Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

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Sheet 5: due on Monday 27 May at 14:00 in Room 1.032.

Let (M, g) be a connected Riemannian manifold. For any piecewise smooth curve $\gamma\colon [a, b] \to M$, its length is defined as

$$L(\gamma) := \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt.$$

The distance function ϱ on is then defined as

$$\varrho(x, y) := \inf_\gamma L(\gamma),$$

where the infimum is taken over all piecewise smooth curves from x to y. A piecewise smooth curve $\gamma\colon [a, b] \to M$ is called *minimal* if $\varrho(\gamma(a), \gamma(b)) = L(\gamma)$. More generally, a curve $\gamma\colon [a, b) \to M$ is called *minimal* if for each $c \in (a, b)$ the restriction $\gamma|_{[a,c]}$ is minimal. We define the open balls as

$$B(x, r) := \{y \in M : \varrho(x, y) < r\}.$$

Exercise 1

Without using exponential coordinates, prove the following statements.

(a) For every $x \in M$ there exists $r > 0$ such that $\overline{B(x, r)}$ is compact.

(b) For all $x \in M$ and $r > 0$ we have the equality

$$\overline{B(x, r)} = \{y \in M : \varrho(x, y) \le r\}.$$

(c) If $\overline{B(x, r)}$ is compact, then for any $y \in B(x, r)$ there exists a minimal curve from x to y.

Exercise 2

Prove (without using exponential coordinates) that the following assertions are equivalent:

(a) every bounded closed subset of M is compact;

(b) M is complete;

(c) there exists a point $p \in M$ such that every minimal curve $\gamma\colon [0, a) \to M$ with $\gamma(0) = p$ can be extended to a continuous path $\gamma\colon [0, a] \to M$.

Hint: assuming (c), one shows as follows that every closed ball is compact. Define

$$R := \sup\{r \mid \overline{B(p, r)} \text{ is compact}\}.$$

By Exercise 1(a), we know that $R > 0$. Assume that $R < \infty$ (i.e. there are noncompact closed balls) and derive a contradiction:

(i) Prove that $B(p, R)$ is pre-compact.

(ii) Using Exercise 1, show that there exists $\epsilon > 0$ such that $\overline{B(p, r + \epsilon)}$ is compact, contradicting the choice of R.

Exercise 3

Show that if M is the Euclidean $\mathbb{R}^m$ one has $\varrho(x, y) = |x - y|$ for all $x, y \in \mathbb{R}^m$.

Exercise 4

Show that if M is complete and $V : M \to [0, \infty)$ is smooth, then $-\Delta + V$ is essentially self-adjoint in $L^2(M)$ when defined on smooth compactly supported functions.