# Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
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**Sheet 5: due on Friday 16 November at 12:00 in Room 1.032.**

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## 1 Partitions of unity [5 points]

Given any open cover $(U_\alpha)_{\alpha\in A}$ of a smooth (paracompact) manifold $X$, a *partition of unity* (subordinate to $(U_\alpha)_{\alpha\in A}$) is a collection of smooth maps $(\phi_\alpha)_{\alpha\in A} \subset C^\infty(X,\mathbb{R})$ such that

- for each $\alpha \in A$ one has $0 \le \phi_\alpha \le 1$ and $\mathrm{supp}(\phi_\alpha) \subset U_\alpha$;
- $(\mathrm{supp}(\phi_\alpha))_{\alpha\in A}$ is a locally finite collection of sets;
- for each $x \in X$ one has $\sum_{\alpha\in A} \phi_\alpha(x) = 1$.

**Claim:** For any open cover $(U_\alpha)_{\alpha\in A}$ of $X$, there exists a partition of unity subordinate to it.

(a) Prove the claim for a *finite* open cover.

(b) Explain (without giving the full proof) the additional difficulty in extending the proof from finite to infinite open covers.

## 2 The sheaf of distributions [5 points]

Consider an open subset $\Omega \subset \mathbb{R}^m$. For any open subsets $V \subset U \subset \Omega$ we consider the restriction map $\mathscr{D}'(U,\mathbb{C}^l) \to \mathscr{D}'(V,\mathbb{C}^l)$ given by $T \mapsto T|_V := T|_{\mathscr{D}(V,\mathbb{C}^l)}$. Show that the assignment $U \mapsto \mathscr{D}'(U,\mathbb{C}^l)$ gives $\mathscr{D}'(\Omega,\mathbb{C}^l)$ the structure of a sheaf.
*Hint:* you may use the claim of Exercise 1.

## 3 Compactly supported distributions [5 points]

Consider a distribution $T \in \mathscr{D}'(\Omega,\mathbb{C}^l)$. Prove that $T$ lies in the subspace $\mathscr{E}'(\Omega,\mathbb{C}^l)$ if and only if $\mathrm{supp}(T)$ is compact.

## 4 An application of the Fourier transform [5 points]

(a) Calculate the Fourier transform of the $L^1$-function $f(x) := e^{-|x|}$.

(b) Use part (a) to prove the identity

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{\pi}{2}.$$

(c) Use part (a) to prove for any $x > 0$ that

$$\int_0^\infty \frac{t\sin(tx)}{1+t^2}\,dt = \frac{\pi}{2}e^{-x}.$$