# Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

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**Sheet 10: due on Monday 8 July at 14:00 in Room 1.032.**

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Let $(M,g)$ be a Riemannian manifold of dimension $n$. The *Levi-Civita connection* is the unique connection $\nabla$ on $TM$ which :

(a) preserves the metric: $\nabla g = 0$, where $(\nabla_X g)(Y,Z) := X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$;

(b) is torsion-free: $\nabla_X Y - \nabla_Y X = [X,Y]$.

The *Riemannian curvature tensor* is defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Ricci tensor is then defined as the trace of the map $X \mapsto R(X,Y)Z$:

$$\mathrm{Ric}(Y,Z) := \sum_{i,j=1}^{n} g\big(\partial_i, R(\partial_j, Y)Z\big),$$

where $\partial_i$ is a local coordinate basis of $\Gamma^\infty(TM)$.
For a function $f \in C^\infty(M)$, define the gradient $\mathrm{grad} f \in \Gamma^\infty(TM)$ as the Riemannian dual of $df$, i.e. $g(X, \mathrm{grad} f) = df(X)$ for all $X \in \Gamma^\infty(TM)$. Then the Hessian of $f$, denoted $\mathrm{Hess}(F) \in \Gamma^\infty(T^*M \otimes T^*M)$, is defined by

$$\mathrm{Hess}(f)(X,Y) := g(\nabla_X \mathrm{grad} f, Y) = X(Yf) - df(\nabla_X Y).$$

In local coordinates $x^i$, define the Christoffel symbols as the components of $\nabla_i \partial_j$:

$$\Gamma_{ij}^k \partial_k := \nabla_i \partial_j.$$

We have the local expressions

$$R_{ijkl} := g\big(\partial_i, R(\partial_k, \partial_l)\partial_j\big), \qquad \mathrm{Ric}_{ik} := \sum_{j,l} g^{jl} R_{ijkl},$$

$$\mathrm{Hess}(f) = \mathrm{Hess}_{ij}(f) dx^i \otimes dx^j, \qquad \mathrm{Hess}_{ij}(f) := \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f.$$

Given a point $p \in M$ and a tangent vector $v \in T_pM$, there exists a unique geodesic $\gamma_v$ satisfying $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. If $v$ is small enough, we define the *exponential map* $\exp_p(v) := \gamma_v(1)$, and then $[0,1] \ni t \mapsto \exp_p(tv) \in M$ is a minimising geodesic from $p$ to $\exp_p(v)$. (Recall from the Hopf-Rinow theorem that $\exp_p$ is defined on all of $T_pM$ if and only if $M$ is complete as a metric space, with respect to the Riemannian distance function $\varrho$.) The cut locus of $T_pM$ consists of all $v \in T_pM$ such that the geodesic $\exp_p(tv)$ is minimising for $t \in [0,1]$ but fails to be minimising for $[0, 1+\varepsilon]$, for any $\varepsilon > 0$. The *cut locus of $p$ in $M$*, denoted $\mathrm{cut}(p)$, is the image of the cut locus of $T_pM$ under the exponential map. We note that the distance function $q \mapsto \varrho(q,p)$ is smooth except at the point $p$ and the cut locus of $p$.

**Exercise 1 (Bochner formula)**

Let $f \in C^\infty(M)$. Show that

$$-\frac{1}{2}\Delta\big(g(\mathrm{grad} f, \mathrm{grad} f)\big) = g(\mathrm{Hess}(f), \mathrm{Hess}(f)) - g(\mathrm{grad}(\Delta f), \mathrm{grad} f) + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f).$$

**Exercise 2**

Fix a point $p \in M$, and define $r \in C(M)$ by

$$r(x) := \varrho(x, p).$$

For every $q \notin \mathrm{cut}(p)$, let $\gamma_q \colon [0, r(q)] \to M$ be a minimizing geodesic from $p$ to $q$ such that $|\dot{\gamma}_q| = 1$.

(a) Show that $g(\mathrm{Hess}(f), \mathrm{Hess}(f)) \geq \frac{1}{n-1}(\Delta r)^2$, and use Exercise 1 to prove that

$$\frac{\big(\Delta(r \circ \gamma_q)\big)^2}{n-1} - \frac{d}{ds}\big(\Delta(r \circ \gamma_q)\big) + \mathrm{Ric}(\dot{\gamma}_q, \dot{\gamma}_q) \leq 0. \tag{1}$$

(b) Consider a function $h \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ such that $h \geq 0$ and $h(0) = 0$. Show that

$$-h^2\big(r(q)\big)\Delta\big(r(q)\big) \leq (n-1)\int_0^{r(q)} (h')^2 ds - \int_0^{r(q)} h^2\, \mathrm{Ric}(\dot{\gamma}_q, \dot{\gamma}_q) ds. \tag{2}$$

If in addition $h(r(q)) = 0$, then we have

$$0 \leq (n-1)\int_0^{r(q)} (h')^2 ds - \int_0^{r(q)} h^2\, \mathrm{Ric}(\dot{\gamma}_q, \dot{\gamma}_q) ds. \tag{3}$$

**Exercise 3**

Consider geodesic polar coordinates $(r, \theta)$ centered at a point $p \in M$. Furthermore, let $\{\theta^j\}$ be a local orthonormal coframe for the standard metric of the sphere $S^{n-1}$ and extend it radially. Show that the metric can be written as

$$g = dr \otimes dr + \sigma_{ij}(r, \theta) d\theta^i \otimes d\theta^j.$$