Covariant Riesz-Transforms and the Calderon-Zygmund inequality

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Interfaces between Geometric Analysis and Mathematical Physics

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- M: complete Riemannian m-manifold
- $d : C^\infty(M) \to \Gamma_{C^\infty}(T^*M)$: exterior derivative on functions
- $d_1 : \Gamma_{C^\infty}(T^*M) \to \Gamma_{C^\infty}(\wedge^2 T^*M)$: exterior derivative on 1-forms
- $T^{r,s}M \to M$: r-times contravariant, s-times covariant tensors
- $\Delta = d^\dagger d$: Laplace-Beltrami-Operator in $L^2(M)$ (e.s.a.!)
- $\Delta_1 = d_1^\dagger d_1 + dd^\dagger$: Laplace-Beltrami-Operator in $\Gamma_{L^2}(T^*M)$ (e.s.a.!)
- $\nabla^{r,s} : \Gamma_{C^\infty}(T^{r,s}M) \to \Gamma_{C^\infty}(T^{r,s+1}M)$: Levi-Civita (LC) connection
- μ: Riemannian volume measure

- Let $1 < p < \infty$. The aim of the talk is to explain the connection between path integrals and the L^p-boundedness of the **covariant Riesz-transform** $CRT(p)$,

$$\boxed{\forall \lambda > 0 : \quad \left\| \nabla^{0,1} (\Delta_1 + \lambda)^{-1/2} \right\|_p < \infty.} \tag{1}$$

- The L^p-boundedness of the 'usual Riesz-transform' $RT(p)$,

$$\left\| d_1 (\Delta_1 + \lambda)^{-1/2} \right\|_p < \infty \tag{2}$$

only needs $\mathrm{Ric} \geq -C$ for some $C > 0$ and is a (by now) classical result by Bakry (1987).

- Proving $CRT(p)$ should be considerably harder than proving $RT(p)$, essentially because the Laplace-Beltrami operator commutes with the exterior differential, but not with the LC connection. In fact, $CRT(p) \Rightarrow RT(p)$ easily.

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- $CRT(p)$ plays a fundamental role in geometric analysis:

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\begin{aligned}
&\|\mathrm{Hess}(f)\|_p = \left\|\nabla^{0,1} d_1 f\right\|_p \\
&= \left\|\nabla^{0,1}(\Delta_1+\lambda)^{-1/2} d_1 (\Delta+\lambda)^{-1/2}(\Delta+\lambda) f\right\|_p \\
&\leq \left\|\nabla^{0,1}(\Delta_1+\lambda)^{-1/2}\right\|_p \left\| d_1 (\Delta+\lambda)^{-1/2}\right\|_p \left(\|\Delta f\|_p + \lambda \|f\|_p \right), \\
&\Rightarrow \boxed{\quad \|\mathrm{Hess}(f)\|_p \leq C(\|\Delta f\|_p + \|f\|_p)},
\end{aligned}
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the L^p-**Calderon-Zygmund inequality** $CZ(p)$.

- $CZ(2)$ is easily seen to hold under $\mathrm{Ric} \geq -C$, and is false in general (G./Pigola, 2015).
- For $p \neq 2$ the inequality $CZ(p)$ is nontrivial even in $\mathbb{R}^m$; the best result so far at a full L^p-scale is under $|\mathrm{Ric}| \leq C$ and $\mathrm{inj}(M) > 0$ (G./Pigola, 2015).

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Typical applications of $CZ(p)$:

- $CZ(p)$ (& a little bit of extra work) implies the global Sobolev inequality

$$\|\mathrm{Hess}(f)\|_p + \|df\|_p \leq C\,\|\Delta f\|_p + C\,\|f\|_p\,.$$

- $CZ(p)$ (& $|\mathrm{Riem}| \leq C$ & some extra work) implies $|\mathrm{Hess}(f)|, |df| \in L^p(M)$ for weak solutions $f \in L^p(M)$ of the Poisson equation $\Delta f = h$, where $h \in L^p(M)$.
- Once one has $CZ(p)$ with a constant depending only on geometric quantities (Ric, inj,...), one can use it to prove L^p-precompactness results for sequences of Riemannian immersions $\Psi_n : M_n \to \mathbb{R}^l$, $n \in \mathbb{N}$: then $\Delta\Psi_n$ is essentially the mean curvature of ψ_n and $\mathrm{Hess}(\Psi_n)$ its second fundamental form!

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How can we prove $CRT(p)$? Approach: reduce $CRT(p)$ to estimates for the heat semigroup on 1-forms, so that we can use probability theory.

