Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
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Sheet 1: due on Friday 19 October at 12:00 in Room 1.032.

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1 Proof of Lemma 0.3 [5 points]

Consider open subsets $U, V \subset \mathbb{R}^m$, a diffeomorphism $\psi \colon U \to V$, and smooth maps $\phi_1 \colon U \to \mathrm{GL}(\mathbb{C}^{l_0})$ and $\phi_2 \colon \mathrm{GL}(\mathbb{C}^{l_1})$. Let $P \colon C^\infty(U, \mathbb{C}^{l_0}) \to C^\infty(U, \mathbb{C}^{l_1})$ be a differential operator of order $\leq k$ (in the sense of Definition 0.1), and consider the operator

$$\widetilde{P} \colon C^\infty(V, \mathbb{C}^{l_0}) \to C^\infty(V, \mathbb{C}^{l_1}), \qquad \widetilde{s} \mapsto \phi_1 \circ P(\phi_2^{-1} \circ \widetilde{s} \circ \psi) \circ \psi^{-1}.$$

Prove that $\widetilde{P}$ is again a differential operator of order $\leq k$ (in the sense of Definition 0.1).

2 Inner products [5 points]

(a) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let X be a $\mathbb{K}$-vector space with an inner product $\langle \cdot | \cdot \rangle$: this means that for all $x, y, z \in X$ and $\lambda \in \mathbb{K}$ we have

(i) $\langle x | y + \lambda z \rangle = \langle x | y \rangle + \lambda \langle x | z \rangle$;

(ii) $\overline{\langle x | y \rangle} = \langle y | x \rangle$;

(iii) $\langle x | x \rangle \geq 0$, and $\langle x | x \rangle = 0$ only if $x = 0$.

Consider the norm $\|x\| := \langle x | x \rangle^{\frac{1}{2}}$. Prove that we have the *polarisation identity*:

$$\langle x | y \rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big), \qquad \text{if } \mathbb{K} = \mathbb{R},$$
$$\langle x | y \rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\big), \qquad \text{if } \mathbb{K} = \mathbb{C}.$$

(b) Let X be a $\mathbb{K}$-vector space with a norm $\|\cdot\|$. Prove that the polarisation identity defines an inner product on X if and only if the norm satisfies the *parallelogram identity*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

3 Fréchet spaces [5 points]

A *Fréchet space* X is a locally convex space whose topology is induced by a countable family of seminorms $(p_i)_{i \in I}$, such that X is Hausdorff and complete. Prove that there exists a translation invariant metric $d \colon X \times X \to [0, \infty)$ which induces the same topology.

4 Formal power series [5 points]

Let $\mathbb{C}[[X_1, \ldots, X_n]]$ be the vector space of formal power series of $X_1, \ldots, X_n$ with complex coefficients, i.e. elements $u \in \mathbb{C}[[X_1, \ldots, X_n]]$ are of the form

$$u = \sum_{\alpha \in \mathbb{N}^n} u_\alpha X^\alpha,$$

where $u_\alpha \in \mathbb{C}$ and $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For a multi-index $\alpha \in \mathbb{N}^n$ we set $|\alpha| := \alpha_1 + \ldots + \alpha_n$. We equip $\mathbb{C}[[X_1, \ldots, X_n]]$ with the topology induced by the family of seminorms $(p_m)_{m \in \mathbb{N}}$ given by

$$p_m(u) := \sup_{|\alpha| \leq m} |u_\alpha|.$$

(a) Prove that $\mathbb{C}[[X_1, \ldots, X_n]]$ is a Fréchet space.

(b) Consider the linear subspace of polynomials (i.e. *finite* power series):

$$P[X_1, \ldots, X_n] := \left\{ u \in \mathbb{C}[[X_1, \ldots, X_n]] : u_\alpha \neq 0 \text{ for only finitely many } \alpha \right\}.$$

Prove that $P[X_1, \ldots, X_n]$ is dense in $\mathbb{C}[[X_1, \ldots, X_n]]$.

(c) Let $T \colon \mathbb{C}[[X_1, \ldots, X_n]] \to \mathbb{C}[[X_1, \ldots, X_n]]$ be the linear map given by

$$T\left(\sum_{\alpha \in \mathbb{N}^n} u_\alpha X^\alpha \right) := \sum_{\alpha \in \mathbb{N}^n} |\alpha| u_\alpha X^{|\alpha|\alpha},$$

where $|\alpha|\alpha = (|\alpha|\alpha_1, \ldots, |\alpha|\alpha_n)$ Show that T is continuous.

(d) Let $S \colon P[X_1, \ldots, X_n] \to \mathbb{C}[[X_1, \ldots, X_n]]$ be the linear map given by

$$S\left(\sum_{\alpha \in \mathbb{N}^n} u_\alpha X^\alpha \right) := \sum_{\alpha \in \mathbb{N}^n} u_\alpha X^0.$$

Show that S can *not* be extended to a continuous linear map on $\mathbb{C}[[X_1, \ldots, X_n]]$.