Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

University of Bonn, Summer Semester 2019
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Sheet 8: due on Monday 24 June at 14:00 in Room 1.032.

Let (M, g) be a Riemannian manifold with geodesic distance ϱ. A function f on a subset $S \subset M$ is called *Lipschitz on* S if there exists a constant C (the 'Lipschitz constant') such that

$$\big|f(x) - f(y)\big| \leq C\varrho(x, y), \qquad \forall x, y \in S.$$

We denote by $\mathrm{Lip}(M)$ the set of functions which are Lipschitz on M, and we equip this set with the *Lipschitz seminorm*

$$\|f\|_{\mathrm{Lip}} := \sup_{x,y\in M,\, x\neq y} \frac{|f(x) - f(y)|}{\varrho(x, y)}.$$

A function f on M is said to be *locally Lipschitz* if f is Lipschitz on any compact subset of M, and the set of locally Lipschitz functions is denoted $\mathrm{Lip}_{\mathrm{loc}}(M)$.

Exercise 1

Let B be an open ball in $\mathbb{R}^m$. Show that $1_B \in W^{1,2}(B)$ but $1_B \notin W^{1,2}(\mathbb{R}^m)$.

Exercise 2

(a) For any non-empty set $E \subset M$ and any point $x \in M$, define

$$\varrho(x, E) := \inf_{y\in E} \varrho(x, y).$$

If M is connected, prove that the function $x \mapsto \varrho(x, E)$ is Lipschitz on M with Lipschitz constant 1.

(b) For $f \in \mathrm{Lip}(M)$, prove that the distributional derivative df is an L^∞-section of T^*M and that

$$\|df\|_{L^\infty} \leq \|f\|_{\mathrm{Lip}}.$$

Hint: you may use Rademacher's Theorem.

Exercise 3

(a) Prove that, for all $f, g \in \mathrm{Lip}_{\mathrm{loc}}(M)$, we have the *product rule*

$$d(fg) = fdg + gdf. \tag{1}$$

(b) Prove that if $f \in \mathrm{Lip}(M) \cap L^\infty(M)$ and $g \in W_0^{1,2}(M)$ then $fg \in W_0^{1,2}(M)$ and the product rule (1) holds.

Exercise 4

For any open set $\Omega \subset M$, denote by $C_b(\Omega)$ the linear space of all bounded continuous functions on Ω with the sup-norm. Let $u(t,x)$ be a continuous function on $I \times M$, where I is an open interval in $\mathbb{R}$, and let the partial derivative $\frac{\partial u}{\partial t}$ be also continuous in $I \times M$. Prove that, for any relatively compact open set $\Omega \subset M$, the path $I \to C_b(\Omega)$, $t \mapsto u(t,\cdot)$, is (norm) differentiable, and its derivative $\frac{du}{dt}$ coincides with the partial derivative $\frac{\partial u}{\partial t}$.