchy-Schwarz. As L^p-convergence implies convergence in the sense of Schwartz distributions, we have shown

$$f * g = \lim_n f_n * g_n = \lim_n F f_n \cdot F g_n = F f \cdot F g \quad \text{in } \mathscr{S}'(\mathbb{R}^m),$$

finishing the proof the proof of (18). The proof of (19) is analogous. ■

Let us note/calculate some Fourier transform of Schwartz distributions:

Example 3.46. 1. The Fourier transform of $\delta_a \in \mathscr{S}'(\mathbb{R}^m)$ is

$$F\delta_a = T_{\zeta \mapsto (2\pi)^{-m/2} e^{-i(a,\zeta)}}.$$

2. The Fourier transform of the Heaviside function $H := 1_{[0,\infty)} \in L^1(\mathbb{R})$ is given by

$$F T_H = (2\pi)^{-1/2} \lim_{\epsilon \to 0+} T_{\zeta \mapsto (\epsilon + i\zeta)^{-1}}.$$

This follows from approximating T_H by T_{H_ϵ} in $\mathscr{S}'(\mathbb{R})$, where $H_\epsilon(x) := 1_{[0,\infty)} e^{-\epsilon x}$.

Let us finally come to the definition of the support of a distribution:

Definition 3.47. 1. Given $T \in \mathscr{D}'(\Omega, \mathbb{C}^l)$, $U \subset \Omega$ open, we set

$$T|_U := T|_{\mathscr{D}(U,\mathbb{C}^l)} \in \mathscr{D}'(U, \mathbb{C}^l)$$

and say that T *is smooth in* U, if $T|_U = T_\psi$ for some smooth $\psi : U \to \mathbb{C}^l$.
2. For $T \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ one defines its support by

$$\mathrm{supp}(T) := \Omega \setminus \bigcup_{\Omega' \subset \Omega,\, T|_{\Omega'} = 0} \Omega',$$

and its singular support by

$$\mathrm{sing-supp}(T) := \Omega \setminus \bigcup_{\Omega' \subset \Omega,\, T|_{\Omega'} \text{ is smooth}} \Omega'.$$

Note that in view of $\mathscr{E}' \hookrightarrow \mathscr{S}' \hookrightarrow \mathscr{D}'$, the above notions are defined in all spaces of distributions. Furthermore, the assignment $U \mapsto \mathscr{D}'(U, \mathbb{C}^l)$, together with the restriction morphisms

$$\mathscr{D}'(V, \mathbb{C}) \longrightarrow \mathscr{D}'(U, \mathbb{C}^l), \quad T \longmapsto T|_U,$$

where $U \subset V \subset \Omega$ are open, is a sheaf over Ω: all sheaf axioms are more or less trivial to check, except locality, which requires the existence of smooth partitions of unity (exercise).

Remark 3.48. 1. For all $T \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ one has $\text{sing} - \text{supp}(T) \subset \text{supp}(T)$, which are both closed subsets of Ω.
2. For all $T \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ one has $T|_{\Omega \setminus \text{supp}(T)} = 0$, and $T|_{\Omega \setminus \text{sing}-\text{supp}(T)}$ is smooth (use sheaf property).
3. For all $T \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ one has $T \in \mathscr{E}'(\Omega, \mathbb{C}^l)$, if and only if $\text{supp}(T)$ is a compact set (exercise).
4. For all $f \in L^1_{\text{loc}}(\Omega, \mathbb{C}^l)$ and all $p \in \Omega$ one has $p \in \text{supp}(T)$, if and only if

$$\int_{B_\epsilon(p)} |f| > 0 \quad \text{for all } \epsilon > 0 \text{ with } B_\epsilon(p) \subset \Omega.$$

This result follows from the fact that the Lebesgue measure is a Borel measure with full support, and it implies that for all $f \in C(\Omega, \mathbb{C}^l)$ one has

$$\text{supp}(f) = \overline{\{f \neq 0\}} = \text{supp}(T_f),$$

that is, in the continuous case the distributional support is equal to the support in the sense of functions on topological spaces.

Example 3.49. 1. One has $\text{sing} - \text{supp}(\delta_a) = \text{supp}(\delta_a) = \{a\}$.
2. For $f := 1_{(0,\infty)} \log \in L^1_{\text{loc}}(\mathbb{R})$ one has

$$\text{sing} - \text{supp}(T_f) = \{0\}, \quad \text{supp}(T_f) = [0, \infty) = \text{supp}(f).$$

3. Although in the case of continuous functions and even in the previous example we had $\text{supp}(T_f) = \text{supp}(f)$, this equality fails in general, e.g. for $f = 1_{\mathbb{Q}}$.

We now turn to Sobolev spaces, the localized versions of which will be the natural mapping spaces for pseudodifferential operators:

Definition 3.50. For all $k \in \mathbb{N}_{\geq 0}$ define $H^k(\Omega, \mathbb{C}^l)$ to be the space of all $f \in \mathscr{D}'(\Omega, \mathbb{C}^l)$ such that $D^\alpha f \in L^2(\Omega, \mathbb{C}^l)$ for all $|\alpha| \leq k$. This space is called the *Sobolev space of order* k and is equipped with the scalar product

$$\langle f, g \rangle^{\Omega}_{H^k} := \sum_{|\alpha| \leq k} \langle D^\alpha f, D^\alpha f \rangle^{\Omega}.$$

Lemma 3.51. *$H^k(\Omega, \mathbb{C}^l)$ is a Hilbert space for all $k \in \mathbb{N}_{\geq 0}$.*

Proof: Assume f_n is a Cauchy in $H^k(\Omega, \mathbb{C}^l)$. Then $D^\alpha f_n$ is Cauchy in $L^2(\Omega, \mathbb{C}^l)$ for all $|\alpha| \leq k$, and we denote its limit by $F^{(\alpha)} \in L^2(\Omega, \mathbb{C}^l)$. Then one can easily prove that $F^{(0,\dots,0)}$ is the limit of f_n in $H^k(\Omega, \mathbb{C}^l)$, and in fact $F^{(\alpha)} = D^\alpha f$. ∎

Although we will not use this result, the *Meyers-Serrin* Theorem states that $H^k(\Omega, \mathbb{C}^l) \cap \mathscr{E}(\Omega, \mathbb{C}^l)$ is dense in $H^k(\Omega, \mathbb{C}^l)$. Note that smooth compactly supported are only dense in $H^k(\Omega, \mathbb{C}^l)$, if $\Omega = \mathbb{R}^m$. This justifies to give the closure of smooth compactly supported functions in $H^k(\Omega, \mathbb{C}^l)$ a new symbol (typically something like $H^k_0(\Omega, \mathbb{C}^l)$).

Example 3.52. If $I = (a, b)$ is an interval, where $-\infty \leq a < b \leq -\infty$, then for all $k \in \mathbb{N}_{\geq 1}$ the space $H^k(I, \mathbb{C}^l)$ is given by all $f \in C^{k-1}(I, \mathbb{C}^l)$, such that $\partial^j f \in L^2(I, \mathbb{C}^l)$ for $j = 0, \dots, k-1$ and such that $\partial T_{\partial^{k-1} f}$ is absolutely continuous and in $L^2(I, \mathbb{C}^l)$. Thus, e.g., $H^1(I, \mathbb{C}^l)$ does not contain any noncontinuous functions. One can show[7] that in fact for all $f \in H^k(I, \mathbb{C}^l)$ the functions $\partial^j f$ have limits as $x \to a$ and $x \to b$ for $j = 0, ..., k-1$.

There is no natural way to define $H^k(\Omega, \mathbb{C}^l)$, if $k \in \mathbb{R}$. However, if $\Omega = \mathbb{R}^m$, we can observe that for $k \in \mathbb{N}_{\geq 0}$ and $f \in \mathscr{D}'(\mathbb{R}^m)$ one has $f \in H^k(\mathbb{R}^m, \mathbb{C}^l)$, if and only $f \in \mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l)$ and $(1 + |\zeta|^2)^{k/2} F f \in L^2(\mathbb{R}^m, \mathbb{C}^l)$, and that the scalar product $\langle f, g \rangle_{H^k}^{\mathbb{R}^m}$ is equivalent to

$$\langle f, g \rangle_{H^k} = \int_{\mathbb{R}^m} (Ff(\zeta), Fg(\zeta))(1 + |\zeta|^2)^k d\zeta.$$

The latter data make sense for $k \in \mathbb{R}$, and justify:

Definition 3.53. For all $s \in \mathbb{R}$ one defines $H^s(\mathbb{R}^l, \mathbb{C}^l)$ to be the space of all $f \in \mathscr{S}'(\mathbb{R}^l, \mathbb{C}^l)$ such that

$$\int_{\mathbb{R}^m} |Ff(\zeta)|^2 (1 + |\zeta|^2)^s d\zeta < \infty.$$

One defines a scalar product on $H^s(\mathbb{R}^l, \mathbb{C}^l)$ by setting

$$\langle f, g \rangle_{H^s} = \int_{\mathbb{R}^m} (Ff(\zeta), Fg(\zeta))(1 + |\zeta|^2)^s d\zeta$$

and calls $H^s(\mathbb{R}^l, \mathbb{C}^l)$ the *Sobolev space of order s*.

Note that it is hidden in the definition that one assumes

$$Ff \in \bigcup_{p \in [1, \infty]} L^p(\mathbb{R}^l, \mathbb{C}^l),$$

keeping in mind that

$$L^p(\mathbb{R}^l, \mathbb{C}^l) \hookrightarrow \mathscr{S}'(\mathbb{R}^l, \mathbb{C}^l).$$

Note that

$$H^0(\mathbb{R}^l, \mathbb{C}^l) = L^2(\mathbb{R}^l, \mathbb{C}^l)$$

and

$$H^s(\mathbb{R}^l, \mathbb{C}^l) \subset H^t(\mathbb{R}^l, \mathbb{C}^l) \quad \text{if } t > s \text{, continuously.}$$

Let us see how the definition works:

Example 3.54. For all $x \in \mathbb{R}^m$ we have seen previously that

$$F\delta_x(\zeta) = (2\pi)^{-m/2} e^{-i(x, \zeta)},$$

a bounded function of $\zeta \in \mathbb{R}^m$. It follows that $\delta_x \in H^s(\mathbb{R}^m)$, if and only if $s < -m/2$, as precisely for such s one has

$$\int_{\mathbb{R}^m} |F\delta_x(\zeta)|^2 (1 + |\zeta|^2)^s d\zeta < \infty.$$

[7]On can also prove with an interpolation inequality that for all $f \in C^{k-1}(I, \mathbb{C}^l) \cap L^2(I, \mathbb{C}^l)$ with $\partial T_{\partial^{k-1} f}$ absolutely continuous and in $L^2(I, \mathbb{C}^l)$, one automatically has $\partial^j f \in L^2(I, \mathbb{C}^l)$ for $j = 0, \dots, k-1$.

One can show easily that for all $s < -m/2$ the map

$$\mathbb{R}^m \longrightarrow H^s(\mathbb{R}^l, \mathbb{C}^l), \quad x \longmapsto \delta_x$$

is continuous.

Some basic properties of the H^s-Sobolev spaces are collected in the proposition below. In its proof we will need:

Lemma 3.55 (Peetre's inequality)**.** *Assume X is a seminormed space and $s \in \mathbb{R}$. Then for all $x, y \in X$ one has*

$$(1 + \|x\|)^s \leq (1 + \|x - y\|)^{|s|}(1 + \|y\|)^s,$$

in particular,

$$(1 + \|x\|^2)^s \leq C_s(1 + \|x - y\|^2)^{|s|}(1 + \|y\|^2)^s,$$

where $C_s > 0$ is a universal constant that only depends on s.

Proof: Both inequalities have only to be proved for $s \geq 0$ (the $s < 0$ then follows from changing the roles of x and y). If we can show that the first inequality holds for all $s \geq 0$, the second one follows from

$$\begin{aligned}(1 + \|x\|^2)^s &\leq (1 + 2\|x\| + \|x\|^2)^s = (1 + \|x\|)^{2s} \\ &\leq (1 + \|x - y\|)^{2s}(1 + \|y\|)^{2s} \leq 2^s 2^s (1 + \|x - y\|^2)^s (1 + \|y\|^2)^s,\end{aligned}$$

where we have used

$$(a + b)^2 \leq 2(a^2 + b^2) \quad \text{for all } a, b \geq 0.$$

Finally, the first inequality follows for $s \geq 0$ simply from applying

$$(1 + \|u - v\|)^s \leq (1 + \|u\|)^s(1 + \|v\|)^s$$

which follows from

$$1 + \|u - v\| \leq 1 + \|u\| + \|v\| \leq (1 + \|u\|)(1 + \|v\|),$$

with $u = x - y$, $v = y$. ■

Proposition 3.56. *Let $s \in \mathbb{R}$.*

1. For all $t \in \mathbb{R}$ the map

$$\Lambda_{s,t} : H^s(\mathbb{R}^l, \mathbb{C}^l) \longrightarrow H^{s-t}(\mathbb{R}^l, \mathbb{C}^l), \quad f \longmapsto F^{-1}(1 + |\zeta|^2)^{t/2} F f$$

is an isomorphism of normed spaces.

2. $H^s(\mathbb{R}^l, \mathbb{C}^l)$ is a Hilbert space.

3. $\mathscr{S}(\mathbb{R}^l, \mathbb{C}^l)$ is dense in $H^s(\mathbb{R}^l, \mathbb{C}^l)$.

4. For all $\alpha \in \mathbb{N}^m_{\geq 0}$ the operator

$$D^\alpha : H^s(\mathbb{R}^l, \mathbb{C}^l) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^l, \mathbb{C}^l)$$

is well-defined and continuous.

5. For all $\psi \in \mathscr{S}(\mathbb{R}^l)$ the operator

$$H^s(\mathbb{R}^l, \mathbb{C}^l) \longrightarrow H^s(\mathbb{R}^l, \mathbb{C}^l), \quad f \longmapsto \psi f$$

is well-defined and continuous.
6. The scalar product $\langle \cdot, \cdot \rangle$ *on* $\mathscr{S}(\mathbb{R}^l, \mathbb{C}^l)$ *extends continuously to an antidual pairing between* $H^s(\mathbb{R}^l, \mathbb{C}^l)$ *and* $H^{-s}(\mathbb{R}^l, \mathbb{C}^l)$*. The pairing incuces a canonical antiisomorphism of Banach spaces* $H^s(\mathbb{R}^l, \mathbb{C}^l)' \cong H^{-s}(\mathbb{R}^l, \mathbb{C}^l)$*.*

Proof: 1. Clear.
2. Follows from 1.
3. The map

$$\Lambda_{0,-t} : L^2(\mathbb{R}^l, \mathbb{C}^l) \longrightarrow H^t(\mathbb{R}^l, \mathbb{C}^l)$$

is an isometric isomorphism. As $\mathscr{S}(\mathbb{R}^l, \mathbb{C}^l)$ is dense in $L^2(\mathbb{R}^l, \mathbb{C}^l)$, it follows that

$$\Lambda_{0,-t}(\mathscr{S}(\mathbb{R}^l, \mathbb{C}^l)) \subset H^s(\mathbb{R}^l, \mathbb{C}^l)$$

is a dense subspace. But we have

$$\Lambda_{0,-t}(\mathscr{S}(\mathbb{R}^l, \mathbb{C}^l)) \subset \mathscr{S}(\mathbb{R}^l, \mathbb{C}^l).$$

4. For all $\psi \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$, $\zeta \in \mathbb{R}^m$, one has

$$|FD^\alpha \psi(\zeta)| = |\zeta^\alpha F\psi(\zeta)| \leq |(1 + |\zeta|^2)^{|\alpha|/2}||F\psi(\zeta)|,$$

so that

$$\|D^\alpha \psi\|^2_{H^{s-|\alpha|}} = \int |FD^\alpha \psi(\zeta)|^2 (1 + |\zeta|^2)^{s-|\alpha|} d\zeta \leq \|\psi\|^2_{H^s}.$$

Now the claim follows from the density of $\mathscr{S}$ in H^s.
5. Assume $g \in \mathscr{S}(\mathbb{R}^l, \mathbb{C}^l)$. Then for all $t > 0$ we have

$$\begin{aligned}
&|F(\psi \cdot g)(x)|^2 = |F\psi * Fg(x)|^2 \leq \left(\int |F\psi(y)||Fg(x-y)| dy \right)^2 \\
&= \left(\int |F\psi(y)|(1 + |y|^2)^{t/2} |Fg(x-y)|(1 + |y|^2)^{-t/2} dy \right)^2 \\
&\leq \|\psi\|^2_{H^t} \int |Fg(x-y)|^2 (1 + |y|^2)^{-t} dy,
\end{aligned}$$

where we have used Cauchy-Schwarz. So

$$\begin{aligned}
&\|\psi g\|^2_{H_s} \leq \|\psi\|^2_{H^t} \int \int |Fg(x-y)|^2 (1 + |y|^2)^{-t} (1 + |x|^2)^s dy dx \\
&\leq \|\psi\|^2_{H^t} \int \int |Fg(x-y)|^2 (1 + |y|^2)^{-t} (1 + |x|^2)^s dx dy \\
&= \|\psi\|^2_{H^t} \int \int |Fg(z)|^2 (1 + |z+y|^2)^s dz (1 + |y|^2)^{-t} dy \\
&\leq C_s \|\psi\|^2_{H^t} \int \int |Fg(z)|^2 (1 + |z|^2)^s dz (1 + |y|^2)^{-t+|s|} dy \\
&= C_s \|\psi\|^2_{H^t} \|g\|^2_{H^s} \int (1 + |y|^2)^{-t+|s|} dy \\
&\leq C_{s,t} \|\psi\|^2_{H^t} \|g\|^2_{H^s},
\end{aligned}$$

where

$$C_{s,t} = C_s \int (1+|y|^2)^{-t+|s|} dy < \infty$$

if $t > 0$ is large enough. As $\mathscr{S}$ is dense in H^s, the proof is complete.
6. This follows from the fact that for Schwartz functions f, g one has

$$\begin{aligned}|\langle f, g\rangle| &= |\langle Ff, Fg\rangle| \\ &= \left| \int \left(Ff(\zeta)(1+|\zeta|^2)^{s/2}, Fg(\zeta)(1+|\zeta|^2)^{-s/2} \right) d\zeta \right| \leq \|f\|_{H^s} \|g\|_{H^{-s}},\end{aligned}$$

in view of Cauchy-Schwarz. That the pairing induces an isomorphism $H^s(\mathbb{R}^l, \mathbb{C}^l)' \cong H^{-s}(\mathbb{R}^l, \mathbb{C}^l)$ is left as an exercise. ■

As a consequence of the fourth and fifth statement of the previous proposition, we immediately get:

Corollary 3.57. *Every partial differential operator* $P = \sum_\alpha P_\alpha D^\alpha$ *of order* $\leq k$ *with* $P_{\alpha ij} \in \mathscr{S}(\mathbb{R}^m)$ *for all* $i = 1, \ldots, l$, $j = 1, \ldots, l'$ *maps*

$$P : H^s(\mathbb{R}^l, \mathbb{C}^l) \longrightarrow H^{s-k}(\mathbb{R}^l, \mathbb{C}^{l'})$$

continuously.

