

Heat kernels on manifolds, University of Bonn, SS 2019,
 Lecturer: Dr. Batu Güneysu,
 Exercise sheet 1: due on Friday, 12 April at 12:00 in Room 1.032

1. Let (X, μ) be a sigma-finite measure space and let $\psi : X \rightarrow \mathbb{C}$ be measurable.
 - a) Show that the maximally defined multiplication operator M_ψ in $L^2(X, \mu)$ is closed.
 - b) Show that ψ is μ -a.e. real-valued, if and only if M_ψ is self-adjoint. *Hint: Calculate the adjoint explicitly.*
 - c) Show that $\text{Ker}(M_\psi) = \{0\}$ if and only if $\mu\{\psi = 0\} = 0$ and $\text{Ran}(M_\psi) = L^2(X, \mu)$ if and only if there exists $\epsilon > 0$ with $|\psi| \geq \epsilon$ μ -a.e. What is the spectrum of M_ψ ?

2. Let (X, μ) be a sigma-finite measure space and let $\psi : X \rightarrow \mathbb{R}$ be measurable.
 - a) Show that the spectral resolution of the induced multiplication operator is given by $E_{M_\psi}(\lambda) = M_{1_{\{\psi \leq \lambda\}}}$, with 1_A the indicator function of a set A .
 - b) Show that if $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is Borel then $\phi(M_\psi) = M_{\phi \circ \psi}$.

3. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A bounded map $V : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is called unitary, if it is bijective and $V^{-1} = V^*$. Given linear two operators T_j in \mathcal{H}_j , T_1 is called unitary equivalent to T_2 , if there exists V as above with $T_2 = VT_1V^*$.
 - a) Show that T_1 is essentially self-adjoint, if and only if T_2 is so.
 - b) Use exercise 1) and exercise 2 a) to show that $-\Delta$ (defined on smooth compactly supported functions) in $L^2(\mathbb{R}^m)$ is essentially self-adjoint. *Hint: consider the operator $T_2f(x) := |x|^2f(x)$ defined on smooth compactly supported functions.*

4. Assume T is a self-adjoint operator in a Hilbert space \mathcal{H} with $T \geq c$ for some constant $c \in \mathbb{R}$ and let $f \in \mathcal{H}$.
 - a) Show that there exists at most one solution $\psi : (0, \infty) \rightarrow \mathcal{H}$ of the following Cauchy problem for the abstract heat equation induced by T :
 - ψ is continuously differentiable with $\psi(t) \in \text{Dom}(T)$ for all $t > 0$,
 - $d/dt\psi(t) = -T\psi(t)$ for all $t > 0$,
 - $\lim_{t \rightarrow 0+} \psi(t) = f$.
 - b) Use the spectral calculus to show that $\psi(t) := e^{-tT}f$ is a solution of the above Cauchy problem.