Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

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Sheet 7: due on Monday 17 June at 14:00 in Room 1.032.

Exercise 1

Consider a chart $V \subset M$ with origin $x_0 = 0$, and a point $x_1 \in V$ such that the straight line segment $[0, x_1]$ is also contained in V. Let $t_1 > 0$, and define $\xi := t_1^{-1}(x_1 - x_0)$. Let U be the ball or radius $r > 0$, and choose r small enough such that $U \subset V$. Consider the tilted cylinder

$$\Gamma := \big\{(t, x) : 0 < t < t_1,\ x \in U + t\xi\big\}.$$

Consider the smooth function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(w) := (r^2 - w)^2$. For some $\alpha > 0$, define $v \in C^2(\overline{\Gamma})$ by

$$v(t, x) := e^{-\alpha t} f\big(|x - \xi t|^2\big).$$

(a) Show that $v = 0$ on $\partial_l \Gamma := \big\{(t, x) : 0 \le t \le t_1,\ x \in \partial(U + t\xi)\big\}$ and that $v > 0$ on $\overline{\Gamma} \backslash \partial_l \Gamma$.

(b) For any $c_1, c_2 > 0$, show that for α large enough we have

$$\alpha f(w) + c_1 f'(w) + c_2 w f''(w) \ge 0, \quad \forall w \in \mathbb{R}.$$

(c) With $w(x, t) := |x - \xi t|^2$, use the chain rule for Δ to show that there exist constants $c, c' > 0$ such that

$$\Delta v \ge e^{-\alpha t}\big(c w f''(w) + c' f'(w)\big).$$

(d) Prove that for α large enough we obtain on Γ the inequality

$$\frac{\partial v}{\partial t} - \Delta v \le 0.$$

Exercise 2

Let u be a non-negative function from $W_0^{1,2}(M)$. Show that there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ of non-negative functions from $C_0^\infty(M)$ such that u_k converges to u in the $W^{1,2}$-norm.

Exercise 3 (The variational principle)

Let H be the Friedrichs realisation of the Laplace operator on M. We denote the bottom of the spectrum by $\lambda_{\min}(M) := \inf \operatorname{spec} H$. Moreover, given any non-zero function $f \in W^{1,2}(M)$, we define its *Rayleigh quotient* by

$$\mathcal{R}(f) := \frac{\int_M |\nabla f|^2 d\mu}{\int_M |f|^2 d\mu}.$$

Let $\mathcal{T}$ be any class of functions such that $C_0^\infty(M) \subset \mathcal{T} \subset W_0^{1,2}(M)$. Prove that

$$\lambda_{\min}(M) = \inf_{f \in \mathcal{T} \backslash \{0\}} \mathcal{R}(f).$$

Exercise 4

Let H be the Friedrichs realisation of the Laplace operator on $\mathbb{R}^m$. Show that one has

$$\sigma(H) = \sigma_{ess}(H) = [0, \infty).$$