If $f \in H^s(\mathbb{R}^m, \mathbb{C}^l)$, one might expect that f should canonically induce an element of $H^s(\mathbb{R}^{m-1}, \mathbb{C}^l)$ by some sort of restriction. However, f need not be a function and if so, it need not have a canonically given representative, so that it doesn't make sense to consider $f(0, \cdot)$. There is a functional analytic way out of this dilemma, the price being that one looses 1/2 of the Sobolev order:

Theorem 3.58 (Trace Theorem)**.** *For all* $s > 1/2$ *the map*

$$\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow \mathscr{S}(\mathbb{R}^{m-1}, \mathbb{C}^l), \quad f \longmapsto f(0, \cdot)$$

extends to a continuous linear map

$$H^s(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow H^{s-1/2}(\mathbb{R}^{m-1}, \mathbb{C}^l).$$

Proof: Let $f \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ and write $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{m-1}$. By the invertibility of the Fourier transform we have

$$\begin{aligned}g(x') := f(0, x') &= C \int_{\mathbb{R}^m} e^{i(x', \zeta')} e^{i(0, \zeta_1)} Ff(\zeta_1, \zeta') d\zeta \\ &= C \int_{\mathbb{R}^{m-1}} e^{i(x', \zeta')} \int_{\mathbb{R}} Ff(\zeta_1, \zeta') d\zeta_1 d\zeta' = CF^{-1}h(x'),\end{aligned}$$

where

$$h(y) := \int_{\mathbb{R}} Ff(\zeta_1, y) d\zeta_1.$$

Thus

$$Fg(\zeta') = C \int_{\mathbb{R}} Ff(\zeta_1, \zeta') d\zeta_1 = C \int_{\mathbb{R}} Ff(\zeta) d\zeta_1.$$

Using Cauchy-Schwarz we estimate as follows,

$$
\begin{aligned}
|Fg(\zeta')|^2 &\leq C\left(\int_{\mathbb{R}} |Ff(\zeta)|^2 (1+|\zeta|^2)^s (1+|\zeta|^2)^{-s} d\zeta_1\right)^2 \\
&\leq C\int_{\mathbb{R}} |Ff(\zeta)|^2 (1+|\zeta|^2)^s d\zeta_1 \int_{\mathbb{R}} (1+|\zeta|^2)^{-s} d\zeta_1 \\
&= C(1+|\zeta'|^2)^{-s+1/2} \int_{\mathbb{R}} (1+t^2)^{-s} dt \cdot \int_{\mathbb{R}} |Ff(\zeta)|^2 (1+|\zeta|^2)^s d\zeta_1 \\
&= C'(1+|\zeta'|^2)^{-s+1/2} \int_{\mathbb{R}} |Ff(\zeta)|^2 (1+|\zeta|^2)^s d\zeta_1.
\end{aligned}
$$

Thus we have

$$
|Fg(\zeta')|^2 (1+|\zeta'|^2)^{s-1/2} \leq C\int_{\mathbb{R}} |Ff(\zeta)|^2 (1+|\zeta|^2)^s d\zeta_1
$$

and by integrating over $\mathbb{R}^{m-1}$ we end up with

$$
\|f(0,\cdot)\|^2_{H^{s-1/2}(\mathbb{R}^{m-1},\mathbb{C}^l)} = \|g\|^2_{H^{s-1/2}(\mathbb{R}^{m-1},\mathbb{C}^l)} \leq \|f\|^2_{H^{s}(\mathbb{R}^{m},\mathbb{C}^l)},
$$

finishing the proof. ■

Theorem 3.59 (Sobolev embedding theorem). *Let $k \in \mathbb{N}_{\geq 0}$. Then for all $s \in \mathbb{R}$ with $s > k + m/2$ the inclusion*

$$
\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l) \subset C_0^k(\mathbb{R}^m, \mathbb{C}^l)
$$

extends to a continuous embedding

$$
H^s(\mathbb{R}^m, \mathbb{C}^l) \hookrightarrow C_0^k(\mathbb{R}^m, \mathbb{C}^l).
$$

Note that this result implies that if $s > k + m/2 > 0$ every

$$
f \in H^s(\mathbb{R}^m, \mathbb{C}^l) \subset L^2(\mathbb{R}^m, \mathbb{C}^l)
$$

has a (of course uniquely determined) Lebesgue representative $\tilde{f} \in C_0^k(\mathbb{R}^{\cdot}\mathbb{C}^l)$ with

$$
\left\|\tilde{f}\right\|_{\infty,k} \leq C\|f\|_{H^s},
$$

where does not depend on f. Indeed, pick a sequence f_n of Schwartz function with $f_n \to f$ in $H^s(\mathbb{R}^m, \mathbb{C}^l)$. Then by the continuity of the above embedding f_n is Cauchy in $C_0^k(\mathbb{R}^m, \mathbb{C}^l)$ and can take $\tilde{f}$ to be the limit.

Proof of Theorem 3.59: We are going to prove the existence of a constant $C = C(k, m, s) > 0$ such that for all $f \in \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ one has

$$
\|f\|_{\infty,k} \leq C\|f\|_{H^s}. \tag{20}
$$

As $\mathscr{S}$ is dense in H^s, this will prove the claim. To see the latter inequality, let $|\alpha| \leq k$. We have

$$
\begin{aligned}
&|FD^\alpha f(\zeta)| = |\zeta^\alpha Ff(\zeta)| = |(1+|\zeta|^2)^{(s-k)/2}\zeta^\alpha Ff(\zeta)(1+|\zeta|^2)^{-(s-k)/2}| \\
&\leq C(1+|\zeta|^2)^{-(s-k)/2}(1+|\zeta|^2)^{s/2}|Ff(\zeta)|.
\end{aligned}
$$

Thus, by the Riemann-Lebesgue Lemma, the formula

$$F^{-1}D^\alpha f(x) = FD^\alpha f(-x),$$

and Cauchy-Schwarz, one gets

$$\begin{aligned}\|D^\alpha f\|_\infty &= \left\|FF^{-1}D^\alpha f\right\|_\infty \le \left\|F^{-1}D^\alpha f\right\|_{L^1}\\ &= \|FD^\alpha f\|_{L^1} \le C' \|f\|_{H^s},\end{aligned}$$

where

$$C' = C\cdot\left(\int_{\mathbb{R}^m}(1+|\zeta|^2)^{-s+k}d\zeta\right)^{1/2} < \infty,$$

as we have assumed $-s+k<m/2$. ■

Theorem 3.60 (Rellich's compactness lemma)**.** *Let $s\in\mathbb{R}$ and let f_n be a bounded sequence in $H^s(\mathbb{R}^m,\mathbb{C}^l)$ with support in a fixed compact set $K\subset\mathbb{R}^m$. Then there is a subsequence f_{n_l} of f_n which for all $t<s$ converges in $H^t(\mathbb{R}^m,\mathbb{C}^l)$.*

Proof: We start by noting that for all $f\in H^s(\mathbb{R}^m,\mathbb{C}^l)$ and all $g\in\mathscr{D}(\mathbb{R}^m)$ the function $F(gf)$ is smooth, and that one has $\partial^\alpha F(gf) = (\partial^\alpha Fg)*Ff$ (exercise). Pick now $g\in\mathscr{D}(\mathbb{R}^m)$ with $g=1$ in K. As we have $gf_n = f_n$, it follows from the above observation (with $\alpha=0$) that

$$Ff_n(\zeta) = Fg*Ff_n(\zeta) = \int Fg(\zeta-x)(1+|x|^2)^{-s/2}Ff_n(x)(1+|x|^2)^{s/2}dx.$$

With Cauchy-Schwarz this leads to the bound

$$|Ff_n(\zeta)| \le \|f_n\|_{H^s}\kappa(\zeta) \le C\kappa(\zeta) := C\left(\int|Fg(\zeta-x)|^2(1+|x|^2)^{-s}dx\right)^{1/2},$$

where κ is a continuous function. Likewise, one sees[8] by replacing Ff_n with $\partial_j Ff_n$ in the above argument and using $\partial_j Ff_n = (\partial_j Fg)*Ff_n$,

$$|\partial_j Ff_n(\zeta)| \le C\kappa_j(\zeta),$$

where κ_j is a continuous function. The first inequality implies that Ff_n is a locally uniformly bounded family of continuous functions. As gradient estimates lead to Lipschitz estimates, the second inequality implies that Ff_n is a locally equicontinuous family. It follows from Arzela-Ascoli that there exists a subsequence Ff_{n_j}, which converges locally uniformly. We now want to prove that f_{n_j} is Cauchy in $H^t(\mathbb{R}^m,\mathbb{C}^l)$ for all $t<s$. To this end, fix $\epsilon>0$ and let $r>0$ be arbitrary (to be chosen later). One has

$$\begin{aligned}&\|f_{n_l}-f_{n_k}\|^2_{H^t}\\ &= \int_{\mathbb{R}^m\setminus B_r(0)}|Ff_{n_l}(\zeta)-Ff_{n_k}(\zeta)|^2(1+|\zeta|^2)^s(1+|\zeta|^2)^{t-s}d\zeta\\ &+ \int_{B_r(0)}|Ff_{n_l}(\zeta)-Ff_{n_k}(\zeta)|^2(1+|\zeta|^2)^t d\zeta.\end{aligned}$$

[8] I forgot in the course to prove the local equicontinuity.

Using Cauchy-Schwarz and the boundedness of f_n in $H^s(\mathbb{R}^m,\mathbb{C}^l)$, one has

$$\int_{\mathbb{R}^m\setminus B_r(0)} |Ff_{n_l}(\zeta)-Ff_{n_k}(\zeta)|^2(1+|\zeta|^2)^s(1+|\zeta|^2)^{t-s}d\zeta \leq C(1+r^2)^{t-s},$$

which is $<\epsilon/2$ for some $r>0$. Fixing the latter r, the second integral is, with

$$C' := \int_{B_r(0)} (1+|\zeta|^2)^t d\zeta$$

one has

$$\int_{B_r(0)} |Ff_{n_l}(\zeta)-Ff_{n_k}(\zeta)|^2(1+|\zeta|^2)^t d\zeta \leq C' \sup_{B_r(0)} |Ff_{n_l}-Ff_{n_k}|^2,$$

which is $<\epsilon/2$ for large k,l, as Ff_{n_j} is uniformly convergent in the closure of $B_r(0)$. This completes the proof. ∎

There is no useful way to define H^s on Ω. However, localized versions of these spaces make sense:

Definition 3.61. Let $s\in\mathbb{R}$.
a) The space $H^s_c(\Omega,\mathbb{C}^l)$ is given by all $T\in H^s(\mathbb{R}^m,\mathbb{C}^l)$ such that T is compactly supported in Ω.
b) The space $H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$ is given by all $T\in\mathscr{D}'(\Omega,\mathbb{C}^l)$ such that $\phi T\in H^s_c(\Omega,\mathbb{C}^l)$ for all $\phi\in\mathscr{D}(\Omega)$.

The space $H^s_K(\Omega,\mathbb{C}^l)$ of all $T\in H^s_c(\Omega,\mathbb{C}^l)$ with support in the compact $K\subset\Omega$ is a Banach space with respect to $\|\cdot\|_{H_s}$, and $H^s_c(\Omega,\mathbb{C}^l)$ becomes an LF space with respect to the family of all seminorms p on $H^s_c(\Omega,\mathbb{C}^l)$ such that $p|_{H^s_K(\Omega,\mathbb{C}^l)}$ is continuous for all compact $K\subset\Omega$.

The space $H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$ becomes an F space with respect to the family of seminorms

$$\|\phi T\|_{H^s}, \quad \phi\in\mathscr{D}(\Omega).$$

We have continuous inclusions

$$H^s_c(\Omega,\mathbb{C}^l)\subset H^t_c(\Omega,\mathbb{C}^l), \quad \text{if } t>s,$$
$$H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l)\subset H^t_{\mathrm{loc}}(\Omega,\mathbb{C}^l)\subset\mathscr{D}'(\Omega,\mathbb{C}^l), \quad \text{if } t>s.$$

Moreover, the isomorphism of linear spaces

$$\mathscr{E}'(\Omega,\mathbb{C}^l)\longrightarrow\{T\in\mathscr{D}'(\mathbb{R}^m,\mathbb{C}^l):\mathrm{supp}(T)\subset\Omega \text{ is compact}\}, \tag{21}$$
$$T\longmapsto\big(\phi\longmapsto T(\phi|_\Omega)\big). \tag{22}$$

induces continuous embeddings

$$H^s_c(\Omega,\mathbb{C}^l)\hookrightarrow\mathscr{E}'(\Omega,\mathbb{C}^l),$$
$$H^s_c(\Omega,\mathbb{C}^l)\hookrightarrow H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l).$$

Proposition 3.56.3 implies that $\mathscr{D}(\Omega,\mathbb{C}^l)$ is dense in $H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$. Moreover, every differential operator

$$P : \mathscr{E}(\Omega,\mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega,\mathbb{C}^{l'})$$

or order $\leq k$ induces continuous linear maps

$$P : H^s_c(\Omega,\mathbb{C}^l) \longrightarrow H^{s-k}_c(\Omega,\mathbb{C}^{l'}),$$
$$P : H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l) \longrightarrow H^{s-k}_{\mathrm{loc}}(\Omega,\mathbb{C}^{l'}),$$

which follows from Corollary 3.57. The Sobolev embedding theorem can be localized as follows: For all $k \in \mathbb{N}_{\geq 0}$, $s \in \mathbb{R}$ with $s > k + m/2$ the inclusion

$$\mathscr{D}(\Omega,\mathbb{C}^l) \subset C^k(\Omega,\mathbb{C}^l)$$

extends to a continuous embedding

$$H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l) \hookrightarrow C^k(\Omega,\mathbb{C}^l).$$

Finally, the scalar product $\langle\cdot,\cdot\rangle^\Omega$ on $\mathscr{D}(\mathbb{R}^l,\mathbb{C}^l)$ extends continuously to an anti-dual pairing between $H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$ and $H^{-s}_c(\mathbb{R}^l,\mathbb{C}^l)$. The pairing induces an anti-isomorphism of locally convex spaces

$$H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l)' \cong H^{-s}_c(\Omega,\mathbb{C}^l).$$

Let us present an important application of this theory: the existence of smooth integral kernels. To this end we record that by the Riesz-Fischer duality theorem, given a Hilbert space X and a continuous linear functional T in X there exists a unique $\psi_T \in X$ such that for all $\phi \in X$ one has $T(\phi) = \langle \psi_T, \phi\rangle$. We also use the fact that the Hilbertian pairing

$$X \times X \longrightarrow \mathbb{K}, \quad (\phi_1,\phi_2) \longmapsto \langle \phi_1,\phi_2\rangle$$

is smooth, and that a map from Ω to a Banach space is weakly C^k, if and only of it is strongly C^{k-1}, where $k \in \mathbb{N}_{\geq 1}$. For $m = 1$ this was shown in an exercise, and since being C^k means the existence and continuity of all partial derivatives of order $\leq k$, the general case follows in fact from the $m = 1$ case.

Theorem 3.62. *Assume we are given a linear operator*

$$T : L^2(\Omega,\mathbb{C}^l) \longrightarrow L^2(\Omega,\mathbb{C}^l)$$

for which there exists continuous linear operators

$$T_1, T_2 : L^2(\Omega,\mathbb{C}^l) \longrightarrow L^2(\Omega,\mathbb{C}^l) \quad \textit{with } T = T_1T_2,$$

and numbers $s \in \mathbb{R}, k \in \mathbb{N}_{\geq 1}$ with

$$s > k + m/2, \quad T_j(L^2(\Omega,\mathbb{C}^l)) \subset H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l) \textit{ for both } j = 1,2.$$

Then one has $T(L^2(\Omega,\mathbb{C}^l)) \subset C^k(\Omega,\mathbb{C}^l)$ and there exists a unique map

$$T(\cdot,\cdot) \in C^{k-1}(\Omega\times\Omega, \mathrm{Mat}_{l\times l}(\mathbb{C}))$$

with

$$\int_\Omega |T(x,y)|^2 dy < \infty, \quad Tf(x) = \int_\Omega T(x,y)f(x)dy \quad \textit{for all } x \in \Omega, f \in L^2(\Omega, \mathbb{C}^l).$$

Proof: I will give a complete proof for the case $l = 1$. Even more, I will assume that the function spaces are defined over $\mathbb{R}$, so that I don't have to care about complex conjugation in the scalar products etc. The general complex and matrix-valued case is left as an exercise.

Note first that T is continuous as the T_j's are. By the local Sobolev embedding theorem we have

$$T_j(L^2(\Omega,\mathbb{R})) \subset C^k(\Omega,\mathbb{R}),$$

thus for all $f \in L^2(\Omega,\mathbb{R})$, one has $Tf = T_1 g$, where $g = T_2$, and so $Tf \in C^k(\Omega,\mathbb{R})$.

Let us now come to $T(\cdot,\cdot)$: clearly, if $T(\cdot,\cdot)$ exists, it unique by the fundamental lemma of distribution theory. For the existence, note that for all open relatively compact $V \subset \Omega$ one has

$$T_j(L^2(\Omega,\mathbb{R})) \subset C^k_b(V,\mathbb{R}).$$

In fact, the induced maps

$$L^2(\Omega,\mathbb{R}) \longrightarrow C^k_b(V,\mathbb{R}), \quad f \mapsto Tf$$

are continuous: Indeed, if $f_n \to f$ in $L^2(\Omega,\mathbb{R})$ and Tf_n converges in $C^k_b(V,\mathbb{R})$ to some $\psi \in C^k_b(V,\mathbb{R})$, then $Tf_n \to Tf$ in $L^2(V,\mathbb{R})$, thus w.l.o.g $Tf_n(x) \to Tf(x)$ for a.e. $x \in V$ and so $\psi = Tf$. This implies the asserted continuity by the closed graph theorem. The latter continuity in turn implies that for all V as above and all $w \in \mathbb{C}^l$ there is a constant $C = C(V,w) > 0$ such that for all $x \in \Omega$, $f \in L^2(\Omega,\mathbb{R})$ one has

$$|T_j f(x)| \leq C \left\| f \right\|^\Omega.$$

Thus, by the Riesz-Fischer duality theorem for Hilbert spaces for all $x \in \Omega$ there exists $T_{j,x} \in L^2(\Omega,\mathbb{R})$ such that for all $f \in L^2(\Omega,\mathbb{R})$ one has

$$T_j f(x) = \int_\Omega T_{j,x}(y) f(y) dy = \langle T_{j,x}, f \rangle^\Omega. \tag{23}$$

Define now

$$T(\cdot,\cdot) : \Omega \times \Omega \longrightarrow \mathbb{R}$$

by

$$T(x,y) := \langle T_{1,x}, T_{2,x} \rangle^\Omega.$$

By (23), the map $x \mapsto T_{j,x}$ is weakly C^k thus strongly C^{k-1} so that by the smoothness of the Hilbertian pairing the map $(x,y) \mapsto T(x,y)$ is C^{k-1} and so its adjoint. On the other hand, (23) one has

$$T(x,y) = T_1[T_{2,x}](y),$$

which is L^2 in y. Finally,

$$Tf(x) = \int_\Omega T(x,y)f(x)dy \quad \text{for all } x \in \Omega,\ f \in L^2(\Omega,\mathbb{R})$$

follows from a simple calculation using Fubini. ■

4. Pseudodifferential operators on open subsets of $\mathbb{R}^m$

In this section I will mainly follow the presentation given by E. van den Ban and M. Crainic in their lecture notes [1]. Classical references are [7, 8]. Whenever there is no danger of confusion, I will denote the distribution T_f which corresponds to a function f with f again.

