Exercises in Advanced Global Analysis I: "Heat kernels on manifolds"

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Sheet 9: due on Monday 1 July at 14:00 in Room 1.032.

Exercise 1

Let $q_1, q_2, q_3 \in [1,\infty]$ satisfy $1/q_1 + 1/q_2 = 1/q_3$. Assume $f_j \colon \mathbb{R} \to L^{q_j}(M)$, $j = 1,2$, are real-valued and strongly differentiable. Show that $f_1 f_2 \colon \mathbb{R} \to L^{q_3}(M)$ is strongly differentiable and satisfies the product rule for d/dt.

Exercise 2

Consider a Riemannian manifold (M,g) (with Riemannian volume measure μ_g) and a ball $B(x,r) \subset M$ (for some $x \in M$ and $r > 0$). For any relatively compact open subset $U \subset B(x,r)$, denote by $\lambda_{\min}(U)$ the smallest eigenvalue of the Friedrichs realisation of the Laplace operator on U. Suppose there exist $\nu, b > 0$ such that

$$\lambda_{\min}(U) \geq a(x,r)\mu_g(U)^{-2/\nu}, \qquad a(x,r) := \frac{b}{r^2}\mu_g\big(B(x,r)\big)^{2/\nu}.$$

Prove that there exists $c = c(\nu) > 0$ such that for any $r \leq R$ we have

$$\mu_g\big(B(x,r)\big) \geq c a(x,R)^{\nu/2} r^{\nu}.$$

Exercise 3

A Riemannian manifold (M,g) (with Riemannian volume measure μ_g) is said to satisfy the *volume doubling property* if there exists a constant $C > 0$ such that for all $x \in M$ and $r > 0$ we have the inequality

$$\mu_g\big(B(x,2r)\big) \leq C\mu_g\big(B(x,r)\big).$$

Now suppose M is connected, complete, non-compact, and satisfies the volume doubling property. Show that there are numbers c, ν such that for all $x \in M$ and $0 < r \leq R$ we have the inequality

$$\frac{\mu_g\big(B(x,R)\big)}{\mu_g\big(B(x,r)\big)} \geq c\left(\frac{R}{r}\right)^{\nu}.$$

Exercise 4

Assume $f_1 \colon M \to \mathbb{R}$ is Lipschitz and compactly supported and

$$f_2 \in W^{1,2}_{\mathrm{loc}}(M) := \{f \in L^2_{\mathrm{loc}}(M) : df \in \Omega^1_{L^2_{\mathrm{loc}}}(M)\}.$$

Show that $f_1 f_2 \in W^{1,2}_0(M)$ and the product rule for 'd' applies.