Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

University of Bonn, Summer Semester 2019
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Sheet 6: due on Monday 3 June at 14:00 in Room 1.032.

Exercise 1 (Weak-to-strong differentiability)

Let X be a Hilbert space, and consider a map $f\colon \mathbb{R} \to X$.

(a) Suppose that for each $\phi \in X^*$ the map $t \mapsto \phi\big[f(t)\big]$ is continuously differentiable. Prove that f is continuous.

(b) Suppose that for each $\phi \in X^*$ the map $t \mapsto \phi\big[f(t)\big]$ is k times continuously differentiable for some integer $k > 1$. Prove that f is $k-1$ times differentiable.

Exercise 2

Show that the inner product $X \times X \to \mathbb{K}$, $(x, y) \mapsto \langle x, y\rangle$ is (jointly) Fréchet smooth.
Recall: A map $T\colon V \to W$ between Banach spaces is called *Fréchet differentiable* at $x \in V$ if there exists a bounded linear operator $DT_x\colon V \to W$ such that

$$\lim_{\|v\|\to 0} \|v\|^{-1}\big(T(x+v) - T(x) - DT_x v\big) = 0.$$

T is called C^1 if T is Fréchet differentiable for each $x \in V$ and the map $DT\colon V \to \mathcal{B}(V, W)$, $x \mapsto DT_x$ is continuous.

Exercise 3

For any smooth manifold M, prove that $C_c^\infty(M)$ is dense in $C_c(M)$ (with respect to its natural locally convex topology).

Exercise 4

Let M be a Riemannian manifold, let H be the Friedrichs realisation of $-\Delta$, and let $P_t f(x) = \int p(t, x, y) f(y) d\mu(y)$ be the smooth version of the heat semigroup for $f \in L^2(M)$. Assume that $f \leq 1$.

(a) Show that $(H + \lambda)^{-1} f \leq 1$ for any $\lambda > 0$.
Hint: Use Lemma 6.2 from the Lecture Notes to show that

$$\int_M \max\big((H+\lambda)^{-1}f - 1, 0\big)^2 d\mu = 0.$$

(b) Show that $P_t f \leq 1$.