We begin by repeating an intuitive calculation from the first lecture: assume we are given a differential operator

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha : \mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'})$$

of order $\leq k$. Then for all $f \in \mathscr{D}(\Omega, \mathbb{C}^l) \subset \mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$, $x \in \Omega$, the Fourier inversion theorem implies

$$Pf(x) = \int_{\mathbb{R}^m} e^{i(x,\zeta)} p(x,\zeta) Ff(\zeta) d\zeta,$$

where

$$p(x,\zeta) = \sum_{|\alpha| \leq k} P_\alpha(x) \zeta^\alpha \in \mathrm{Mat}_{l \times l'}(\mathbb{C}) \tag{24}$$

is the (full) symbol of P.

Lemma 4.1. *In the above situation, for all $\alpha, \beta \in \mathbb{N}^m$, $K \subset \Omega$, there exists a constant $C = C(K, \alpha, \beta) > 0$ such that*

$$|\partial_x^\alpha \partial_\zeta^\beta p(x,\zeta)| \leq C(1 + |\zeta|)^{k - |\beta|} \quad \textit{for all } x \in K,\ \zeta \in \mathbb{R}^m.. \tag{25}$$

Proof: From formula (24). ■

Thus, searching for a natural generalization of the theory of differential operators one is naturally lead to:

Definition 4.2. For $k \in \mathbb{R}$ the space $S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ of *symbols of order* $k \in \mathbb{R}$ is defined to be the space of all smooth maps

$$p : \Omega \times \mathbb{R}^m \longrightarrow \mathrm{Mat}_{l \times l'}(\mathbb{C})$$

which satisfy the bounds (25).

In case $l = l' = 1$, we will simply write $S^k(\Omega)$ and likewise in analogous situations in the sequel.
The space $S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ becomes an F space with respect to the seminorms

$$\sup_{(x,\zeta) \in K \times \mathbb{R}^n} |\partial_x^\alpha \partial_\zeta^\beta p(x,\zeta)| (1 + |\zeta|)^{|\beta| - k}, \tag{26}$$

where α, β runs through $\mathbb{N}^m$ and K runs through all compact subsets of $\mathbb{R}^m$. We have continuous inclusions

$$S^{k_1}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})) \subset S^{k_2}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})), \quad \text{if } k_1 \leq k_2.$$

One sets

$$S^{\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) := \bigcup_{k\in\mathbb{R}} S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$
$$S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) := \bigcap_{k\in\mathbb{R}} S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

Letting k run through $\mathbb{R}$ in (26), the space $S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ becomes a locally convex space again.

The following examples are illustrative:

Example 4.3. 1. We have seen that all matrix-valued polynomials of the form

$$p(x,\zeta) = \sum_{|\alpha|\leq k} P^{\alpha}(x)\zeta_{\alpha}$$

are in $S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$.
2. The function

$$p(x,\zeta) := (1+|\zeta|^2)^{k/2}$$

is in $S^k(\Omega)$, for all $k \in \mathbb{R}$.
3. The function

$$p(x,\zeta) := \Big(1+|\zeta|^2 + \sum_{i,j=1}^{m} x^i x^j \zeta_i\zeta_j\Big)^{-1}$$

is in $S^{-2}(\Omega)$.
4. The pointwise multiplication

$$S^a(\Omega,\dots)\times S^b(\Omega,\dots) \longrightarrow S^{a+b}(\Omega,\dots), \quad (v,w)\longmapsto vw$$

is well-defined for all $a, b \in \mathbb{R}$.

For the following definition, note that every map

$$A \in \mathscr{E}(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

determines a continuous linear operator

$$Q_A : \mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'}), \quad Q_A f(x) = \int_{\Omega} A(x,y) f(y) dy.$$

Definition 4.4. An operator of the form Q_A as above is called a *smoothing operator*. The linear space of smoothing operators is denoted by $\Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$.

Remark 4.5. 1. The map

$$\mathscr{E}(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{L}(\mathscr{D}(\Omega,\mathbb{C}^l), \mathscr{E}(\Omega,\mathbb{C}^{l'})), \quad A\longmapsto Q_A$$

is an embedding by the fundamental lemma of distribution theory.
2. Nontrivial smoothing operators are never local, that is, for all Q_A as above

$$\mathrm{supp}(Q_A f) \subset \mathrm{supp}(f) \quad \text{for all } f\in\mathscr{D}(\Omega,\mathbb{C}^l) \ \Rightarrow\ A = 0.$$

We can now give the definition of pseudodifferential operators:

Definition 4.6. 1. Let $k \in \mathbb{R} \cup \{-\infty\}$. For every $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ the continuous linear operator

$$\mathrm{Op}(p) : \mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'})$$

is defined by

$$\mathrm{Op}(p)f(x) = \int_{\mathbb{R}^m} e^{i(x,y)} p(x,\zeta) Ff(\zeta) d\zeta.$$

2. Let $k \in \mathbb{R}$. A linear operator

$$P : \mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'})$$

is called a *pseudodifferential operator of order* k, if there exists $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ and $Q \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ such that $P = \mathrm{Op}(p) + Q$. The linear space of such operators is denoted by

$$\Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \subset \mathscr{L}(\mathscr{D}(\Omega, \mathbb{C}^l), \mathscr{E}(\Omega, \mathbb{C}^{l'})).$$

At this point, we only remark that requiring $P = \mathrm{Op}(p) + Q$ rather than just $P = \mathrm{Op}(p)$ is needed to prove coordinate invariance later on, while smoothing operators play of course no role in the local regularity theory (so the price we pay in order to obtain for coordinate invariance is little).

Remark 4.7. 1. Note that every differential operator of order $\leq k$ is a pseudodifferential operator of order k.
2. In view of Remark 4.5.2, pseudodifferential operators need not be local, while of course differential operators are local. In fact these are essentially all local operators, as Peetre's theorem states that a (not necessarily continuous) local linear map

$$\mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'})$$

is automatically a differential operator, when restricted to $\mathscr{D}(U, \mathbb{C}^l)$, where $U \subset \Omega$ is open and relatively compact. However the order of this operator can go to ∞ as $U \to \Omega$. This result implies, e.g, that local differential operators on *compact* manifolds are automatically differential operators!

The map $p \mapsto \mathrm{Op}(p)$ seems to be not injective for general Ω's (although I am not aware of a counterexample). For $\Omega = \mathbb{R}^m$ we can use the Fourier transform to obtain:

Proposition 4.8. *The map*

$$S^k(\mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{L}(\mathscr{D}(\mathbb{R}^m, \mathbb{C}^l), \mathscr{E}(\mathbb{R}^m, \mathbb{C}^{l'})), \quad p \longmapsto \mathrm{Op}(p)$$

is injective.

Proof: Assume $\mathrm{Op}(p)f = 0$ for all $f \in \mathscr{D}(\mathbb{R}^m, \mathbb{C}^l)$. Fix $x \in \mathbb{R}^m$. Then we have

$$\int_{\mathbb{R}^m} e^{i(x,\zeta)} p(x,\zeta) g(\zeta) d\zeta = 0$$

for all $g \in F(\mathscr{D}(\mathbb{R}^m, \mathbb{C}^l))$. As $\mathscr{D}(\mathbb{R}^m, \mathbb{C}^l)$ is dense in $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ and F is an isomorphism of locally convex spaces from $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ to $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$, it follows that $F(\mathscr{D}(\mathbb{R}^m, \mathbb{C}^l))$ is dense[9] in $\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$. Thus, given $\phi\in\mathscr{D}(\mathbb{R}^m,\mathbb{C}^l)$ arbitrary, we can pick a sequence ϕ_n in $F(\mathscr{D}(\mathbb{R}^m,\mathbb{C}^l))$ with $\phi_n\to\phi$ in $\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$. Then we have

$$\int_{\mathbb{R}^m} e^{i(x,\zeta)}p(x,\zeta)\phi(\zeta)d\zeta=\lim_n\int_{\mathbb{R}^m} e^{i(x,\zeta)}p(x,\zeta)\phi_n(\zeta)d\zeta=0,$$

thus by the fundamental lemma of distribution theory we have $e^{i(x,y)}p(x,\zeta)=0$ for all $x,\zeta\in\mathbb{R}^m$, thus $p=0$. ■

Given $f:\Omega\to\mathbb{C}^l$, $g:\Omega\to\mathbb{C}^{l'}$ let

$$f\otimes g:\Omega\longrightarrow \mathrm{Mat}_{l\times l'}(\mathbb{C})$$

be the function defined by $(f\otimes g)_{ij}(x,y):=\overline{f^{(i)}}(x)g^{(j)}(y)$. Then *Schwartz's kernel theorem* states that that in fact for *every* continuous linear operator

$$P:\mathscr{D}(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{D}'(\Omega,\mathbb{C}^{l'})$$

there exists a unique distribution $K_P\in\mathscr{D}'(\Omega\times\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C}))$, *its so called Schwartz kernel*, such that for all $f\in\mathscr{D}(\Omega,\mathbb{C}^l)$, $g\in\mathscr{D}(\Omega,\mathbb{C}^{l'})$ one has

$$Pf(g)=K_P(f\otimes g).$$

The proof of this result is very technical and can be found in Hoermander's book. We will only need this statement for pseudodifferential operators for which we will provide a sketch of proof:

Theorem 4.9. *a) Assume $k\in\mathbb{R}$ and $P\in\Psi^k(\Omega;\mathbb{C}^l,\mathbb{C}^{l'})$. Then there exists a unique distribution $K_P\in\mathscr{D}'(\Omega\times\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C}))$ such that for all $f\in\mathscr{D}(\Omega,\mathbb{C}^l)$, $g\in\mathscr{D}(\Omega,\mathbb{C}^{l'})$ one has*

$$Pf(g)=K_P(f\otimes g).$$

The distribution K_P is called the Schwartz kernel *of P. In case $P=Q_A\in\Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^{l'})$ one has $K_{Q_A}=A$, where the continuous embedding*

$$L^1_{\mathrm{loc}}(\Omega\times\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C}))\hookrightarrow\mathscr{D}'(\Omega\times\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

is given by

$$A\longmapsto\Big(\phi\longmapsto\int_{\Omega\times\Omega}\mathrm{tr}(A(x,y)^*\phi(x,y))dxdy\Big).$$

b) Let

$$P:\mathscr{D}(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{E}'(\Omega,\mathbb{C}^{l'})$$

be a continuous linear operator. Then the following statements are equivalent:
i) one has $P(\mathscr{E}'(\Omega,\mathbb{C}^l))\subset\mathscr{E}(\Omega,\mathbb{C}^{l'})$ and the induced map

$$P:\mathscr{E}'(\Omega,\mathbb{C}^l)\longrightarrow\mathscr{E}(\Omega,\mathbb{C}^{l'})$$

[9] This is the place where we use $\Omega=\mathbb{R}^m$: $F(\mathscr{D}(\Omega,\mathbb{C}^l))$ is not dense in $\mathscr{S}(\mathbb{R}^m,\mathbb{C}^l)$ and of course not even a subspace of $\mathscr{D}(\mathbb{R}^m,\mathbb{C}^l)$.

is continuous.
ii) For all $r, s \in \mathbb{N}$ *one has* $P(H_c^{-s}(\Omega, \mathbb{C}^l)) \subset H_{\mathrm{loc}}^t(\Omega, \mathbb{C}^{l'})$ *and the induced map*

$$P : H_c^{-s}(\Omega, \mathbb{C}^l) \longrightarrow H_{\mathrm{loc}}^t(\Omega, \mathbb{C}^{l'})$$

is continuous.
iii) One has $P \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$.

Proof: a) The uniqueness is seen by showing the density of functions of the form $f \otimes g$ in $\mathscr{D}(\mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. For the existence, assume first $P = Q_A$ is smoothing. Then one has

$$\begin{aligned} Q_A f(g) &= \int_\Omega (Pf(x), g(x))dx = \int_\Omega \int_\Omega (A(x,y)f(y), g(x))dydx \\ &= \int_{\Omega\times\Omega} \mathrm{tr}\big(A(x,y)^*(f\otimes g)(x,y)\big)dxdy. \end{aligned}$$

It remains to consider the case $P = \mathrm{Op}(p)$, where $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\Omega)$. We get the linear map

$$F_2^{(k)} : S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{D}'(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

defined by

$$F_2^{(k)} q(\phi) := \int_{\Omega\times\mathbb{R}^m} \mathrm{tr}\left(q(x,\zeta)^* F_2\phi(x,\zeta)\right) d\zeta dx,$$

where F_2 denotes the Fourier transform acting on the second variable. Then with the diffeomorphism

$$L : \Omega \times \mathbb{R}^n \longrightarrow \Omega \times \mathbb{R}^n, \quad L(x,y) = (x, y-x),$$

the distribution

$$K_P := ((L^{-1})^*)' F_2^{(k)} p \,\,|_{\Omega\times\Omega} \in \mathscr{D}'(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

does the job, where

$$(L^{-1})^* : \mathscr{D}(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{D}(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

denotes the pullback with respect to L^{-1}, and

$$((L^{-1})^*)' : \mathscr{D}'(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{D}'(\Omega\times\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

its dual.
b) We are going to show iii) $\Leftrightarrow$ ii) $\Leftrightarrow$ i).
iii) $\Rightarrow$ ii): Note that for $r \in \mathbb{N}$ one has

$$H_{\mathrm{loc}}^r(\Omega, \mathbb{C}^l) = \{f \in L_{\mathrm{loc}}^2(\Omega, \mathbb{C}^l) : \partial^\alpha f \in L_{\mathrm{loc}}^2(\Omega, \mathbb{C}^l) \quad \text{for all } |\alpha| \le r\}$$

and that the F topology on $H_{\mathrm{loc}}^r(\Omega, \mathbb{C}^l)$ is equivalently given by the family of seminorms

$$\sum_{|\alpha|\le k} \|1_K \partial^\alpha f\|^\Omega, \qquad \text{where } K \subset \Omega \text{ is compact.}$$

Likewise, one has

$$H_c^r(\Omega, \mathbb{C}^l) = \{f \in L_c^2(\Omega, \mathbb{C}^l) : \partial^\alpha f \in L_c^2(\Omega, \mathbb{C}^l) \quad \text{for all } |\alpha| \le r\}$$

and that the LF topology on $H^r_c(\Omega,\mathbb{C}^l)$ is equivalently given by the family of seminorms p on $H^r_c(\Omega,\mathbb{C}^l)$, such that $p|_{H^r_K(\Omega,\mathbb{C}^l)}$ is compact for all compact $K\subset\Omega$. Now let $\tilde{P}\in\Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^{l'})$. Then using that $\partial^\alpha\tilde{P}$ is smoothing with

$$K_{\partial^\alpha\tilde{P}}(x,y)=\partial^\alpha_x K_{\tilde{P}}(x,y)$$

and Cauchy-Schwartz one can easily show that $\partial^\alpha P$, where $|\alpha|\leq s$, induces a continuous linear map

$$\partial^\alpha\tilde{P}:L^2_c(\Omega,\mathbb{C}^l)\longrightarrow L^2_{\mathrm{loc}}(\Omega,\mathbb{C}^l),$$

so that $\tilde{P}$ induces a continuous linear map

$$\tilde{P}:L^2_c(\Omega,\mathbb{C}^l)\longrightarrow H^s_{\mathrm{loc}}(\Omega,\mathbb{C}^l).$$

By duality,

$$\tilde{P}:H^{-s}_c(\Omega,\mathbb{C}^l)\longrightarrow L^2_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$$

continuously. Applying this with $\tilde{P}=\partial^\beta P$, where $|\alpha|\leq s$, shows that

$$P:H^{-s}_c(\Omega,\mathbb{C}^l)\longrightarrow H^t_{\mathrm{loc}}(\Omega,\mathbb{C}^l)$$

continuously.

ii) $\Rightarrow$ iii): Let $k\in\mathbb{N}$ be arbitrary. The map

$$\Omega\longrightarrow H^{-t}_c(\Omega,\mathbb{C}^l),\quad x\longmapsto\big(\,\phi\longmapsto\delta_x\otimes e_j(\phi):=\phi^{(j)}(x)\,\big)$$

is C^k, if we pick $t>m/2+k$, and one has

$$f=\sum_j\int_\Omega f^{(j)}(x)\delta_x\otimes e_j dx\quad\text{ for all } f\in\mathscr{D}(\Omega,\mathbb{C}^l),$$

where the integral converges in the strong sense (exercise), and thus commutes with continuous linear operators. By assumption, the map

$$H^{-t}_c(\Omega,\mathbb{C}^l)\times H^{-t}_c(\Omega,\mathbb{C}^l)\longrightarrow\mathrm{Mat}_{l\times l'}(\mathbb{C}),\quad(f,g)\longmapsto\langle Pf,g\rangle^\Omega_{H^{-t}_c,H^{-t}_{\mathrm{loc}}}$$

is the composition of a pairing with a continuous linear map, thus smooth. It follows that the map

$$A:\Omega\times\Omega\longrightarrow\mathrm{Mat}_{l\times l'}(\mathbb{C}),\quad\overline{A(x,y)_{ij}}:=\langle P\delta_y\otimes e_i,\delta_x\otimes e_j\rangle^\Omega_{H^{-t}_c,H^{-t}_{\mathrm{loc}}}$$

is C^k for all k, so smooth. Finally, we can calculate

$$\begin{aligned}\int_\Omega(Pf,g)&=\langle Pf,g\rangle^\Omega_{H^{-t}_c,H^{-t}_{\mathrm{loc}}}=\sum_j\int_\Omega\langle Pf,\delta_x\otimes e_j\rangle^\Omega_{H^{-t}_c,H^{-t}_{\mathrm{loc}}}g^{(j)}(x)dx\\&=\sum_{ij}\int_\Omega\overline{f^{(i)}(y)\overline{A(x,y)_{ij}}}dy g^{(j)}(x)dx\\&=\int_\Omega\int_\Omega(A(x,y)f(y),g(x))dydx.\end{aligned}$$

Thus, we have shown $P = Q_A$.
ii) $\Rightarrow$ i): The essential observation is that by the local Sobolev embedding theorem one has

$$\mathscr{E}(\Omega, \mathbb{C}^l) \cong \bigcap_{r\in\mathbb{Z}} H^r_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$$

as locally convex spaces, so that by duality

$$\mathscr{E}'(\Omega, \mathbb{C}^l) \cong \bigcup_{r\in\mathbb{Z}} H^r_c(\Omega, \mathbb{C}^l)$$

as locally convex spaces (exercise). Thus, if $f_n \to f$ in $\mathscr{E}'(\Omega, \mathbb{C}^l)$, then $f_n \to f$ in $H^{s'}_c(\Omega, \mathbb{C}^l)$ for some $s' \in \mathbb{Z}$ and so $f_n \to f$ in $H^{-s}_c(\Omega, \mathbb{C}^l)$ for some $s \in \mathbb{N}$, and by assumption $Pf_n \to Pf_n$ in $H^t_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$ for all $t \in \mathbb{N}$. This implies $Pf_n \to Pf$ in $H^{t'}_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$ for all $t' \in \mathbb{Z}$ and so $Pf_n \to Pf$ in $\mathscr{E}'(\Omega, \mathbb{C}^l)$.
i) $\Rightarrow$ ii) If $s, t \in \mathbb{Z}$ and $f_n \to f$ in $H^s_c(\Omega, \mathbb{C}^l)$, then $f_n \to f$ in $\mathscr{E}'(\Omega, \mathbb{C}^l)$ and $Pf_n \to Pf$ in $\mathscr{E}'(\Omega, \mathbb{C}^l)$ and so $Pf_n \to Pf_n$ in $H^t_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$. ■

As an immediate consequence of part b) we get:

Corollary 4.10. *There exists a canonical isomorphism of linear spaces*

$$\Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \cong \mathscr{L}(\mathscr{E}'(\Omega, \mathbb{C}^l), \mathscr{E}(\Omega, \mathbb{C}^{l'})).$$