Indeed:

- In view of the Laplace-transform

$$\nabla^{0,1}(\Delta_1+\lambda)^{-1/2}=\int_0^\infty \nabla^{0,1}e^{-t\Delta_1}t^{-1/2}e^{-t\lambda}dt$$

and a highly sophisticated machinery from harmonic analysis on metric measure spaces by Auscher/Coulhon/Doung/Hofmann (2004), the estimate $CRT(p)$ follows from the semigroup estimate $SG(p)$

$$\boxed{\exists C>0\ \forall t>0: \quad \left\|\nabla^{0,1}e^{-t\Delta_1}\right\|_{p,p}\leq Ce^{Ct}t^{-1/2}.}$$

- Whatever it is, this machinery should be sophisticated:

$$\left\|\int_0^\infty \nabla^{0,1}e^{-t\Delta_1}t^{-1/2}e^{-t\lambda}dt\right\|_{p,p}\leq C\int_0^\infty e^{Ct}t^{-1}dt=\infty\ldots$$

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- Is there a probabilistic formula for $\nabla^{0,1}e^{-t\Delta_1}$ which is explicit enough to prove $SG(p)$?
- Some hope: there are probabilistic path integral formulae for $e^{-t\Delta_1}$ ('covariant Feynman-Kac formula'), and for $d_1 e^{-t\Delta_1}$ by Bismut (1984), Elworthy/Li (1998), and Thalmaier (1997) ('BELT formula').

- $(\Omega, \mathbb{P})$: a probability space. Notation: $\mathbb{E}[\cdots] = \int \cdots d\mathbb{P}$.
- $X(x) : [0,\infty) \times \Omega \to M$: Brownian motion (BM) starting from $x \in M$; paths $X(x)(\omega) : [0,\infty) \to M$ continuous, for all $k \in \mathbb{N}$, $0 < t_1 < \cdots < t_k$, $A_1, \dots, A_k \subset M$,

$$\mathbb{P}\{X_{t_1}(x) \in A_1, \dots, X_{t_k}(x) \in A_k\} = \int_{A_1} \cdots \int_{A_k} e^{-t_1\Delta}(x, x_1) e^{-(t_2-t_1)\Delta}(x_1, x_2) \cdots e^{-(t_k - t_{k-1})\Delta}(x_{k-1}, x_k) d\mu(x_1) \cdots d\mu(x_k).$$

- $//^{r,s}(x) : [0,\infty) \times \Omega \to \mathrm{Hom}(T_x^{r,s}M, T_{X(x)}^{r,s}M)$: parallel transport along $X(x)$ with respect to $\nabla^{r,s}$
- $Q(x) : [0,\infty) \times \Omega \to \mathrm{End}(T_x^* M)$ is defined pathwise by

$$\frac{d}{dt} Q_t(x) = -\frac{1}{2} Q_t(x) //_t^{0,1}(x)^{-1} \mathrm{Ric}^T_{X_t(x)} //_t^{0,1}(x), \quad Q_0(x) = \mathrm{id}_{T_x^* M}.$$

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- These processes are precisely the ingredients of the **covariant Feynman-Kac formula** (Malliavin 1978, Driver/Thalmaier 2001, G. 2012)

$$e^{-t\Delta_1}\alpha(x) = \mathbb{E}\left[Q_t//_t^{0,1}(x)^{-1}\alpha(X_t(x))\right],$$

valid for all $\alpha \in \Gamma_{C_c^\infty}(T^*M)$, $t \geq 0$, $x \in M$, if Ric is (sufficiently) bounded from below.

- How can one prove the covariant Feynman-Kac formula? Using some stochastic analysis (Itô's formula) one finds that for fixed $t > 0$, the process

$$Y := Q(x)//^{0,1}(x)^{-1}e^{-(t-\bullet)\Delta_1}\alpha(X(x)) : [0,t] \times \Omega \to T_x^*M,$$

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On the other hand, being a local martingale, Y_s has a constant expectation in $s \in [0, t]$, if

$$\mathbb{E}\left[\sup_{s\in[0,t]} |Y_s|\right] < \infty,$$

which is the case, as under say $\mathrm{Ric} \geq -C$, we have $|Q_s(x)| \leq e^{Ct}$ $\mathbb{P}$-a.s. (Gronwall), and

$$\sup_{u\in[0,t],y\in M} |e^{-u\Delta_1}\alpha(y)| < \infty \quad \text{(Kato-Simon)}.$$

Therefore:

$$e^{-t\Delta_1}\alpha(x) = \mathbb{E}\left[Y_0\right] = \mathbb{E}\left[Y_t\right] = \mathbb{E}\left[Q_t(x)//_t^{0,1}(x)^{-1}\alpha(X_t(x))\right],$$

completing the proof of the covariant Feynman-Kac formula.

Let us now prepare our attack on the path integral for $\nabla^{0,1} e^{t\Delta}$...

