Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

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Sheet 3: due on Monday 6 May at 14:00 in Room 1.032.

Exercise 1

Assume $E \to M$ is a (smooth) vector bundle with rank l. Show that there exists a Borel measurable globally defined frame $e_1, \dots e_l : M \to E$, which can be chosen orthonormal if $E \to M$ carries a metric.

Exercise 2

Assume $E, F \to M$ are vector bundles and that h_E, h'_E are metrics on $E \to M$, that h_F, h'_F are metrics on $F \to M$ and that g, g' are Riemann metrics on M. Given a differential operator

$$P : \Gamma_{C^\infty}(M, E) \longrightarrow \Gamma_{C^\infty}(M, F),$$

find a relation between the formal adjoints P^{g', h'_E, h'_F} and P^{g, h_E, h_F}.

Exercise 3

This exercise aims to show that one can define the action of a differential operator to a locally integrable function without using any distribution theory (the manuscript has been updated accordingly!):
Assume $E, F \to M$ are vector bundles and that

$$P : \Gamma_{C^\infty}(M, E) \longrightarrow \Gamma_{C^\infty}(M, F)$$

is a differential operator. Given $f \in \Gamma_{L^1_{\mathrm{loc}}}(X, E)$ and a subspace $A \subset \Gamma_{L^1_{\mathrm{loc}}}(X, F)$ we write $Pf \in A$, if there exists $h \in A$ such that for all triples of metrics (g, h_E, h_F) it holds that

$$\int_X h_E \left(P^{\rho, h_E, h_F} \psi, f\right) \mathrm{d}\mu_g = \int_X h_F (\psi, h) \, \mathrm{d}\mu_g \quad \text{for all } \psi \in \Gamma_{C^\infty_c}(X, F) \,. \tag{1}$$

Show that h is uniquely determined (so that we can set $Pf := h$) and that $Pf \in A$ is equivalent to (1) being true for *some* triple (g, h_E, h_F) (thus the assumption $Pf \in A$ is independent of the metrics).
Hint: Use exercise 2.