Symbols in $S^{-\infty}$ correspond to smoothing operators and vice versa, showing that we could have allowed $k = -\infty$ in Definition 4.6.2.:

Proposition 4.11. *a) For every $p \in S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ there exists a unique*

$$A_p \in \mathscr{D}(\Omega \times \Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

with $\mathrm{Op}(p) = Q_{A_p}$*, in particular,* $\mathrm{Op}(p) \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$*. If* $\mathrm{Op}(p)|_{\mathscr{D}(\Omega\setminus K; \mathbb{C}^l)} = 0$ *for some compact $K \subset \Omega$, then one has* $\mathrm{supp}(A) \subset \Omega \times K$*.*
b) If

$$A \in \mathscr{E}(\Omega \times \Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})))$$

satisfies $\mathrm{supp}(A) \subset \Omega \times K$ *for some compact $K \subset \Omega$, then there exists a*

$$p \in S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

such that $Q_A = \mathrm{Op}(p)$*.*

Proof: Exercise. The key observation is that elements of $S^{-\infty}$ are Schwartz functions with respect to their second variable, and that the Fourier transform on the second variable

$$F_2^{(-\infty)} : S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

is an isomorphism of locally convex spaces. ■

Next, we show that pseudodifferential operators can be localized:

Proposition 4.12. *Given $\psi_1, \psi_2 \in \mathscr{D}(\Omega)$, $k \in \mathbb{R} \cup \{-\infty\}$, $P \in \Psi^d(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$, one has*

$$\psi_1 P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}), \quad P\psi_2 \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}), \quad \psi_1 P \psi_2 \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}).$$

Proof: Clearly it suffices to show

$$\psi_1 P \in \Psi^k(\Omega;\mathbb{C}^l,\mathbb{C}^{l'}),\quad P\psi_2 \in \Psi^k(\Omega;\mathbb{C}^l,\mathbb{C}^{l'}).$$

Case $k=-\infty$: For $P=Q_A$ one has

$$\psi_1 Q_A \psi_2 = Q_{\tilde{A}},$$

where $\tilde{A}(x,y)=\psi_1(x)A(x,y)\psi_2(y)$.
Case $k\in\mathbb{R}$: in view of the $k=-\infty$ case, it remains to show that

$$\psi_1 \mathrm{Op}(p), \mathrm{Op}(p)\psi_2 \in \Psi^d(\Omega;\mathbb{C}^l,\mathbb{C}^{l'}) \quad \text{for all } p\in S^k(\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

CLearly, $\psi_1\mathrm{Op}(p)=\mathrm{Op}(\tilde{p})$, where

$$\tilde{p}\in S^k(\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

is given by $\psi_1(x)p(x,\zeta)$. It remains to prove

$$\mathrm{Op}(p)\psi_2 \in \Psi^d(\Omega;\mathbb{C}^l,\mathbb{C}^{l'}).$$

For all $f\in\mathscr{D}(\Omega,\mathbb{C}^l)$, $x\in\Omega$ we calculate using the convolution theorem,

$$\begin{aligned}
\mathrm{Op}(p)(\psi_2 f)(x) &= \int_{\mathbb{R}^m} e^{i(x,\zeta)}p(x,\zeta)F(\psi_2 f)(\zeta)d\zeta\\
&= \int_{\mathbb{R}^m} e^{i(x,\zeta)}p(x,\zeta)F\psi_2(\zeta)Ff(\zeta)d\zeta\\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i(x,\zeta)}p(x,\zeta)F\psi_2(\zeta-y)Ff(y)dyd\zeta\\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i(x,\zeta)}e^{i(x,y)}p(x,\zeta+y)F\psi_2(\zeta)Ff(y)d\zeta dy\\
&= \int_{\mathbb{R}^m} e^{i(x,\zeta)}q(x,\zeta)Ff(y)d\zeta,
\end{aligned}$$

where

$$q(x,\zeta):=\int_{\mathbb{R}^m} e^{i(x,\zeta)}Ff(y)p(x,\zeta+y)dy.$$

It is tedious but straightforward from the definition of the S^k-topology to check that

$$q\in S^k(\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

■

Next, we show that the Schwartz kernel of a pseudodifferential operator can only be singular on the diagonal:

Proposition 4.13. *Assume $k\in\mathbb{R}\cup\{-\infty\}$ and $P\in\Psi^k(\Omega;\mathbb{C}^l,\mathbb{C}^{l'})$. Then one has*

$$K_P|_{(\Omega\times\Omega)\setminus\mathrm{diag}(\Omega)} \in \mathscr{E}((\Omega\times\Omega)\setminus\mathrm{diag}(\Omega),\mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

in other words,

$$\mathrm{sing}-\mathrm{supp}(K_P)\subset \mathrm{diag}(\Omega).$$

Proof: As the Schwartz kernels of smoothing operators are smooth, we can assume $P = \mathrm{Op}(p)$ for some $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Consider again the maps

$$F_2^{(k)} : S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{D}'(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

and

$$L : \Omega \times \mathbb{R}^m \longrightarrow \Omega \times \mathbb{R}^m.$$

from the proof of Theorem 4.9. As we have

$$K_P = ((L^{-1})^*)' F_2^{(k)} p \mid_{\Omega\times\Omega},$$

and

$$\mathrm{diag}(\Omega) = L^{-1}(\Omega \times \{0\})$$

it suffices to show that $F_2^{(k)} p$ is smooth away from $\Omega \times \{0\}$, in other words, it suffices to show that for all $\psi \in \mathscr{D}(\mathbb{R}^m)$ with $0 \neq \mathrm{supp}(\psi)$ one has that $\psi_{(0)} F_2^{(k)} p$ is smooth, where for all $n \in \mathbb{N}$ the function $\psi_{(n)} \in \mathscr{D}(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ is given by

$$\psi_{(n)}(x, \zeta)_{ij} := \delta_{ij} |\zeta|^{-2n} \psi(\zeta).$$

To this end, one checks as usual that for all $n \in \mathbb{N}$ one has

$$\psi_{(0)} F_2^{(k)} p = \psi_{(n)} F_2^{(k-2n)} (\Delta_2^n p)$$

where Δ_2 is the n-th power of the Laplace-operator $\Delta = \sum_{j=1}^m \partial_j^2$ acting on the second variable. Let $s \in \mathbb{N}$ be arbitrary. We are going to prove in a moment that for n large enough such that $k - 2n < -m - s$ the map $F_2^{(k-2n)}$ maps into $C^s(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, which will complete the proof. In order to prove the latter mapping property, we assume for notational convenience $l = l' = 1$. Then for all $q \in S^{k-2n}(\Omega)$, $\phi \in \mathscr{D}(\Omega \times \mathbb{R}^m)$, one has

$$\begin{aligned} F_2^{(k-2n)} q(\phi) &= \int_{\Omega\times\mathbb{R}^m} \overline{q(x,\zeta)} F_2 \phi(x,\zeta) d\zeta dx \\ &= \int_\Omega \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \overline{q(x,\zeta)} e^{-i(\zeta',\zeta)} d\zeta \phi(x,\zeta') dx d\zeta' \\ &= \int_{\Omega\times\mathbb{R}^m} \overline{r(x,\zeta)} \phi(x,\zeta') dx d\zeta', \end{aligned}$$

where

$$r(x,\zeta') := \int_{\mathbb{R}^m} q(x,\zeta) e^{i(\zeta',\zeta)} d\zeta.$$

Let $N \subset \Omega$ be a compact set with $\mathrm{supp}(\phi) \subset N \times N$. Using the Leibnitz rule and (25) we get for all $\zeta, \zeta' \in \mathbb{R}^m$, $x \in N$, $|\alpha| + |\beta| \leq s$ the estimates

$$\begin{aligned} &|\partial_x^\alpha \partial_{\zeta'}^\beta q(x,\zeta) e^{i(\zeta',\zeta)}| \\ &\leq C_{\alpha,\beta,m} |\partial_x^\alpha q(x,\zeta)| + C_{\alpha,\beta,m} |q(x,\zeta)| |\partial_{\zeta'}^\beta e^{i(\zeta',\zeta)}| \\ &\leq C_{\alpha,\beta,m,q,N} (1 + |\zeta|)^{k-2n} + C_{\alpha,\beta,m,q,N} (1 + |\zeta|)^{k-2n} |\partial_{\zeta'}^\beta e^{i(\zeta',\zeta)}| \\ &\leq C_{\alpha,\beta,m,q,N} (1 + |\zeta|)^{k-2n} + C_{\alpha,\beta,m,q,N} (1 + |\zeta|)^{k-2n+s}. \end{aligned}$$

This shows that one may differentiate under the integral to prove that r is a C^s map. ■

Let $\mathrm{pr}_j : \Omega \times \Omega \to \Omega$, $j = 1, 2$, denote the projection map onto the j-th slot.

Definition 4.14. Assume $k \in \mathbb{R} \cup \{-\infty\}$ and $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$. Then P is called *properly supported*, if the induced projection maps

$$\mathrm{pr}_j|_{\mathrm{supp}(K_P)} : \mathrm{supp}(K_P) \longrightarrow \Omega, \quad \mathrm{j=1,2}$$

are proper, that is, $\mathrm{pr}_j^{-1}(N) \subset \mathrm{supp}(K_P)$ is compact for all compact $N \subset \Omega$.

Remark 4.15. 1. P is above is properly supported, if and only if for all compact $N \subset \Omega$ one has that

$$(N \times \Omega) \cap \mathrm{supp}(K_P), \quad (\Omega \times N) \cap \mathrm{supp}(K_P)$$

are compact. Note here a subset $N \subset \mathrm{supp}(K_P)$ is compact (in the subspace topology), if and only if it is compact as a subset of Ω.
2. Every differential operator is properly supported.
3. If P is a pseudodifferential operator and $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$, then $\phi_1 P \phi_2$ is properly supported.

Properly supported pseudodifferential operators will turn out to have many technical advantages. It is thus important to know that every pseudodifferential operator can be approximated by a properly supported one, whose kernel is supported arbitrarily closely to $\mathrm{diag}(\Omega)$:

Theorem 4.16. *Assume $k \in \mathbb{R} \cup \{-\infty\}$, $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ and that $U \subset \Omega \times \Omega$ is an open neighbourhood of $\mathrm{diag}(\Omega)$. Then there is a $q \in S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ such that*

- $\mathrm{Op}(q)$ *is properly supported,*
- $\mathrm{supp}(K_{\mathrm{Op}(q)}) \subset U$,
- $P - \mathrm{Op}(q) \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$.

The proof relies on:

Lemma 4.17. *Assume $U \subset \Omega \times \Omega$ is an open neighbourhood of $\mathrm{diag}(\Omega)$. Then for every open covering $(V_i)_{i \in I}$ of Ω there exists a smooth partition of unity $(\phi_i)_{i \in I}$ subordinate to $(V_i)_{i \in I}$ such that for all $i, j \in I$ with*

$$\mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) \neq \emptyset$$

one has

$$\mathrm{supp}(\phi_i) \times \mathrm{supp}(\phi_j) \subset U.$$

Proof: Exercise. One shows that there exists a refinement $(V_i)_{i \in I}$ of $(V'_j)_{j \in J}$ which is locally finite such that for all $A, B \in (V'_i)_{j \in J}$ with $A \cap B$ nonempty one has $A \times B \subset U$. ■

Proof of Theorem 4.16: By the Lemma we can pick a family of functions $(\phi_i)_{i \in I} \subset \mathscr{D}(\Omega)$ such that $(\mathrm{supp}(\phi_i))_{i \in I}$ is a locally finite collection of sets, and

$$\sum_{i \in I} \phi_i = 1,$$

and for all $i, j \in I$ with

$$\operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset$$

one has

$$\operatorname{supp}(\phi_i) \times \operatorname{supp}(\phi_j) \subset U.$$

With $P_{ij} := \phi_i P \phi_j$, one finds that for all i, j there exists $q_{ij} \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ such that $P_{ij} = \mathrm{Op}(q_{ij})$. One has

$$K_{ij} := K_{P_{ij}} = (\phi_i \otimes \phi_j) K_P,$$

so that

$$\operatorname{supp}(K_{ij}) \subset \operatorname{supp}(\phi_i) \times \operatorname{supp}(\phi_j).$$

If

$$\operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) = \emptyset,$$

then K_{ij} is smooth away from $\operatorname{diag}(\Omega)$ and supported away from $\operatorname{diag}(\Omega)$, so smooth on $\Omega \times \Omega$. If

$$\operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset,$$

then

$$\operatorname{supp}(K_{ij}) \subset U.$$

Set

$$J := \{(i, j) \in I \times I : \operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset\}.$$

Then one has

$$\mathrm{pr}_1(\operatorname{supp}(q_{ij})) \subset \operatorname{supp}(\phi_i),$$

so that

$$\overline{\mathrm{pr}_1(\operatorname{supp}(q_{ij}))}, \quad (i, j) \in J$$

is a locally finite collection of sets. Thus

$$q := \sum_{(i,j)\in J} q_{ij} \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

while

$$K := \sum_{(i,j)\in(I\times I)\setminus J} K_{ij} \in \mathscr{E}(\Omega \times \Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

Then one easily finds $P = \mathrm{Op}(q) + Q_K$, and

$$K_{\mathrm{Op}(q)} = \sum_{(i,j)\in J} K_{ij},$$

in particular,

$$\operatorname{supp}(K_{\mathrm{Op}(q)}) \subset U.$$

Finally, it remains to show that $\mathrm{Op}(q)$ is properly supported: given $A \subset \Omega$ let

$$J_A := \{(i, j) \in J : \operatorname{supp}(\phi_i) \cap A \text{ is nonempty and finite}\}.$$

Then

$$(A \times \Omega) \cap \operatorname{supp}(K_{\mathrm{Op}(q)}) \subset \bigcup_{(i,j)\in J_A} \operatorname{supp}(\phi_i) \times \operatorname{supp}(\phi_j),$$

so that $(A \times \Omega) \cap \mathrm{supp}(K_{\mathrm{Op}(q)})$ is compact. The compactness of $(\Omega \times A) \cap \mathrm{supp}(K_{\mathrm{Op}(q)})$ is seen analogously. ■

Definition 4.18. Given a countable set N, a function $N \to \mathbb{R}$, $n \mapsto k_n$, with $k_n \to -\infty$ as $n \to \infty$[10], $k \in \mathbb{R}$, a symbol $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, and for each $n \in N$ a symbol $p_n \in S^{k_n}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Then one writes

$$p \sim \sum_{n\in N} p_n,$$

if for all $k' \in \mathbb{R}$ there exists a finite set $F_0 \subset N$ such that for all finite subsets $F \subset N$ with $F_0 \subset F$ one has

$$p - \sum_{n\in F} p_n \in S^{k'}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

One then calls $\sum_{n\in N} p_n$ an *asymptotic expansion* of p.

Remark 4.19. If $p, q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ and

$$p \sim \sum_{n\in N} p_n, \quad q \sim \sum_{n\in N} p_n,$$

then one has $p - q \in S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Thus any asymptotic expansion of p determines p modulo $S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$.

Proposition 4.20. *Let $k \in \mathbb{R}$, $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, $\phi \in \mathscr{D}(\Omega)$ and define $q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ by*

$$q(x, \zeta) := \int_{\mathbb{R}^m} e^{i(x,y)} p(x, y + \zeta) F\phi(y) dy.$$

Then one has

$$q \sim \sum_{\alpha\in\mathbb{N}^m} D^\alpha_x \phi \partial^\alpha_y p.$$

Proof: Exercise. The only idea one needs is to use the multidimensional Taylor formula with remainder to rewrite $p(x, y + \zeta)$. ■

Given $k \in \mathbb{R} \cup \{-\infty\}$ and a symbol $q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ and $\Omega' \subset \Omega$ open we define $q_{\Omega'} \in S^k(\Omega', \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ by $q_{\Omega'} := q|_{\Omega'\times\mathbb{R}^m}$.

Corollary 4.21. *Let $k \in \mathbb{R}$, $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, let $\Omega' \subset \Omega$ open with $\overline{\Omega'} \subset \Omega$ compact, $\phi \in \mathscr{D}(\Omega)$ with $\phi = 1$ in an open neighbourhood of $\overline{\Omega'}$. Then there is a $q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ with*

$$\mathrm{Op}(q) = \mathrm{Op}(p)\phi$$

and

$$q_{\Omega'} - p_{\Omega'} \in S^{-\infty}(\Omega', \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

[10] that is, for all $r \in \mathbb{R}$ there exists a finite set $F \subset N$ such that for all $n \in N \setminus F$ one has $k_n < r$.

Proof: We can assume $k \in \mathbb{R}$. Define q as in Proposition 4.20. Then $\mathrm{Op}(q) = \mathrm{Op}(p)\phi$ by the proof of Proposition 4.12, and Proposition 4.20 implies

$$q_{\Omega'} \sim \sum_{\alpha \in \mathbb{N}^m} (D_x^\alpha \phi \partial_y^\alpha p)_{\Omega'},$$

but in view of $D^\alpha \phi = 0$ for all $\alpha \in \mathbb{N}^m \setminus \{0\}$ and $\phi p = p$ in Ω' one trivially

$$p_{\Omega'} \sim \sum_{\alpha \in \mathbb{N}^m} (D_x^\alpha \phi \partial_y^\alpha p)_{\Omega'},$$

thus

$$q_{\Omega'} - p_{\Omega'} \in S^{-\infty}(\Omega', \mathrm{Mat}_{l \times l'}(\mathbb{C}))$$

by Remark 4.19. ■

Now we can prove the following structure theorem:

Theorem 4.22. *Let $k \in \mathbb{R}$. Then the map $p \mapsto \mathrm{Op}(p)$ induces an isomorphism*

$$S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))/S^{-\infty}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})) \longrightarrow \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})/\Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \tag{27}$$

of linear spaces.