- Given a continuous process $A : [0,\infty) \times \Omega \to \mathbb{R}^1$ and a Euclidean BM $B : [0,\infty) \times \Omega \to \mathbb{R}^1$ we can define another continuous process

$$\int_0^{\bullet} A_s dB_s : [0,\infty) \times \Omega \to \mathbb{R}^1,$$

the Itô integral, by approximating $\int_0^t A_s dB_s(\omega)$ with 'left-point (!) Lebesgue-Stieltjes Riemann sums' (but the convergence is *not* for $\mathbb{P}$-a.e. $\omega \in \Omega$).

- In general, $\int_0^{\bullet} A_s dB_s$ will only be local martingale; however, there is the Burkholder-Davis-Gundy inequality, which states that for all $q \in [1,\infty)$, there exists $C_q < \infty$ s.t. for all $t \geq 0$,

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\Big|\int_0^t A_s dB_s\Big|^q\Big] \leq C_q \mathbb{E}\Big[\Big(\int_0^t |A_s|^2 ds\Big)^{q/2}\Big] \in [0,\infty].$$

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Define the section

$$\widetilde{\mathrm{Ric}} \in \Gamma_{C^\infty}(\mathrm{End}(\otimes^2 T^*M)) = \Gamma_{C^\infty}\big(\mathrm{End}\big(\mathrm{Hom}(TM, T^*M)\big)\big)$$

on $A \in \mathrm{Hom}(T_xM, T_x^*M)$, $v \in T_xM$, by

$$\widetilde{\mathrm{Ric}}(A)(v) = \mathrm{Ric}^T(Av) - 2\sum_{j=1}^m \mathrm{Riem}^T(e_i, v)(Ae_j) \in T_x^*M,$$

and the section

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on $\alpha \in T_x^*M$, $v \in T_xM$ by

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$$\widetilde{Q}(x):[0,\infty)\times\Omega\to\mathrm{End}(\otimes^2 T_x^* M),$$
$$(d/dt)\widetilde{Q}_t(x)=-\frac{1}{2}\widetilde{Q}_t(x)(//_t^{0,2})^{-1}\widetilde{\mathrm{Ric}}_{X_t}//_t^{0,2}|_x,\quad \widetilde{Q}_0(x)=\mathrm{id}_{\otimes^2 T_x^* M},$$
$$B(x):[0,\infty)\times\Omega\longrightarrow T_xX\quad \text{anti-dev. of } X(x) \text{ w.r.t. } \nabla^{1,0} \text{ (BM!)},$$

and for fixed $t>0$, $\xi\in\otimes^2 T_x M$ further

$$\ell(\zeta,t):=\frac{(t-\bullet)}{t}\zeta:[0,t]\times\Omega\to\otimes^2 T_x M,$$
$$\ell^{(1)}(\xi,t):=-\int_0^\bullet Q_s^{T,-1}dB_s\widetilde{Q}_s^T\dot{\ell}_s(\xi,t)|_x:[0,t]\times\Omega\to T_xM,$$
$$\ell^{(2)}(\xi,t):=\frac{1}{2}\int_0^\bullet Q_s^{T,-1}\left((//_s^{0,2})^{-1}\rho(X_s)//_s^{0,2}\right)^T\widetilde{Q}_s^T\ell_s(\zeta,t)ds|_x$$
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Theorem (Baumgarth/G. 2018)

Assume $|\mathrm{Riem}|, |\nabla^{1,3}\mathrm{Riem}| \leq A$ *für some* $A < \infty$. *Then for all* $\alpha \in \Gamma_{C^\infty_c}(T^*M)$, $t > 0$, $x \in M$, $\xi \in \otimes^2 T_x M$ *one has*

$$(\nabla e^{-t\Delta_1}\alpha(x), \xi)$$
$$= -\mathbb{E}\left[\left(Q_t(x)//_t^{0,1}(x)^{-1}\alpha(X_t(x)), \ell_t^{(1)}(\xi,t) + \ell_t^{(2)}(\xi,t)\right)\right].$$

Proof: Using the Itô formula one finds (a long calculation) that the process

$$Y := (\tilde{Q}(x)//^{0,2}(x)^{-1}\nabla e^{-(t-\bullet)\Delta_1}\alpha(X(x)), \ell(\xi,t))$$
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&: [0, t] \times \Omega \longrightarrow \mathbb{R}
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is a local martingale (without any restriction on the geometry of M).

