Heat kernels on manifolds, University of Bonn, SS 2019,
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Exercise sheet 3: due on Monday, 13 May in Room 1.032

1. Let M be a Riemannian manifold. Prove the following formulae for all smooth functions f, f_1, f_2 on M, all smooth functions u on $\mathbb{R}$ and all smooth 1-forms α on M:

$$d(f_1 f_2) = f_1 df_2 + f_2 df_1, \tag{1}$$
$$d^\dagger(f\alpha) = f d^\dagger\alpha - (df, \alpha), \tag{2}$$
$$\Delta(f_1 f_2) = f_1 \Delta f_1 2 + f_2 \Delta f_1 + 2\Re(df_1, df_2), \tag{3}$$
$$\Delta(u \circ f) = (u'' \circ f) \cdot |df|^2 + (u' \circ f) \cdot \Delta f. \tag{4}$$

2. Using the abstract characterization of the closure of a nonnegative closable form, and the abstract correspondence between nonnegative closed forms and nonngeative self-adjoint operators (Theorem 2.13), give a detailed proof of the fact that the closure Q of the form

$$Q'(f_1, f_2) = \int (df_1, df_2) d\mu,$$

defined on $C_c^\infty(M)$, is given by $\mathrm{Dom}(Q) = W_0^{1,2}(M)$ and

$$Q(f_1, f_2) = \int (df_1, df_2) d\mu,$$

and that the induced self-adjoint operator H is given by

$$\mathrm{Dom}(H) = \{f \in W_0^{1,2}(M) : \Delta f \in L^2(M)\}, \quad Hf = -\Delta f.$$

3. Assume N_1 and N_2 are manifolds (note that $T_{(x,y)}(N_1 \times N_2) = T_x N_1 \oplus T_y N_2$ for all $x \in N_1, y \in N_2$!), and that g_j is a Riemann metric on N_j. Show that for every smooth strictly postive function ψ on N_1 the volume measure μ_ψ with respect to the warped product metric $g_\psi = g_1 + \psi^2 g_2$ on $N_1 \times N_2$ is given by

$$d\mu_\psi(x, y) = \psi^{\dim(N_1)}(y) d\mu_{g_1}(x) d\mu_{g_2}(y),$$

and the Laplace-Beltrami is given by

$$\Delta_\psi f = \Delta_{g_1} f + \dim(N_2) g_1(d\log(\psi), df) + \frac{1}{\psi^2}\Delta_{g_2} f.$$