Proof: Clearly the map $p \mapsto \mathrm{Op}(p)$ induces a surjective linear map

$$S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})) \longrightarrow \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})/\Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}).$$

Thus it remains to prove that the kernel of the latter map is $S^{-\infty}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$, which follows if we can show that given an arbitrary $p \in S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ with $\mathrm{Op}(p) \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ one has $p \in S^{-\infty}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$. Pick

$$A \in \mathscr{E}(\Omega \times \Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$$

with $\mathrm{Op}(p) = Q_A$. We will assume for the moment that there exists a compact set $N \subset \Omega$ with

$$\mathrm{supp}(p) \subset N \times \mathbb{R}^m, \quad \mathrm{supp}(A) \subset N \times \mathbb{R}^m. \tag{28}$$

Assuming (28), we are going to prove that for an arbitrary $\Omega' \subset \Omega$ open with $\overline{\Omega'} \subset \Omega$ compact, one has $p_{\Omega'} \in S^{-\infty}(\Omega', \mathrm{Mat}_{l \times l'}(\mathbb{C}))$. To this end, note that by (28) we have $p^{\mathbb{R}^m} \in S^k(\mathbb{R}^m, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$, where $p^{\mathbb{R}^m}$ denotes the extension by zero of p to $\mathbb{R}^m \times \mathbb{R}^m$. In particular, $\mathrm{Op}(p^{\mathbb{R}^m}) \in \Psi^k(\mathbb{R}^m; \mathbb{C}^l, \mathbb{C}^{l'})$ coincides with $\mathrm{Op}(p)$ on $\mathscr{D}(\Omega, \mathbb{C}^l)$. Pick $\phi \in \mathscr{D}(\Omega)$ with $\phi = 1$ in an open neighbourhood of $\overline{\Omega'}$. Then one has

$$\mathrm{Op}(p^{\mathbb{R}^m})\phi = \mathrm{Op}(p)\phi = Q_{A(1 \otimes \phi)} \in \Psi^{-\infty}(\mathbb{R}^m; \mathbb{C}^l, \mathbb{C}^{l'}), \tag{29}$$

which, as $A(1 \otimes \phi)$ is compactly supported in $\Omega \times \Omega$, implies by Proposition 4.11 b) the existence of some

$$r \in S^{-\infty}(\mathbb{R}^m, \mathrm{Mat}_{l \times l'}(\mathbb{C})) \tag{30}$$

with

$$Q_{A(1 \otimes \phi)} = \mathrm{Op}(r). \tag{31}$$

Applying Corollary 4.21 shows

$$\mathrm{Op}(p^{\mathbb{R}^m})\phi = \mathrm{Op}(q), \tag{32}$$

where

$$q \in S^k(\mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

and

$$q_{\Omega'} - p^{\mathbb{R}^m}_{\Omega'} \in S^{-\infty}(\mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

It follows from (29), (31), (32) that $\mathrm{Op}(r) = \mathrm{Op}(q)$ and so $p = q$, as the quantization map is injective on $\mathbb{R}^m$. Thus

$$p_{\Omega'} - r_{\Omega'} = p_{\Omega'} - q_{\Omega'} \in S^{-\infty}(\Omega', \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

which completes the proof in view of (30) under (28).
It remains to remove (28): so assume $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ with $\mathrm{Op}(p) \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ and pick

$$A \in \mathscr{E}(\Omega \times \Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

with $\mathrm{Op}(p) = Q_A$. Given $\Omega' \subset \Omega$ open with $\overline{\Omega'} \subset \Omega$ compact, pick $\chi \in \mathscr{D}(\Omega)$ with $\chi = 1$ in an open neighbourhood of $\overline{\Omega'}$. Then χp and $(\chi \otimes 1)A$ satisfy (28) and $\mathrm{Op}(\chi p) = Q_{(\chi\otimes 1)A}$ is smoothing, too. Thus by the previous case we have

$$p_{\Omega'} = (\chi p)_{\Omega'} \in S^{-\infty}(\Omega', \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

This completes the proof. ■

Definition 4.23. Let $k \in \mathbb{R}$.
1. The *full symbol map* of order k is the isomorphism of linear spaces

$$\sigma : \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})/\Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \longrightarrow S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))/S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \tag{33}$$

given by the inverse of the map (27).
2. The *principal symbol map* of order k is the isomorphism of linear spaces

$$\sigma_k : \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})/\Psi^{k-1}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \longrightarrow S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))/S^{k-1}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \tag{34}$$

which is induced by the map (33).

Later on, an essential observation will be that the full symbol map does not make sense on manifolds, while the principal symbol map does.

The following 'completeness result' will be the key tool in the construction of parametrices for elliptic pseudodifferential operators, for it will replace the typical Neumann-series argument from functional analysis:

Theorem 4.24 (Asymptotic completeness lemma)**.** *Let $(k_j) \subset \mathbb{R}$ be a sequence with $k_j \searrow -\infty$ as $j \to \infty$, and assume for every j we are given a symbol $p_j \in S^{k_j}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Then there exists a symbol $p \in S^{k_0}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ with the following properties:*

- *one has*

$$\mathrm{pr}_1(\mathrm{supp}(p)) \subset \overline{\bigcup_{j=0}^{\infty} \mathrm{pr}_1(\mathrm{supp}(p_j))}$$

- *one has*

$$p \sim \sum_{j=0}^{\infty} p_j.$$

Moreover, by Remark 4.19, such a p is uniquely determined modulo $S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$.

Lemma 4.25. *Let $(k_j) \subset \mathbb{R}$ be a sequence and assume for every j we are given a symbol $p_j \in S^{k_j}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Assume furthermore that $\chi \in \mathscr{E}(\mathbb{R}^m)$ with $\chi = 0$ on $\overline{B_1(0)}$ and $\chi = 1$ on $\mathbb{R}^m \setminus \overline{B_2(0)}$ and for all $t > 0$ define*

$$\chi_t \in \mathscr{E}(\Omega \times \mathbb{R}^m), \quad \chi_t(x, \zeta) := \chi(t\zeta).$$

Then for all $j \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^m$, $N \subset \Omega$, there exists $C_{j,\alpha,\beta,N} > 0$ with

$$\left|\partial_x^\alpha \partial_\zeta^\beta (\chi_t p_j)(x,\zeta)\right| \le C_{j,\alpha,\beta,N}(1+|\zeta|)^{k_j-|\beta|} \quad \textit{for all } x \in N,\ \zeta \in \mathbb{R}^m,\ 0 < t \le 1 \tag{35}$$

Proof: Fix $x \in N$, $0 < t \le 1$.
For all $\zeta \in \mathbb{R}^m$ with $|\zeta| \le t^{-1}$ one has $\chi_t(\zeta) = 0$ and the asserted estimate holds trivially.
For all $\zeta \in \mathbb{R}^m$ with $|\zeta| \ge 2t^{-1}$ one has $\chi_t(\zeta) = 1$ and the asserted estimate follows from

$$p_j \in S^{-k_j}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}).$$

For all $\zeta \in \mathbb{R}^m$ with $t^{-1} < |\zeta| < 2t^{-1}$ we infer from the Leibnitz formula and

$$p_j \in S^{-k_j}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})$$

the following inequalities,

$$\begin{aligned}
&\left|\partial_x^\alpha \partial_\zeta^\beta (\chi_t p_j)(x,\zeta)\right| \\
&\le \sum_{\gamma\le\beta} \left|\partial_x^\alpha \partial_\zeta^\gamma \chi_t(x,\zeta) \partial_\zeta^{\beta-\gamma} p_j(x,\zeta)\right| \\
&= \sum_{\gamma\le\beta} t^{|\gamma|} \left|\partial_\zeta^\gamma \chi(t\zeta) \partial_\zeta^{\beta-\gamma} p_j(x,\zeta)\right| \\
&\le \left(\max_{\gamma\le\beta} \left\|\partial_\zeta^\gamma \chi\right\|_\infty\right) \sum_{\gamma\le\beta} t^{|\gamma|} C_{j,\alpha,\gamma,N}(1+|\zeta|)^{k_j-|\beta|+|\gamma|} \\
&\le \left(\max_{\gamma\le\beta} \left\|\partial_\zeta^\gamma \chi\right\|_\infty\right) \sum_{\gamma\le\beta} t^{|\gamma|} C_{j,\alpha,\gamma,N}(1+|\zeta|)^{k_j-|\beta|+|\gamma|}.
\end{aligned}$$

Because of $t \le 1$, $|\zeta| \le 2t^{-1}$ one has

$$t^{|\gamma|} = 3^{|\gamma|}(3t^{-1})^{-\gamma} \le 3^{|\gamma|}(1+2t^{-1})^{-\gamma} \le 3^{|\gamma|}(1+|\zeta|)^{-\gamma},$$

so that the above can be further estimated by

$$\leq C_{j,\alpha,\beta,\chi,N}(1+|\zeta|)^{k_j-|\beta|},$$

completing the proof.

■

Lemma 4.26. *Let $(k_j)\subset\mathbb{R}$ be a sequence, and assume for every j we are given a symbol $p_j\in S^{k_j}(\Omega,\mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Assume furthermore that $\chi\in\mathscr{E}(\mathbb{R}^m)$ with $\chi=0$ on $\overline{B_1(0)}$ and $\chi=1$ on $\mathbb{R}^m\setminus\overline{B_2(0)}$. Then there exists a sequence $(t_j)\subset\mathbb{R}$ such that*

- *$t_j\to 0$ as $j\to\infty$,*
- *for all $l\in\mathbb{N},\alpha,\beta\in\mathbb{N}^m$ the series*

$$\sum_{j\geq l}(1+|\zeta|)^{|\beta|-k_l}\partial_x^\alpha\partial_\zeta^\beta(\chi_t p_j)$$

converges absolutely and uniformly in any subset of $\Omega\times\mathbb{R}^m$ of the form $N\times\mathbb{R}^m$, where $N\subset\Omega$ is compact.

Proof: Pick a sequence of open subsets (Ω_j) of Ω with compact closure such that

$$\bigcup_{j\in\mathbb{N}}\Omega_j=\Omega,\quad \overline{\Omega_j}\subset\Omega_{j+1}.$$

Pick $C_{j,\alpha,\beta,\overline{\Omega_j}}$ as in the above lemma and t_j with

$$\max(C_{j,\alpha,\beta,\overline{\Omega_j}},1)(1+t_j^{-1})^{k_j-k_l}<2^{-j},$$

so that $t_j\to 0$ as $j\to\infty$. Let $x\in\Omega_j$, $\zeta\in\mathbb{R}$.
If $|\zeta|\geq t_j^{-1}$, then we have

$$\begin{aligned}
&\left|\partial_x^\alpha\partial_\zeta^\beta(\chi_{t_j}p_j)(x,\zeta)\right|\\
&\leq \max(C_{j,\alpha,\beta,\overline{\Omega_j}},1)(1+|\zeta|)^{k_j-|\beta|}\\
&\leq \max(C_{j,\alpha,\beta,\overline{\Omega_j}},1)(1+|\zeta|)^{k_j-k_l}(1+|\zeta|)^{k_l-|\beta|}\\
&\leq \max(C_{j,\alpha,\beta,\overline{\Omega_j}},1)(1+t_j^{-1})^{k_j-k_l}(1+|\zeta|)^{k_l-|\beta|}\\
&\leq 2^{-j}(1+|\zeta|)^{k_l-|\beta|}.
\end{aligned}$$

If $|\zeta|<t_j^{-1}$, then we have $\chi_{t_j}(x,\zeta)=0$ so the above estimate remains trivially valid.
If $x\in N$, $\zeta\in\mathbb{R}^m$, we can pick j_0 with $N\subset\Omega_{j_0}$. Thus for all $j\geq j_0$ one has

$$\left|\partial_x^\alpha\partial_\zeta^\beta(\chi_{t_j}p_j)(x,\zeta)\right|\leq 2^{-j}(1+|\zeta|)^{k_l-|\beta|},$$

which completes the proof. ■

Proof of Theorem 4.24: Pick χ, (t_j) as in the previous lemma. Because of

$$\mathrm{supp}(\chi_{t_j})\cap(\Omega\times B_{1/t_j}(0))=\emptyset$$

and $1/t_j \to \infty$ as $j \to \infty$, the sum

$$p(x,\zeta) := \sum_{j}^{\infty} \chi_{t_j}(x,\zeta) p_j(x,\zeta)$$

is locally finite and thus defines a smooth function

$$p : \Omega \times \mathbb{R}^m \to \mathrm{Mat}_{l\times l'}(\mathbb{C}).$$

Because of the previous lemma, for all $l \in \mathbb{N}$ the series

$$r_l := \sum_{j\geq l} \chi_{t_j} p_j$$

converges absolutely with respect to all seminorms on $S^{k_l}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ so that by the completeness[11] of the latter space one gets

$$r_l \in S^{k_l}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})), \tag{36}$$

in particular,

$$p = r_0 \in S^{k_0}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) = S^{k}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

It remains to show

$$p \sim \sum_{j=0}^{\infty} p_j \tag{37}$$

To this end, note that

$$p - \sum_{j=0}^{l-1} p_j = \sum_{j=0}^{l-1} (\chi_{t_j} - 1) p_j + r_l.$$

As $\sum_{j=0}^{l-1}(\chi_{t_j} - 1)p_j$ is a smooth function on $\Omega \times \mathbb{R}^m$ with compact support in the ζ-variable, this function is an element of

$$S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \subset S^{k_l}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

which together with (36) implies

$$p - \sum_{j=0}^{l-1} p_j \in S^{k_l}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

This shows (37), as $k_l \to -\infty$ as $l \to \infty$. ■

We now prove that the formal adjoint of a pseudodifferential operator is again in that class:

[11] A sequence $\sum_j a_j$ in a complete locally convex space X converges, if for all continuous seminorms ν on X one has $\sum_j \nu(a_j) < \infty$. Completeness is essential here.

Theorem 4.27. *Let $k \in \mathbb{R} \cup \{-\infty\}$. Then for every $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ there exists a unique $P^\dagger \in \Psi^k(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ such that*

$$\int (Pf, g) = \int (f, P^\dagger g) \quad \text{for all } f \in \mathscr{D}(\Omega, \mathbb{C}^l),\ g \in \mathscr{D}(\Omega, \mathbb{C}^{l'}). \tag{38}$$

Moreover, if $\sigma(P)$ is represented by a and $\sigma(P^\dagger)$ is represented by b, then

$$b \sim \sum_{\alpha \in \mathbb{N}^m} D_x^\alpha \partial_\zeta^\alpha a^\dagger.$$

Note that (38) defines a linear operator

$$P^\dagger : \mathscr{D}(\Omega, \mathbb{C}^{l'}) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^l)$$

which is clearly uniquely determined by its defining property. The point of Theorem 4.27 is to show that $P^\dagger$ is a pseudodifferential operator of order k and to determine its full symbol. We prepare the proof with the following auxiliary result, which contains the whole algebra behind the argument:

Lemma 4.28. *Define*

$$T := F^{-1} e^{i(x,\zeta)} F : \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

and let $k \in \mathbb{R} \cup \{-\infty\}$. Then the following statements hold true:
i) Upon restriction, T induces a map

$$T : S_c^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

where S_c^k is understood to be the space of symbols having a compact support in the x-variable.
ii) If $p \in S_c^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, then one has $\mathrm{Op}(p)^\dagger = \mathrm{Op}(q)$, where $q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ is given by $q = Tp^\dagger$ (T applied to the ajoint of the matrix p), and one has

$$q \sim \sum_{\alpha \in \mathbb{N}} \frac{1}{\alpha} D_x^\alpha \partial_\zeta^\alpha p^\dagger.$$

Proof: Exercise. ■

Given the lemma, the general case reduces to a partition of unity argument (which is, however, a little subtle):

Proof of Theorem 4.27: Since smoothing operators obviously have formal adjoints ($Q_A^\dagger$ is induced by $A(y,x)^\dagger$) and since every pseudodifferential operator is equal to a properly supported quantization up to a smoothing error, we can assume that P is properly supported and that there exists $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ with $P = \mathrm{Op}(p)$. If p was compactly supported we could apply the previous lemma directly. To do so, we pick a partition of unity $(\psi_j)_{j\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ and for each j set

$$P_j := \psi_j P, \quad K_j := K_{P_j}.$$

Then P_j is properly supported, too and $P_j = \mathrm{Op}(p_j)$, where now

$$p_j := (\psi_j \otimes 1)p \in S^k_c(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

Now we can use the lemma to pick $q_j \in S^k(\Omega, \mathrm{Mat}_{l'\times l}(\mathbb{C}))$ with $\mathrm{Op}(p_j)^\dagger = \mathrm{Op}(q_j)$ and

$$q_j \sim \sum_{\alpha\in\mathbb{N}} \frac{1}{\alpha} D^\alpha_x \partial^\alpha_\zeta p_j^\dagger.$$

As we have (q_j corresponds to the adjoint of P_j)

$$\mathrm{pr}_1(\mathrm{supp}(q_j)) \subset \mathrm{pr}_2(\mathrm{supp}(K_j)),$$

the family $(\mathrm{pr}_1(\mathrm{supp}(q_j)))_j$ is locally finite, so that we can define

$$q := \sum_j q_j \in S^k(\Omega, \mathrm{Mat}_{l'\times l}(\mathbb{C}))$$

Using $p = \sum_j p_j$ it is now easily checked that

$$q \sim \sum_{\alpha\in\mathbb{N}} \frac{1}{\alpha} D^\alpha_x \partial^\alpha_\zeta p^\dagger,$$

and

$$P^\dagger = \mathrm{Op}(p)^\dagger = \mathrm{Op}(q),$$

finishing the proof. ■

It follows that for every $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ we can define the continuous linear map

$$P : \mathscr{E}'(\Omega, \mathbb{C}^{l'}) \longrightarrow \mathscr{D}'(\Omega, \mathbb{C}^l)$$

by

$$Pf(g) = f(P^\dagger g).$$

Lemma 4.29. *a) Assume $k \in \{-\infty\} \cup \mathbb{R}$ and $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$. The P is properly supported, if and only if P and $P^\dagger$ map*

$$P : \mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{D}(\Omega, \mathbb{C}^{l'}), \quad P^\dagger : \mathscr{D}(\Omega, \mathbb{C}^{l'}) \longrightarrow \mathscr{D}(\Omega, \mathbb{C}^l).$$

continuously. In particular, P is properly supported, if and only if $P^\dagger$ is properly supported.
b) Assume $k, r \in \mathbb{R} \cup \{-\infty\}$ and that $P \in \Psi^k(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l''})$, $Q \in \Psi^r(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ are properly supported. Then PQ is properly supported.

Proof: Exercise. ■

By density, it follows that every properly supported $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ also maps

$$P : \mathscr{E}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'}).$$

In particular, we can define the continuous linear maps

$$P : \mathscr{E}'(\Omega, \mathbb{C}^{l'}) \longrightarrow \mathscr{E}'(\Omega, \mathbb{C}^l), \quad P : \mathscr{D}'(\Omega, \mathbb{C}^{l'}) \longrightarrow \mathscr{D}'(\Omega, \mathbb{C}^l),$$

both by

$$Pf(g) = f(P^{\dagger}g).$$

Theorem 4.30. *Assume $k, s \in \mathbb{R}$ and $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$.*
a) P maps

$$P : H^s_c(\Omega, \mathbb{C}^l) \longrightarrow H^{s-k}_{\mathrm{loc}}(\Omega, \mathbb{C}^{l''})$$

continuously.
b) If P is properly supported, then P maps

$$P : H^s_{\mathrm{loc}/c}(\Omega, \mathbb{C}^l) \longrightarrow H^{s-k}_{\mathrm{loc}/c}(\Omega, \mathbb{C}^{l''})$$

continuously.

Proof: All asserted results can be deduced from the following result: assume $k \in \mathbb{R}$, $p \in S^k_c(\mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Then

$$\mathrm{Op}(p) : H^s(\mathbb{R}^m, \mathbb{C}^l) \longrightarrow H^{s-k}_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{C}^{l''})$$

continuously, which can be proved with some efforts using Schur's lemma. Detailed proofs can be found in [5, 8].