The following estimates will entail that under our assumptions on the geometry of M, the process Y is even a martingale:

Lemma ($\ell^{(j)}$-estimates)

Assume $|\mathrm{Riem}|, |\nabla^{1,3}\mathrm{Riem}| \leq A$ *for some* $A < \infty$, *and let* $q \in [1, \infty)$, $t > 0$, $x \in M$, $\xi \in T_x M$.

a) One has:

$$\mathbb{E}\left[\sup_{s \in [0,t]} |\ell_s^{(1)}(\xi, t)|^q\right]^{1/q} \leq C_{q,m} t^{-1/2} e^{t C_{A,q,m}} |\xi|,$$

b) One has:

$$\mathbb{E}\left[\sup_{s \in [0,t]} |\ell_s^{(2)}(\xi, t)|^q\right]^{1/q} \leq C e^{C_{A,m} t} |\xi|.$$

Proof: By Gronwall

$$|Q_s(x)|, |Q_s(x)^{-1}|, |\widetilde{Q}_s(x)|, \widetilde{Q}_s(x)^{-1} \leq e^{C_{m,A}s} \text{ } \mathbb{P}\text{-f.s. for all } s \in [0,t],$$

so that

$$\mathbb{E}\Big[\sup_{s\in[0,t]} |\ell_s^{(2)}(\xi,t)|^q\Big] \leq e^{qC_{m,A}t}|\xi|^q,$$

and using the Burkholder-Davis-Gundy inequality, we find

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\in[0,t]} |\ell_s^{(1)}(\xi,t)|^q\Big] &\leq C_{q,m}\mathbb{E}\Big[\Big(\int_0^t |Q_s^{T,-1}|^2|\widetilde{Q}_s^T|^2|\dot{\ell}_s(\xi,t)|^2|_x ds\Big)^{q/2}\Big] \\ &\leq C_{q,m}t^{-q/2}e^{tC_{A,q,m}}|\xi|^q,\end{aligned}$$

completing the proof of the Lemma.

Using these results for $q = 1$ and

$$\sup_{u\in[0,t],y\in M} |e^{-u\Delta_1}\alpha(y)| < \infty, \sup_{u\in[0,t],y\in M} |\nabla e^{-u\Delta_1}\alpha(y)| < \infty \text{ w.l.o.g.,}$$

we have

$$\mathbb{E}\Big[\sup_{s\in[0,t]} |Y_s|\Big] < \infty,$$

so Y is a martingale and

$$\begin{aligned}(\nabla e^{-t\Delta_1}\alpha(x), \xi) &= \mathbb{E}[Y_0] = \mathbb{E}[Y_t] \\ &= -\mathbb{E}\left[\Big(Q_t(x)//_t^{0,1}(x)^{-1}\alpha(X_t(x)), \ell_t^{(1)}(\xi,t) + \ell_t^{(2)}(\xi,t)\Big)\right],\end{aligned}$$

completing the proof of the path integral formula.

Theorem (Baumgarth/G. 2018)

Assume $|\mathrm{Riem}|, |\nabla^{1,3}\mathrm{Riem}| \leq A$ *for some* $A < \infty$. *Then for all* $p \in (1,\infty)$ *there is constant* $C = C_{A,p,m} > 0$, *so that for all* $t > 0$ *one has*

$$\left\| \nabla^{0,1} e^{-t\Delta_1} \right\|_{p,p} \leq Ce^{Ct} t^{-1/2}.$$

In particular, one has $CRT(p)$ and $CZ(p)$, with constants depending only on A, p, m.

Theorem (Baumgarth/G. 2018)

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Proof: The formula for $\nabla^{0,1} e^{-t\Delta_1}\alpha(x)$ together with

$$|Q_t(x)|, |Q_t(x)^{-1}|, |\widetilde{Q}_t(x)|, \widetilde{Q}_t(x)^{-1} \leq e^{C'''t} \quad \mathbb{P}\text{-f.s.},$$

Hölder for $\mathbb{E}$, and the $\ell^{(j)}$-estimates for $q = p^*$ shows

$$\begin{aligned}|\nabla^{0,1} e^{-t\Delta_1}\alpha(x)| &\leq C'' e^{C''t}\mathbb{E}\left[|\alpha(X_t(x))|^p\right]^{1/p} \\ &= C'' e^{C''t}\left(e^{-t\Delta}|\alpha|^p(x)\right)^{1/p},\end{aligned}$$

so

$$\begin{aligned}&\int_M |\nabla^{0,1} e^{-t\Delta_1}\alpha(x)|^p d\mu(x) \leq C' e^{C't}\int_M e^{-t\Delta}|\alpha|^p(x) d\mu(x) \\ &\leq C' e^{C't}\int_M |\alpha|^p(x) d\mu(x),\end{aligned}$$

as $e^{-t\Delta}$ is a contraction in $L^r(M)$ for all $r \in [1,\infty]$ (without any assumptions on the geometry of M). Done!

Thank you for listening!