■

Theorem 4.27 has a simple but nevertheless important consequence: pseudofifferential operators are psuedo-local:

Corollary 4.31. *Let $k \in \mathbb{R} \cup \{-\infty\}$. Then every $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ is pseudo-local, that is, one has*

$$\mathrm{sing}-\mathrm{supp}(Pu) \subset \mathrm{sing}-\mathrm{supp}(u) \quad \textit{for all } u \in \mathscr{E}'(\Omega, \mathbb{C}^{l'}).$$

Proof: The only essential observation is that $P^{\dagger}$ has a Schwartz kernel $K_{P^{\dagger}}$ which is smooth away from the diagonal. The rest is routine: let

$$x \in \Omega \setminus \mathrm{sing}-\mathrm{supp}(\phi),$$

let Ω_1 be an open nbh of x_1 and let Ω_2 be an open nbh of $\mathrm{sing}-\mathrm{supp}(\phi)$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. It now suffices to show that for all $\phi_1 \in \mathscr{D}(\Omega_1)$ and all $\phi_2 \in \mathscr{D}(\Omega_2)$ with $\phi_2 = 1$ in an open nbh of $\mathrm{sing}-\mathrm{supp}(\phi)$ one has that $\phi_1(P\phi_2 u)$ is smooth. To this end, define a function

$$A : \Omega \times \Omega \longrightarrow \mathrm{Mat}_{l'\times l}(\mathbb{C}),$$
$$A(x, y) := \phi_1(x)\phi_2(y) K_{P^{\dagger}}|_{(\Omega\times\Omega)\setminus\mathrm{diag}(\Omega)}(x, y),$$

which is indeed a well-defined smooth function, as $\phi_1 \otimes \phi_2$ is supported away from $(\Omega \times \Omega) \setminus \mathrm{diag}(\Omega)$, the region where $K_{P^{\dagger}}$ is smooth. Now by a simple calculation one finds $\phi_1(P\phi_2 u) = Q_{A^{\dagger}}u$, the latter being a smooth function. ■

Our next aim is to prove that the composition of two properly supported pseudodifferential operators is a pseudodifferential operator. The main technical tool behind this result is:

Proposition 4.32. *Let $k, r \in \mathbb{R}\cup_{-\infty}$, $l, l', l'' \in \mathbb{N}$, and*

$$p \in S^k(\Omega, \mathrm{Mat}_{l'\times l''}(\mathbb{C})), \quad q \in S^r_c(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

Then one has $\mathrm{Op}(p)\mathrm{Op}(q) = \mathrm{Op}(r)$, *where (cf. Lemma 4.28)*

$$r \in S^{r+k}(\Omega, \mathrm{Mat}_{l\times l''}(\mathbb{C}))$$

is given by

$$r(x,\zeta) = e^{D_y,\partial_\zeta} p(x,\zeta)q(y,\eta)|_{y=x,\eta=\zeta}.$$

In particular, one has $\mathrm{pr}_1(\mathrm{supp}(r)) \subset \mathrm{pr}_1(\mathrm{supp}(p))$ *and (cf. exercise sheet 11)*

$$r \sim \sum_{\alpha\in\mathbb{N}^m} \frac{1}{\alpha!}\partial_\zeta^\alpha p D_x^\alpha q.$$

Proof: I will sketch a proof. A detailed proof can be found in section 7.1 of [1].
Step 1: Let us first assume

$$p \in \mathscr{D}(\Omega\times\mathbb{R}^m, \mathrm{Mat}_{l'\times l''}(\mathbb{C})), \quad q \in \mathscr{D}(\Omega\times\mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

a case which in fact shows the whole algebra behind the result. Considering p and q as smooth compactly supported functions on $\mathbb{R}^{2m}$ in the obvious way, for all $f \in \mathscr{D}(\Omega, \mathbb{C}^l)$ one can calculate, using Fubini,

$$\mathrm{Op}(p)\mathrm{Op}(q)f(x) = \int\int e^{i(\zeta,x-y)}p(x,\zeta)\mathrm{Op}(q)f(x)dyd\zeta \tag{39}$$

$$= \int\int\int e^{i(\zeta,x-y)}e^{i(y,\zeta)}p(x,\zeta)q(y,\eta)Ff(\eta)d\eta dyd\zeta \tag{40}$$

$$= \int e^{i(\eta,x)}r(x,\eta)Ff(\eta)d\eta, \tag{41}$$

where

$$r(x,\eta) := \int\int e^{i(\zeta-\eta,x-y)}p(x,\zeta)q(y,\eta)Ff(\eta)d\zeta dy. \tag{42}$$

For fixed $(x,\eta)\in\mathbb{R}^{2m}$ define $R \in \mathscr{D}(\mathbb{R}^m\times\mathbb{R}^m, \mathrm{Mat}_{l\times l''}(\mathbb{C}))$ by

$$R_{x,\eta}(y,\zeta) = p(x,\zeta)q(y,\eta).$$

Using that for the bounded smooth function $u(y,\zeta) := e^{-i(\zeta,y)}$ on $\mathbb{R}^{2m}$ one has $Fu(y,\zeta) = e^{i(\zeta,y)}$, using the convolution theorem, and using the definition of the operator $e^{(D_y,\partial_\zeta)}$, we get

$$Fe^{(D_y,\partial_\zeta)}p(x,\zeta)q(y,\eta)|_{y=x,\eta=\zeta} = Fe^{(D_y,\partial_\zeta)}R_{x,\eta}(x,\eta) = e^{i(x,\eta)}FR_{x,\eta}(x,\eta)$$
$$= Fu\cdot FR_{x,\eta}(x,\eta) = F(u * R_{x,\eta})(x,\eta),$$

and so by applying F^{-1} to this equation,

$$e^{D_y,\partial_\zeta}p(x,\zeta)q(y,\eta)|_{y=x,\eta=\zeta} = e^{-i(x,\zeta)} * R_{x,\eta}(x,\eta) = r(x,\eta),$$

where the latter formula follows immediately from (42). This proves the claim in this case (the asymptotic expansion follows from the formula for r, exercise sheet 11).

Step 2: If

$$p \in S^k_c(\Omega, \mathrm{Mat}_{l' \times l''}(\mathbb{C})), \quad q \in S^r_c(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})),$$

we use that the map

$$\rho : S^k_c(\Omega, \mathrm{Mat}_{l' \times l''}(\mathbb{C})) \times S^r_c(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})) \longrightarrow S^{k+r}_c(\Omega, \mathrm{Mat}_{l \times l''}(\mathbb{C}))$$

given by

$$\rho(p', q')(x, \zeta) := e^{(D_y, D_\zeta)} p'(x, \zeta) q'(y, \eta)|_{y=x, \eta=\zeta}$$

is bilinear and continuous, and that for fixed $f \in \mathscr{D}(\Omega, \mathbb{C}^l)$, the map

$$S^{k+r}_c(\Omega, \mathrm{Mat}_{l \times l''}(\mathbb{C})) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^l), \quad r' \longmapsto \mathrm{Op}(r')f$$

is linear and continuous. Then the asserted formula for r follows from the density of $\mathscr{D}(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l' \times l''}(\mathbb{C}))$ in $S^k_c(\Omega, \mathrm{Mat}_{l \times l''}(\mathbb{C}))$ in the S^{k+1}_c (!) topology, the density of the density of $\mathscr{D}(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ in $S^r_c(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ in the S^{r+1}_c topology, and Step 1.

Step 3: In the general case, with a partition of unity (ψ_j) associated to a relatively compact open cover of Ω we can write p as a locally finite sum $\sum_j \psi_j p$ with

$$\psi_j p \in S^k_c(\Omega, \mathrm{Mat}_{l' \times l''}(\mathbb{C})) \quad \text{for all } j.$$

By step 2, for all j we have $\mathrm{Op}(\psi_j p)\mathrm{Op}(q) = \mathrm{Op}(r_j)$, where

$$r_j(x, \zeta) = e^{(D_y, \partial_\zeta)} (\psi_j p)(x, \zeta) q(y, \eta)|_{y=x, \eta=\zeta}.$$

The latter formula implies that $(\mathrm{pr}_1(\mathrm{supp}(r_j)))_j$ is a locally finite collection of subsets of Ω, so that $r := \sum \psi_j r$ defines an element

$$r \in S^{k+r}_c(\Omega, \mathrm{Mat}_{l \times l''}(\mathbb{C}))$$

which satisfies

$$r(x, \zeta) = e^{(D_y, \partial_\zeta)} p(x, \zeta) q(y, \eta)|_{y=x, \eta=\zeta}.$$

■

We continue with auxiliary results:

Lemma 4.33. *For all $k \in \mathbb{R} \cup \{-\infty\}$ and all properly supported $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ there exists a symbol*

$$p \in S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$$

with $P = \mathrm{Op}(p)$.

Proof: Exercise. One first assumes $k = -\infty$ and uses Proposition 4.11 together with a partition of unity argument. The general case then follows easily from Proposition 4.16. ■

Theorem 4.34. *Assume $k, r \in \mathbb{R} \cup \{-\infty\}$ and that $P \in \Psi^k(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l''})$, $Q \in \Psi^r(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ are properly supported. Then one has $PQ \in \Psi^{k+r}(\Omega; \mathbb{C}^l, \mathbb{C}^{l''})$. If $\sigma(P)$ is represented by a, $\sigma(Q)$ is represented by b and $\sigma(PQ)$ by c, then one has*

$$c \sim \sum_{\alpha \in \mathbb{N}} \frac{1}{\alpha!} \partial^\alpha_\zeta a D^\alpha_x b,$$

in particular,

$$\sigma^{k+r}(PQ) = \sigma^k(P)\sigma^r(Q).$$

Proof: By Lemma 4.33, we can pick

$$p \in S^k(\Omega, \mathrm{Mat}_{l'\times l''}(\mathbb{C})), \quad q \in S^r(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

with $P = \mathrm{Op}(p)$ and $Q = \mathrm{Op}(q)$. Pick also a partition of unity (ψ_j) associated to an open relatively compact cover of Ω and set $p_j := \psi_j p$ so that with $P_j := \mathrm{Op}(p_j)$ one has $P = \sum_j P_j$. As P is properly supported, the sets

$$A_j := \mathrm{pr}_2\big(\mathrm{pr}_1^{-1}(\mathrm{supp}()) \cap \mathrm{supp}(K_P)\big) \subset \Omega$$

are compact, so that we can pick $\chi_j \in \mathscr{D}(\Omega)$ with $\chi_j = 1$ in A_j. Since

$$K_{P_j\chi_j} = (\psi_j \otimes \chi_j)K_P = (\psi_j \otimes 1)K_P$$

it follows that $P_j\chi_j = P_j$ and so $P_jQ = P_jQ_j$, where $Q_j = \mathrm{Op}(q_j)$ with

$$q_j := \chi_j q \in S^r_c(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

It follows from the above proposition that $P_jQ = \mathrm{Op}(r_j)$, where

$$r_j \in S^{r+k}(\Omega, \mathrm{Mat}_{l\times l''}(\mathbb{C}))$$

is given by

$$r_j(x,\zeta) = e^{D_y,\partial_\zeta} p_j(x,\zeta)q_j(y,\eta)|_{y=x,\eta=\zeta}$$

and

$$r_j \sim \sum_{\alpha\in\mathbb{N}^m} \frac{1}{\alpha!}\partial_\zeta^\alpha p_j D_x^\alpha q_j.$$

It follows that

$$\mathrm{pr}_1(\mathrm{supp}(r_j)) \subset \mathrm{supp}(\psi_j),$$

so that

$$r := \sum_j r_j \in S^{r+k}(\Omega, \mathrm{Mat}_{l\times l''}(\mathbb{C}))$$

is well-defined. Now one has

$$\mathrm{Op}(r) = \sum_j P_jQ_j = \sum_j P_jQ = PQ.$$

Moreover, $q_j = q$ in an open nbh of $\mathrm{supp}(\psi_j)$, so that

$$\partial_\zeta^\alpha p_j D_x^\alpha q_j = \partial_\zeta^\alpha p_j D_x^\alpha q,$$

and so

$$r \sim \sum_{\alpha \in \mathbb{N}^m} \sum_j \frac{1}{\alpha!} \partial_\zeta^\alpha p_j D_x^\alpha q_j$$

$$\sum_{\alpha \in \mathbb{N}^m} \sum_j \frac{1}{\alpha!} \partial_\zeta^\alpha p_j D_x^\alpha q$$

$$\sum_{\alpha \in \mathbb{N}^m} \frac{1}{\alpha!} \partial_\zeta^\alpha p D_x^\alpha q,$$

completing the proof. ■

Definition 4.35. Assume $k \in \mathbb{R} \cup \{-\infty\}$.
a) A symbol

$$p \in S^k(\Omega, \mathrm{Mat}_{l \times l'}(\Omega))$$

is called *elliptic*, if
(43)

$$pq - 1 \in S^{-1}(\Omega, \mathrm{Mat}_{l' \times l'}(\mathbb{C})), \quad qp - 1 \in S^{-1}(\Omega, \mathrm{Mat}_{l \times l}(\mathbb{C})) \quad \text{for some } q \in S^{-k}(\Omega, \mathrm{Mat}_{l' \times l}(\mathbb{C}))\ .$$

b) An equivalence class

$$[p]_k \in S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))/S^{k-1}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$$

is called *elliptic*, if p is elliptic.
c) $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ is called *elliptic*, if $\sigma^k(P)$ is elliptic.

As the pointwise multiplication

$$S^a(\Omega, \dots) \times S^b(\Omega, \dots) \longrightarrow S^{a+b}(\Omega, \dots), \quad (v, w) \longmapsto vw$$

is well-defined, it follows that part b) of the above definition is well-defined. Indeed, if

$$p, p' \in S^k(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$$

are given with

$$p' - p \in S^{k-1}(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$$

and (43), then one has

$$p'q - 1 = (p' - p)q + pq - 1 \in S^{-1}(\Omega, \mathrm{Mat}_{l' \times l'}(\mathbb{C})),$$

and likewise

$$qp' - 1 = q(p' - p) + qp - 1 \in S^{-1}(\Omega, \mathrm{Mat}_{l' \times l'}(\mathbb{C})),$$

proving the claim.

As the multiplication factors

$$\Big(S^a(\Omega, \dots)/S^{a-1}(\Omega, \dots)\Big) \times \Big(S^b(\Omega, \dots)/S^{b-1}(\Omega, \dots)\Big) \to S^{a+b}(\Omega, \dots)/S^{a+b-1}(\Omega, \dots),$$
$$([v]_a, [w]_b) \mapsto [vw]_{a+b},$$

we can state that

$$p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

is elliptic, if and only if there exists

$$q \in S^{-k}(\Omega, \mathrm{Mat}_{l'\times l}(\mathbb{C}))$$

with

$$[p]_d[q]_{-d} = [1]_0, \quad [q]_{-d}[p]_d = [1]_0.$$

The above notion of ellipticity is equivalent to the standard notion of (local) ellipticity of partial differential operators:

Lemma 4.36. *Assume $k \in \mathbb{N}$ and that*

$$P = \sum_\alpha P_\alpha D^\alpha : \mathscr{D}(\Omega, \mathbb{C}^l) \longrightarrow \mathscr{E}(\Omega, \mathbb{C}^{l'})$$

is a differential operator of order $\leq k$. Then P is elliptic, if and only if

$$\sum_{|\alpha|=k} P_\alpha(x)\zeta^\alpha \in \mathrm{GL}_{l\times l'}(\mathbb{C}) \quad \textit{for all } x \in \Omega,\ \zeta \in \mathbb{R}^m \setminus \{0\}, \tag{44}$$

in particular, one necessarily has $l = l'$.

Proof: Note first that

$$\sigma^k(P) = \left[\sum_{|\alpha|=k} P_\alpha \zeta^\alpha\right]_k,$$

where with the usual abuse of notation understand $P_\alpha\zeta^\alpha$ to be the function $(x,\zeta) \mapsto P_\alpha(x)\zeta^\alpha$.
Assume now (44) and pick a smooth function $\chi \in \mathscr{E}(\Omega \times \mathbb{R}^m)$ such that $\chi = 1$ in a nbh of $\Omega \times \{0\}$ and such that χ is compactly supported in its ζ-slot. Define

$$q : \Omega \times \mathbb{R}^m \longrightarrow \mathrm{Mat}_{l\times l}(\mathbb{C}), \quad q(x,\zeta) := (1 - \chi(x,\zeta)) \left(\sum_{|\alpha|=k} P_\alpha(x)\zeta^\alpha\right)^{-1}.$$

Then one has $q \in S^{-k}(\Omega, \mathrm{Mat}_{l\times l}(\mathbb{C}))$ and

$$q(x,\zeta) \sum_{|\alpha|=k} P_\alpha(x)\zeta^\alpha - 1 = -\chi(x,\zeta)1, \quad \sum_{|\alpha|=k} P_\alpha(x)\zeta^\alpha - 1 = -\chi(x,\zeta)1.$$

The right hand sides define smoothing symbols, showing that P is elliptic.
Assume now P is elliptic. Pick

$$q \in S^{-k}(\Omega, \mathrm{Mat}_{l\times l}(\mathbb{C})), \quad r \in S^{-1}(\Omega, \mathrm{Mat}_{l\times l}(\mathbb{C}))$$

with $qp = 1 + r$. Fix $x \in \Omega$. Then one has

$$q(x,\zeta)p(x,\zeta) - 1 = O_x(1/(1 + |\zeta|))$$

as $|\zeta| \to \infty$. This shows that there exists $R > 0$ with $p(x, \zeta)$ invertible for all $\zeta \in \mathbb{R}^m$ with $|\zeta| > R$. Since $p(x, \zeta)$ is a polynomial in its ζ-slot, ultimately $p(x, \zeta)$ is invertible for all $\zeta \in \mathbb{R}^m \setminus \{0\}$, by a simple scaling argument.

■

Proposition 4.37. *Assume $k \in \{-\infty\} \cup \mathbb{R}$ and that $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ is properly supported and elliptic. Then there exists a properly supported $Q \in \Psi^{-k}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ such that*

$$QP - 1 \in \Psi^{-1}(\Omega; \mathbb{C}^l, \mathbb{C}^l), \quad PQ - 1 \in \Psi^{-1}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l'}).$$

Proof: Pick

$$q \in S^{-k}(\Omega, \mathrm{Mat}_{l \times l}(\mathbb{C}))$$

such that

$$\sigma^k(P)[q]_{-k} = [1]_0, \quad [q]_{-k}\sigma^k(P) = [1]_0.$$

Pick $Q \in \Psi^{-k}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ with $\sigma^{-k}(Q) = [q]_{-k}$. We can even make sure that Q is properly supported by Theorem 4.16. It follows that

$$\sigma^0(QP) = \sigma^0(Q)\sigma^0(P) = [1]_0, \quad \sigma^0(PQ) = [1]_0.$$

In view of $\sigma^0(1) = [1]_0$, it follows from applying $(\sigma^0)^{-1}$ to the last formulae that

$$QP - 1 \in \Psi^{-1}(\Omega; \mathbb{C}^l, \mathbb{C}^l), \quad PQ - 1 \in \Psi^k(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l'}).$$

■

Definition 4.38. Let (k_j) be a sequence of real numbers with $k_j \searrow -\infty$ as $j \to \infty$. Assume that for every j we are given $Q_j \in \Psi^j(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ and that we are given $Q \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ where $k := \max_j k_j \in \mathbb{R}$. Then one writes

$$Q \sim \sum_j Q_j,$$

if for all $k' \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has

$$Q - \sum_{j=0}^{n} Q_j \in \Psi^{k'}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}).$$

Lemma 4.39. *Assume $s, k \in \mathbb{R}$ and that $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$. Then*

$$P : H^s_c(\Omega, \mathbb{C}^l) \longrightarrow H^{s-k}_{\mathrm{loc}}(\Omega, \mathbb{C}^l).$$

In particular,

$$\bigcap_{k \in \mathbb{Z}} \Psi^{-k}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \subset \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}).$$

Proof: Exercise. ■

The following the the asymptotic completeness lemma at the level of operators:

Theorem 4.40. *Let* (k_j) *be a sequence of real numbers with* $k_j \searrow -\infty$ *as* $j \to \infty$ *and set* $k := \max_j k_j$. *Assume that for every* j *we are given a properly supported* $Q_j \in \Psi^j(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$. *Then there exists* $Q \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$, *such that*

$$Q \sim \sum_j Q_j.$$

The operator Q *is uniquely determined modulo* $\Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$.

Proof: Uniqueness: If Q' is another operator with the above properties, then by Lemma 4.39,

$$Q - Q' \in \bigcap_{k \in \mathbb{R}} \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}) \subset \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'}).$$

Existence: This follows from the asymptotic summation lemma for symbols: we can pick

$$q_j \in S^{k_j}(\Omega, \mathrm{Mat}_{l \times l}(\mathbb{C}))$$

with $\mathrm{Op}(q_j) = Q_j$ by Lemma 4.33 and

$$q \in S^k(\Omega, \mathrm{Mat}_{l \times l}(\mathbb{C}))$$

with $q \sim \sum_j q_j$. Then $Q := \mathrm{Op}(q)$ does the job. ■

Finally we can prove:

Theorem 4.41. *Assume* $k \in \{-\infty\} \cup \mathbb{R}$ *and that* $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ *is properly supported and elliptic. Then there exists* $Q \in \Psi^{-k}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ *such that*

$$QP - 1 \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^l), \quad PQ - 1 \in \Psi^{-\infty}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l'}).$$

Moreover, Q *is uniquely determined modulo* $\Psi^{-\infty}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$.

Remark 4.42. An operator Q as above is called a *parametrix* for P. As the proof below shows, every properly supported $Q \in \Psi^{-k}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ with

$$QP - 1 \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^l)$$

automatically satisfies

$$PQ - 1 \in \Psi^{-\infty}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l'}).$$

Proof of Theorem 4.41: Existence: We are going to bootstrap Proposition 4.37 using the last Theorem: by Proposition 4.37 we can pick a properly supported $Q' \in \Psi^{-k}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ such that

$$Q'P - 1 \in \Psi^{-1}(\Omega; \mathbb{C}^l, \mathbb{C}^l), \quad PQ' - 1 \in \Psi^{-1}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^{l'}).$$

Then

$$R := 1 - Q'P \in \Psi^{-1}(\Omega; \mathbb{C}^l, \mathbb{C}^l)$$

and Theorem 4.34 implies

$$R^j \in \Psi^{-j}(\Omega; \mathbb{C}^l, \mathbb{C}^l) \quad \text{for all } j \in \mathbb{N},$$

and these are properly supported. Then Theorem 4.40 implies the existence of $A \in \Psi^0(\Omega;\mathbb{C}^l,\mathbb{C}^l)$ such that

$$A \sim \sum_{j=0}^{\infty} R^j.$$

In particular one has

$$A(1-R)-1 \in \bigcap_{n\in\mathbb{N}} \Psi^{-n}(\Omega;\mathbb{C}^l,\mathbb{C}^l) \subset \Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^l).$$

Defining the properly supported

$$Q := AQ' \in \Psi^{-k}(\Omega;\mathbb{C}^{l'},\mathbb{C}^l),$$

one now finds

$$QP-1 = AQ'P-1 = A(1-R)-1 \in \Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^l).$$

On the other hand, -

$$B := QP-1 \in \Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^l)$$

is properly supported.
Analogously, one finds a properly supported $P' \in \Psi^{-k}(\Omega;\mathbb{C}^l,\mathbb{C}^{l'})$ such that

$$P'Q-1 \in \Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^l).$$

Then

$$C := P'Q-1 \in \Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^l)$$

is properly supported. We have

$$P'QP = P(1+B) = P+PB, \quad P'QP = (1+C)P' = P'+CP',$$

so that (by Theorem 4.9),

$$S := P-P' = PB+CP' \in \Psi^{-\infty}(\Omega;\mathbb{C}^l,\mathbb{C}^{l'})$$

and

$$D := PQ-1 = SQ+P'Q-1 \in \Psi^{-\infty}(\Omega;\mathbb{C}^{l'},\mathbb{C}^{l'}).$$

This proves the existence of a parametrix.
Uniqueness: assume

$$\tilde{Q} \in \Psi^{-k}(\Omega;\mathbb{C}^{l'},\mathbb{C}^l)$$

is another parametrix for P, so that in particular

$$E := \tilde{Q}P-1 \in \Psi^{-\infty}(\Omega;\mathbb{C}^{l'},\mathbb{C}^{l'})$$

Then we have

$$\tilde{Q}-Q = \tilde{Q}PQ-\tilde{Q}D-Q = EQ-\tilde{Q}D \in \Psi^{-\infty}(\Omega;\mathbb{C}^{l'},\mathbb{C}^l),$$

completing the proof. ■

Corollary 4.43 (Elliptic regularity). *Assume $k \in \mathbb{R}$ and that $P \in \Psi^k(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$ is properly supported and elliptic. Then for all $s \in \mathbb{R}$ one has the implication*

$$Pf \in H^s_{\mathrm{loc}}(\Omega, \mathbb{C}^{l'}) \ \Rightarrow \ f \in H^{s+k}_{\mathrm{loc}}(\Omega, \mathbb{C}^l) \quad \textit{for all } f \in \mathscr{D}'(\Omega, \mathbb{C}^l).$$

In particular, one has the implication

$$Pf \in \mathscr{E}(\Omega, \mathbb{C}^{l'}) \ \Rightarrow \ f \in \mathscr{E}(\Omega, \mathbb{C}^l) \quad \textit{for all } f \in \mathscr{D}'(\Omega, \mathbb{C}^l).$$

Proof: Pick a parametrix $Q \in \Psi^{-k}(\Omega; \mathbb{C}^{l'}, \mathbb{C}^l)$ for P. Then $QP - 1$ is smoothing, so that

$$QPf - f =: h \in \mathscr{E}(\Omega, \mathbb{C}^l) = \bigcap_{r \in \mathbb{R}} H^r_{\mathrm{loc}}(\Omega, \mathbb{C}^l).$$

But the mapping properties of pseudodifferential operators show $QPf \in H^{s+k}_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$, so $f = QPf - h \in H^{s+k}_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$. ■

5. Function spaces and distributions on manifolds

Assume in the sequel that M is a smooth m-dimensional (without boundary), so for example an open subset $\Omega \subset \mathbb{R}^m$. We remark that M admits smooth partitions of unity, that is, given an open cover $(U_i)_{i \in I}$ we can pick $(\phi_i)_{i \in I} \subset C^\infty(M)$ such that $0 \leq \phi_i \leq 1$ for all i and $(\mathrm{supp}(\phi_i))_i$ is locally finite and $\sum_i \phi = 1$. This can be proved by starting from a continuous partition of unity (which one gets from topology) and applying Friedrichs mollifiers (the details are tedious any can be found in Lee's book on differentiable manifolds).

We also fix a smooth complex vector bundle $E \to M$ with rank l.

We denote for all open $U \subset M$ the space of smooth sections in $E \to M$ which are defined on U by $\Gamma_{C^\infty}(U, E)$, and its subspace of smooth compactly supported sections by $\Gamma_{C^\infty_c}(U, M)$. Then a local frame for $E \to M$ over U is by definition a collection of local sections $b_1, \dots b_l \in \Gamma_{C^\infty}(U, M)$ such that $b_1(x), \dots b_l(x) \in E_x$ is a basis for all $x \in U$.

By a *local control datum* for $E \to M$, we shall understand a pair $\big((U, \mathbf{x}), \{b_1, \dots, b_l\}\big)$ given by a chart

$$\mathbf{x} = (x^1, \dots, x^m) :\longrightarrow V \subset \mathbb{R}^m$$

and a local frame $b_1, \dots b_l \in \Gamma_{C^\infty}(U, M)$ for $E \to M$.

We want turn $\Gamma_{C^\infty}(M, E)$ into a F space. To this end, assume we are given a family $\mathscr{A}$ of local control data for $E \to M$ which also form an atlas for M (in short, an atlas of local control data for $E \to M$). For all $\big((U, \mathbf{x}), \{b_1, \dots, b_l\}\big) \in \mathscr{A}$, all $K \subset U$ compact, and all $n \in \mathbb{N}$ we define a seminorm $p_{U, \mathbf{x}, K, \{b_1, \dots b_l\}}$ on $\Gamma_{C^\infty}(M, E)$ by setting

$$p_{U, \mathbf{x}, \{b_1, \dots b_l\}, K, n}(\psi) := \max_{K, |\alpha| \leq n, j = 1, \dots, l} \frac{\partial^{|\alpha|} \psi^\alpha}{\partial x^\alpha},$$

where we have written $\psi = \sum_{j=1}^l \psi^j b_j$ for (uniquely determined) $\psi^1, \dots, \psi^l \in C^\infty(U)$. In this way we obtain a locally convex topology on $\Gamma_{C^\infty}(M, E)$, and it is rather easy to show that this topology does not depend on the choice of an atlas $\mathscr{A}$ of local control data for

$E \to M$. It is easy to check that this topology turns $\Gamma_{C^\infty}(M,E)$ into an F space, which is denoted by $\Gamma_{\mathscr{E}}(M,E)$.

Lemma 5.1. *The embedding* $\Gamma_{\mathscr{D}}(M,E) \hookrightarrow \Gamma_{\mathscr{E}}(M,E)$ *is continuous and has a dense image. In particular, there is a canonical continuous embedding* $\Gamma_{\mathscr{E}'}(M,E) \hookrightarrow \Gamma_{\mathscr{D}'}(M,E)$.

Proof: The continuity of $\Gamma_{\mathscr{D}}(M,E) \hookrightarrow \Gamma_{\mathscr{E}}(M,E)$ is checked easily using the $p_{U,\mathbf{x},\{b_1,\dots b_l\},K,r}$'s.To see the density, pick a compact exhaustion (K_n) of M and for each n some $\chi_n \in \mathscr{D}(M)$ with $\chi_n = 1$ in K_n and $\mathrm{supp}(\chi_n) \subset K_{n+1}$. Then given $\psi \in \Gamma_{\mathscr{E}}(M,E)$ one has $\chi_n\psi \to \psi$ with respect to any $p_{U,\mathbf{x},\{b_1,\dots b_l\},K,r}$, which follows from the Leibniz rule. ■

Let us now come to the LC topology on $\Gamma_{C_c^\infty}(M,E)$. Denote by $\Gamma_{\mathscr{D}_K}(M,E)$ the space of smooth sections with support in $K \subset M$ compact, and equip $\Gamma_{C_c^\infty}(M,E)$ with all seminorms p on $\Gamma_{C_c^\infty}(M,E)$ such that the restriction of p to $\Gamma_{\mathscr{D}_K}(M,E)$ is continuous for all compact K.

It is easily checked that this produces an LF space (by picking an atlas of local control data and the induced seminorms, and on each local control datum with domain U a compact exhaustion of U). We denote this F space which we denote by $\Gamma_{\mathscr{D}}(M,E)$.

Definition 5.2. We call the LC space $\Gamma_{\mathscr{D}'}(M,E) := \Gamma_{\mathscr{D}}(M,E)'$ the *space of distributions on* $E \to M$, and the LC space $\Gamma_{\mathscr{E}'}(M,E) := \Gamma_{\mathscr{E}}(M,E)'$ the *space of compactly supported distributions on* $E \to M$.

Let us produce some subspaces of these spaces: Assume $p \in [1,\infty]$. We call two Borel measurable sections f_1, f_2 of $E \to M$ *equivalent*, if for some (and then automatically any!) Borel measure μ on M which locally has a smooth density with respect to the Lebesgue measure (in short: a smooth meausre), one has $f_1 = f_2$ μ-a.e. Then we get the LF space $\Gamma_{L^p_{\mathrm{loc}}}(M,E)$ of equivalence classes f of Borel sections in $E \to M$ such that for all local control data $((U,\phi),\{b_1,\dots b_l\})$ one has

$$(f^1,\dots,f^l) \in L^p_{\mathrm{loc}}(\phi(U),\mathbb{C}^l), \quad \text{if } f = \sum_{j=1}^{l} f^j b_j \text{ in } U.$$

In addition we get the F space $\Gamma_{L^p_c}(M,E)$ of all $f \in \Gamma_{L^p_{\mathrm{loc}}}(M,E)$ such that ψ has a compactly supported representative. In both cases, the LC structure is defined by letting $((U,\phi),\{b_1,\dots b_l\})$ run through an atlas and by checking that this definition does not depends on the atlas.
For $f_1, f_2 \in \Gamma_{L^1_{\mathrm{loc}}}(M,E)$ we have $f_1 = f_2$, if and only if for some/all pairs (μ,h) such that μ is a smooth measure on M and h a metric on $E \to M$, and all $\psi \in \Gamma_{\mathscr{D}}(M,E)$ one has

$$\int (f_1,\psi)_h d\mu = \int (f_2,\psi)_h d\mu,$$

and there are canonical continuous embeddings

$$\Gamma_{\mathscr{D}}(M,E) \hookrightarrow \Gamma_{L^p_c}(M,E) \hookrightarrow \Gamma_{L^p_{\mathrm{loc}}}(M,E), \quad \Gamma_{\mathscr{E}}(M,E) \hookrightarrow \Gamma_{L^p_{\mathrm{loc}}}(M,E).$$

There is no way to produce a canonical embedding of $\Gamma_{L^1_{\mathrm{loc}}}(M,E)$ into $\Gamma'_{\mathscr{D}}(M,E)$. On the other hand, any (μ,h) produces the continuous embedding

$$T^{\mu,h}:\Gamma_{L^1_{\mathrm{loc}}}(M,E)\hookrightarrow\Gamma'_{\mathscr{D}}(M,E),\quad \psi\longmapsto T^{\mu,h}_f(\psi):=\int_M (f,\psi)_h d\mu,$$

and likewise

$$T^{\mu,h}:\Gamma_{L^1_c}(M,E)\hookrightarrow\Gamma'_{\mathscr{E}}(M,E).$$

Definition 5.3. 1. Given $T\in\Gamma_{\mathscr{D}'}(M,E)$, $U\subset\Omega$ open, we set

$$T|_U:=T|_{\Gamma_{\mathscr{D}}(U,E)}\in\Gamma_{\mathscr{D}'}(U,E)$$

and say that T *is smooth in* U, if there exists/for all (μ,h) as above, there exists $\psi\in\Gamma_{\mathscr{E}}(U,E)$ such that $T|_U=T^{\mu,h}_{\psi}$.
2. The support of T is defined by

$$\mathrm{supp}(T):=M\setminus\bigcup_{U\subset M,\,T|_U=0}U.$$

The assignment $U\mapsto\Gamma_{\mathscr{D}'/\mathscr{E}'}(U,E)$, together with the restriction morphisms

$$\Gamma_{\mathscr{D}'/\mathscr{E}'}(V,E)\longrightarrow\Gamma_{\mathscr{D}'/\mathscr{E}'}(U,E),\quad T\longmapsto T|_U,$$

where $U\subset V\subset\Omega$ are open, is a linear sheaf over M.

Remark 5.4. 1. For all $T\in\Gamma_{\mathscr{D}'}(M,E)$ the set $\mathrm{supp}(T)$ is closed.
2. For all $T\in\mathscr{D}'(M,E)$ one has $T\in\Gamma_{\mathscr{E}'}(M,E)$, if and only if $\mathrm{supp}(T)$ is a compact set.
3. For all $f\in\Gamma_{L^1_{\mathrm{loc}}}(M,E)$ and all $x\in M$ one has $x\in\mathrm{supp}(T)$, if and only if there exists/for all (μ,h) one has

$$\int_U |f|_h d\mu>0\quad\text{for all open neighbourhoods } U \text{ of } x.$$

In particular, for all continuous sections $f\in\Gamma_C(M,\mathbb{C}^l)$ one has

$$\mathrm{supp}(f)=\overline{\{f\neq 0\}}=\mathrm{supp}(T_f),$$

that is, in the continuous case the distributional support is equal to the support in the sense of functions on topological spaces.

We can define local Sobolev spaces as follows:

Definition 5.5. Let $s\in\mathbb{R}$. Then $T\in\Gamma_{\mathscr{D}'}(M,E)$ is by definition in $\Gamma_{H^s_{\mathrm{loc}}}(M,E)$, if for all local control data $((U,\phi),\{b_1,\dots b_l\})$ the linear form defined by the diagram below is in $H^s_{\mathrm{loc}}(\phi(U),\mathbb{C}^l)$,

$$\begin{array}{ccc} \Gamma_{\mathscr{D}}(U,E) & \xrightarrow{T|_U} & \mathbb{C} \\ {\scriptstyle (\psi^1,\dots,\psi^l)\mapsto\sum_{j=1}^l(\psi^j\circ\phi)b_j}\uparrow & \nearrow & \\ \mathscr{D}(\phi(U),\mathbb{C}^l) & & \end{array}\tag{45}$$

One sets $\Gamma_{H^s_c}(M,E) := \Gamma_{H^s_{\mathrm{loc}}}(M,E) \cap \Gamma_{\mathscr{E}}(M,E)$.

The space $\Gamma_{H^s_c}(M,E)$ canonically becomes an F space, and $\Gamma_{H^s_c}(M,E)$ becomes an LF space (on $((U,\phi),\{b_1,\dots b_l\})$ as above take the induced topology from $H^s_{\mathrm{loc}/c}(\phi(U),\mathbb{C}^l)$, let $((U,\phi),\{b_1,\dots b_l\})$ run through an atlas and check that this topology does not depend on an atlas, a fact which is technical but doable).
One has

$$\Gamma_{H^0_{c/\mathrm{loc}}}(M,E) = \Gamma_{L^2_{c/\mathrm{loc}}}(M,E),$$

and for all $t \geq s$ there are canonical coninuous embeddings

$$\Gamma_{H^s_{c/\mathrm{loc}}}(M,E) \hookrightarrow \Gamma_{H^s_{c/loc}}(M,E), \tag{46}$$

$$\Gamma_{H^s_{\mathrm{loc}}}(M,E) \hookrightarrow \Gamma_{\mathscr{D}'}(M,E), \tag{47}$$

$$\Gamma_{H^s_c}(M,E) \hookrightarrow \Gamma_{\mathscr{E}'}(M,E), \tag{48}$$

$$\Gamma_{H^s_c}(M,E) \hookrightarrow \Gamma_{H^s_{\mathrm{loc}}}(M,E). \tag{49}$$

In addition by the local Sobolev embedding theorem we canonically have an isomorphism of LCS's

$$\bigcap_{s\in\mathbb{R}} \Gamma_{H^s_{\mathrm{loc}}}(M,E) \cong \Gamma_{\mathscr{E}}(M,E),$$

and every (μ,h) induces an isomorphism of LCS's

$$\Gamma_{H^s_{\mathrm{loc}}}(M,E)' \cong_{\mu,h} \Gamma_{H^{-s}_c}(M,E),$$

so that

$$\bigcup_{s\in\mathbb{R}} \Gamma_{H^s_c}(M,E) \cong_{\mu,h} \Gamma_{\mathscr{E}'}(M,E),$$

6. Pseudodifferential operators on manifolds

Fix now a second smooth complex vector bundle $F \to M$ of rank l'.

Definition 6.1. Let $k \in \mathbb{R} \cup \{-\infty\}$. A linear map

$$P : \Gamma_{\mathscr{D}}(M,E) \longrightarrow \Gamma_{\mathscr{E}}(M,E)$$

is called a *pseudodifferential operator of order k (properly supported, elliptic)*, if for all local control data $(U,\phi,\{b_1,\dots,b_l\})$ for $E \to M$ and all local frames $c_1,\dots,c_{l'} \in \Gamma_{\mathscr{E}}(U,F)$ the induced map

$$\begin{array}{ccc} \Gamma_{\mathscr{D}}(U,E) & \xrightarrow{\quad P|_U \quad} & \Gamma_{\mathscr{E}}(U,E) \\ {\scriptstyle (\psi^1,\dots,\psi^l)\mapsto\sum_{j=1}^l(\psi^j\circ\phi)b_j}\Big\uparrow & & \Big\downarrow{\scriptstyle \sum_{j=1}^l \psi^j b_j\mapsto(\psi^1\circ\phi^{-1},\dots,\psi^l\circ\phi^{-1})} \\ \mathscr{D}(\phi(U),\mathbb{C}^l) & \longrightarrow & \mathscr{E}(\phi(U),\mathbb{C}^{l'}) \end{array} \tag{50}$$

is in $\Psi^k(\phi(U);\mathbb{C}^l,\mathbb{C}^{l'})$ (properly supported, elliptic).

The linear space of such operators is denoted with $\Psi^k(M;E,F)$. Note that every pseudodifferential operator is automatically continuous from $\Gamma_{\mathscr{D}}$ to $\Gamma_{\mathscr{E}}$.

With some efforts one can show [1] that the above definition coincides with our previous one in case M is an open subset of $\mathbb{R}^m$ and $E = M\times\mathbb{C}^l \to M$ is the trivial bundle. One has to check the coordinate invariance (a rather complicated issue) and the frame invariance (rather straight forward).
We have the following result which is central for geometry, and which follows straightforwardly from a partition of unity argument. Note for its formulation that every properly supported $P\in\Psi^k(M;E,F)$ maps

$$P:\Gamma_{\mathscr{D}}(M,E)\longrightarrow\Gamma_{\mathscr{D}}(M,F)$$

continuously and thus by density induces a continous linear map

$$P:\Gamma_{\mathscr{E}}(M,E)\longrightarrow\Gamma_{\mathscr{E}}(M,F),$$

and that the composition of properly supported pseudifferential operators is again a properly supported differential operator. This follows immediately from the Euclidean results.

Theorem 6.2. *Let $k\in\mathbb{R}\cup\{-\infty\}$ and assume $P\in\Psi^k(M;E,F)$ is elliptic and properly supported. Then there exists $Q\in\Psi^{-k}(M;F,E)$ properly supported, such that*

$$QP-1\in\Psi^{-\infty}(M;E,E),\quad PQ-1\in\Psi^{-\infty}(\Omega;F,E).$$

Any such Q is uniquely determined modulo $\Psi^{-\infty}(M;F,E)$, and called a parametrix for P.

From now on, let M be a *Riemannian manifold* and let $E,F\to M$ be *metric vector bundles*, that is, we will omit the underlying metrics in the notation and simply denote all metrics with $(\cdot,\cdot)$ and the induced fiberwise norms by $|\cdot|$, and the Riemannian volume measure with μ. This abuse of notation is common in geometric analysis, and the idea behind this is that one considers these data as given in applications. For example, M could be a Riemannian submanifold of Euclidean space, or a Riemannian spin manifold, or the Hyperbolic space. The bundle E could be $\wedge^k T^*M$, so that its sections are k-forms $\Omega^k(M)$ on M, or the spinor bundle, if M is spin and so on. We get the global L^p-spaces of sections $\Gamma_{L^p}(M,E)$ given by all $f\in\Gamma_{L^p_{\mathrm{loc}}}(M,E)$ such that

$$\int_M |f|^p d\mu<\infty,\qquad \text{if } p<\infty,$$

and analogously

$$\inf\{C\geq 0: |f|\leq C\ \ \mu\text{-a.e.}\}<\infty,\qquad \text{if } p=\infty.$$

In particular $\Gamma_{L^2}(M,E)$ is a Hilbert space according to

$$\langle f_1,f_2\rangle := \int_M (f_1,f_2)d\mu.$$

Given $P \in \Psi^k(M;E,F)$ there is a unique $P^\dagger \in \Psi^k(M;F,E)$ such that

$$\int_M (P\psi_1, \psi_2)\, d\mu = \int_M \big(\psi_1, P^\dagger \psi_2\big)\, d\mu$$

for all $\psi_1 \in \Gamma_{\mathscr{D}}(M,E)$, $\psi_2 \in \Gamma_{\mathscr{D}}(M,F)$. The above defined embedding

$$\Gamma_{L^1_{\mathrm{loc}}}(M,E) \hookrightarrow \Gamma_{\mathscr{D}}(M,E)$$

is considered as canonically given now, and can safely be ommited in the notation. Then P induces a continuous linear map

$$P : \Gamma_{\mathscr{E}'}(M,E) \longrightarrow \Gamma_{\mathscr{D}'}(M,F)$$

given by $Pf_1(f_2) := f_1(P^\dagger f_2)$, $f_1 \in \Gamma_{\mathscr{E}'}(M,F)$, $f_2 \in \Gamma_{\mathscr{D}}(M,F)$. If $k > -\infty$ then by restriction one gets a continuous linear map

$$P : \Gamma_{H^s_c}(M,E) \longrightarrow \Gamma_{H^{s-k}_{\mathrm{loc}}}(M,F), \tag{51}$$

If P is properly supported, then the assignment $Pf_1(f_2) := f_1(P^\dagger f_2)$, $f_1 \in \Gamma_{\mathscr{D}'/\mathscr{E}'}(M,F)$, $f_2 \in \Gamma_{\mathscr{D}/\mathscr{E}}(M,F)$ also defines continuous linear maps

$$P : \Gamma_{\mathscr{D}'/\mathscr{E}'}(M,E) \longrightarrow \Gamma_{\mathscr{D}'/\mathscr{E}'}(M,F)$$

which by restriction induces continuous linear maps

$$P : \Gamma_{H^s_{\mathrm{loc}/c}}(M,E) \longrightarrow \Gamma_{H^{s-k}_{\mathrm{loc}/c}}(M,F). \tag{52}$$

It suffices to prove all these facts locally (cf. Theorem 4.30).
Precisely as in the Euclidean case one now shows:

Corollary 6.3 (Elliptic regularity, strong form)**.** *Assume $k \in \mathbb{R}$ and that $P \in \Psi^k(M;E,F)$ is properly supported and elliptic. Then for all $s \in \mathbb{R}$ one has the implication*

$$Pf \in \Gamma_{H^s_{\mathrm{loc}}}(M,F) \;\Rightarrow\; f \in \Gamma_{H^{s+k}_{\mathrm{loc}}}(M,F) \quad \text{for all } f \in \Gamma_{\mathscr{D}'}(M,E).$$

In particular, one has the implication

$$Pf \in \Gamma_{\mathscr{E}}(M,F) \;\Rightarrow\; f \in \Gamma_{\mathscr{E}}(M,F) \quad \text{for all } f \in \Gamma_{\mathscr{D}'}(M,E).$$

We know come to functional analytic properties of pseudodifferential operators on compact manifolds, where we refer the reader to Section 2 for the basic facts of this theory.
If $k \in \mathbb{R}$ and $P \in \Psi^k(M;E,F)$ is properly supported (for example, if M is compact or if P is a differential operator, this assumption is automatically satisfied), then P maps

$$P : \Gamma_{\mathscr{D}}(M,E) \longrightarrow \Gamma_{\mathscr{D}}(M,F)$$

continuously and thus induces a densely defined linear operator from $\Gamma_{L^2}(M,E)$ to $\Gamma_{L^2}(M,E)$. This operator $P|_{\Gamma_{\mathscr{D}}(M,E)}$, that is, $\mathrm{Dom}(P) = \Gamma_{\mathscr{D}}(M,E)$, is closable and its closure is denoted by $\overline{P}$, which is usually an unbounded operator if $k > 0$. One closed extension of $P|_{\Gamma_{\mathscr{D}}(M,E)}$ is given by $P^{\max}$, whose domain of definition is given by

$$\mathrm{Dom}(P^{\max}) = \{f \in \Gamma_{L^2}(M,E) : Pf \in \Gamma_{L^2}(M,E)\}.$$

In particular we have

$$\mathrm{Dom}(\overline{P}) \subset \mathrm{Dom}(P^{\max}).$$

If $P \in \Psi^k(M; E, F)$ is elliptic, then by elliptic regularity one has

$$\mathrm{Dom}(P^{\max}) \subset \Gamma_{H^k_{\mathrm{loc}}}(M, E).$$

The importance of the following result stems from the abstract results Theorem 2.5 and Theorem 2.6:

Theorem 6.4. *Assume M is compact, that $k \in [0, \infty)$ and that $P \in \Psi^k(M; E, F)$ is elliptic.*
a) One has

$$\mathrm{Dom}(P^{\max}) = \Gamma_{H^k_{\mathrm{loc}}}(M, E) = \Gamma_{H^k_c}(M, E) =: \Gamma_{H^k}(M, E)$$

and $\overline{P} = P^{\max}$.
b) $\overline{P}$ is invertible modulo compact operators.
c) For all[12] *$\lambda \in \mathrm{Res}(\overline{P})$ the resolvent $(\overline{P} - \lambda)^{-1}$ is compact. All eigenvectors of $\overline{P}$ are smooth.*
d) If $E = F$ is a complex vector bundle and $P = P^\dagger$, then $\overline{P}$ is self-adjoint, in particular $\mathrm{spec}(\overline{P}) \subset \mathbb{R}$ and $\mathrm{Res}(\overline{P})$ is not empty).

Proof: By the compactness of M one has

$$\Gamma_{\mathscr{E}}(M, E) = \Gamma_{\mathscr{D}}(M, E), \quad \Gamma_{H^k_{\mathrm{loc}}}(M, E) = \Gamma_{H^k_c}(M, E)$$

as locally convex space.
a) We already know $\mathrm{Dom}(P^{\max}) \subset \Gamma_{H^k}(M, E)$ as P is elliptic. If $f \in \Gamma_{H^k}(M, E)$, then $f \in \Gamma_{L^2}(M, E)$ by definition and $Pf \in \Gamma_{L^2}(M, F)$ by the mapping properties of pseudos, and so $f \in \mathrm{Dom}(P^{\max})$.
As $\mathscr{S}$ is dense in H^s and $\mathscr{D} \subset \mathscr{S}$ we have that $\mathscr{D}$ is dense in H^s. Given $f \in \Gamma_{L^2}(M, E)$, using a partition of unity which is subordinate to a *finite* atlas of control data, we can pick thus pick a sequence

$$(f_n) \subset \Gamma_{\mathscr{E}}(M, E)$$

such that $f_n \to f$ in $\Gamma_{H^k}(M, E)$, so that $Pf_n \to Pf$ in $\Gamma_{L^2}(M, E)$, showing that $P^{\max} = \overline{P}$.
b) As we can pick a parametrix Q for P, so that

$$QP = 1 + R_1, \quad PQ = 1 + R_2$$

where R_j is smoothing, the Fredholm property follows since

$$Q : \Gamma_{L^2}(M, F) \longrightarrow \Gamma_{H^k}(M, E) \hookrightarrow \Gamma_{L^2}(M, E)$$

is bounded (here we use $k \geq 0$).
c) $f \in \mathrm{Dom}(\overline{P})$, $(\overline{P} - \lambda)f = 0$, implies $(P - \lambda)f = 0$ as a distribution, so f is smooth by

[12] In general $\mathrm{Res}(\overline{P})$ could be empty; in case $\overline{P}$ is self-adjoint, however, we have $\mathrm{Spec}(\overline{P}) \subset \mathbb{R}$ and so $\mathbb{C} \setminus \mathbb{R} \subset \mathrm{Res}(\overline{P})$.

elliptic regularity.
To show that $(\overline{P}-\lambda)^{-1}$ is compact, pick a parametrix Q for $P-\lambda$, so that

$$Q(P-\lambda)=1+R,$$

where R is smoothing. This implies

$$(P-\lambda)^{-1}=Q-R(P-\lambda)^{-1}.$$

It follows that $(P-\lambda)^{-1}$ maps

$$(P-\lambda)^{-1}:\Gamma_{L^2}(M,E)\longrightarrow\Gamma_{H^k}(M,F)$$

continuously, but Theorem 3.60 implies that the embedding $\Gamma_{H^k}(M,F)\hookrightarrow\Gamma_{L^2}(M,F)$ is compact, so as a map from $\Gamma_{L^2}(M,E)$ to $\Gamma_{L^2}(M,F)$, the map $(P-\lambda)^{-1}$ is compact.
d) Clearly $P|_{\Gamma_{\mathscr{E}}(M,E)}$ is symmetric, and so it remains to show

$$\mathrm{Ker}(P|^*_{\Gamma_{\mathscr{E}}(M,E)}\pm i)=\{0\}.$$

The operator $P|^*_{\Gamma_{\mathscr{E}}(M,E)}$ is easily seen to be equal to $P^{\max}$. Thus we have to show that $f\in\Gamma_{L^2}(M,E)$, $Pf\in\Gamma_{L^2}(M,F)$, $(P\pm i)f=0$ as a distribution implies $f=0$. But by elliptic regularity $(P\pm i)f=0$ implies that f is smooth, so in case $f\neq 0$, f would be an eigenvector of $P|_{\Gamma_{\mathscr{E}}(M,E)}$ corresponding to a complex eigenvalue, contradicting the symmetry of P. ■

Recalling $m=\dim M$, let $j\in\{0,\dots,m\}$ and let us denote the various spaces of (to be consistent) j-forms on M with

$$\Omega^j_{\times}(M):=\Gamma_{\times}(M,\wedge^jT^*M),$$

and let

$$d_j:=d|_{\Omega^j_{\mathscr{E}}(M)}:\Omega^j_{\mathscr{E}}(M)\longrightarrow\Omega^{j+1}_{\mathscr{E}}(M)$$

be exterior derivative acting on smooth j-forms, a first order differential operator. Then the Laplace-Beltrami operator acting on j-forms is given by

$$\Delta_j:=d_j^{\dagger}d_j+d_{j-1}d_{j-1}^{\dagger}:\Omega^j_{\mathscr{E}}(M)\longrightarrow\Omega^j_{\mathscr{E}}(M).$$

This is a second order elliptic differential operator with $\Delta_j^{\dagger}=\Delta_j$. Define

$$\widehat{\Omega}^j_{d^{\dagger}}(M):=\mathrm{Ran}\left(\overline{d_j^{\dagger}}\right),\quad \Omega^j_{d^{\dagger}}(M):=\widehat{\Omega}^j_{d^{\dagger}}(M)\cap\Omega^j_{\mathscr{E}}(M)$$

$$\widehat{\Omega}^j_{d}(M):=\mathrm{Ran}\left(\overline{d_{j-1}}\right),\quad \Omega^j_{d}(M):=\widehat{\Omega}^j_{d}(M)\cap\Omega^j_{\mathscr{E}}(M)$$

$$\widehat{\Omega}^j_{\mathrm{har}}(M):=\mathrm{Ker}\left(\overline{\Delta_j}\right),\quad \Omega^j_{\mathrm{har}}(M):=\widehat{\Omega}^j_{\mathrm{har}}(M)\cap\Omega^j_{\mathscr{E}}(M).$$

Note that by Theorem 6.4 d) the operator $\overline{\Delta_j}$ is self-adjoint, if M is compact (in fact, by a Theorem of Chernoff this self-adjointness only requires M to be geodesically complete).

Theorem 6.5 (Hodge)**.** *Assume M is compact.*
a) One has

$$\Omega^j_{\mathrm{har}}(M)=\widehat{\Omega}^j_{\mathrm{har}}(M),$$

and this space is finite dimensional and called the space of harmonic j-forms *on* M.
b) One has

$$\Omega^j_{L^2}(M) = \widehat{\Omega}^j_{\mathrm{har}}(M) \oplus \widehat{\Omega}^j_{d^\dagger}(M) \oplus \widehat{\Omega}^j_{d}(M). \tag{53}$$

c) One has

$$\Omega^j_{\mathscr{E}}(M) = \Omega^j_{\mathrm{har}}(M) \oplus \Omega^j_{d^\dagger}(M) \oplus \Omega^j_{d}(M).$$

d) One has

$$\mathrm{Ker}\left(d_j|_{\Omega^j_{\mathscr{E}}(M)}\right) = \Omega^j_{\mathrm{har}}(M) \oplus \Omega^j_{d}(M).$$

e) Let $\mathsf{H}^j(M)$ *denote the* j*-th singular cohomology group of the topological space (!) underlying* M*. Then one has* $\mathsf{H}^j(M) \cong \Omega^j_{\mathrm{har}}(M)$ *naturally as groups, in particular, the Betti numbers* $b_j(M) := \mathrm{rank}\mathsf{H}^j(M)$ *are finite.*

Proof: a) The equality follows from elliptic regularity, and the finite dimensionality from Theorem 6.4 b).
b) The spaces on the RHS of (53) are easily seen to pairwise orthogonal, because of $d^2 = 0 = (d^\dagger)^2 = 0$. It remains to show that every $\alpha \in \Omega^j_{L^2}(M)$ can be written as a sum of this form. As $\overline{\Delta_j}$ has a closed range and is self-adjoint we have

$$\Omega^j_{L^2}(M) = \mathrm{Ran}(\overline{\Delta_j})^\perp \oplus \mathrm{Ran}(\overline{\Delta_j}) = \mathrm{Ker}(\overline{\Delta_j}^*) \oplus \mathrm{Ran}(\overline{\Delta_j}) \tag{54}$$

$$= \mathrm{Ker}(\overline{\Delta_j}) \oplus \mathrm{Ran}(\overline{\Delta_j}), \tag{55}$$

and so because of

$$\overline{\Delta_j} = \overline{d_j^\dagger}\,\overline{d_j} + \overline{d_{j-1}}\,\overline{d_{j-1}^\dagger}$$

we have

$$\alpha = \alpha_1 + \overline{\Delta_j}\alpha_2 = \alpha_1 + \overline{d_j^\dagger}\,\overline{d_j}\alpha_2 + \overline{d_{j-1}}\,\overline{d_{j-1}^\dagger}\alpha_2,$$

where $\alpha_1 \in \mathrm{Ker}\left(\overline{\Delta_j}\right)$. This proves b).
c) Let $\alpha \in \Omega^j_{\mathscr{E}}(M)$ and decompose $\alpha = \alpha_1 + \overline{\Delta_j}\alpha_2$ according to (54). Then

$$\overline{\Delta_j}\alpha_2 = \alpha - \alpha_1$$

is smooth and so α_2 is smooth and the proof is completed upon writing

$$\alpha = \alpha_1 + \overline{\Delta_j}\alpha_2 = \alpha_1 + \Delta_j\alpha_2 = \alpha_1 + d_j^\dagger d_j\alpha_2 + d_{j-1}d_{j-1}^\dagger\alpha_2.$$

d) This follows by combining

$$\Omega^j_{\mathrm{har}}(M) \oplus \Omega^j_{d}(M) \subset \mathrm{Ker}\left(d_j|_{\Omega^j_{\mathscr{E}}(M)}\right), \quad \Omega^j_{d^\dagger}(M) \cap \mathrm{Ker}\left(d_j|_{\Omega^j_{\mathscr{E}}(M)}\right) = \{0\}$$

with c).
e) By de Rham's Theorem and part d) of the above Theorem we have

$$\mathsf{H}^j(M) \cong \mathrm{Ker}\left(d_j|_{\Omega^j_{\mathscr{E}}(M)}\right)/\Omega^j_d(M) \cong \left(\Omega^j_{\mathrm{har}}(M) \oplus \Omega^j_d(M)\right)/\Omega^j_d(M) \cong \Omega^j_{\mathrm{har}}(M).$$

■

References

[1] Crainic, M. & Van den Ban, E.: *Analysis on manifolds.* Lecture notes, available online.
[2] Cycon, H.L. & Froese, R.G. & Kirsch, W. & Simon, B.: Schrödinger Operators: With Applications to Quantum Mechanics and Global Geometry. (book).
[3] Kato, T.: *Perturbation theory for linear operators.* (book).
[4] Köthe, G.: *Topological vector spaces 1, 2* (books).
[5] Lesch, M.: *Global Analysis 2*, Lecture notes from 2011.
[6] Rudin, W.: *Functional analysis* (book).
[7] Shubin, M.: *Pseudodifferential operators* (book).
[8] Treves, F.: *Fourier integral operators, part 1* (book).

-