

# Scattering theory of the Hodge-Laplacian

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Spectral theory and its applications

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This talk is about the paper:

**Francesco Bei & Batu Güneysu & Jörn Müller:** *Scattering theory of the Hodge-Laplacian under a conformal perturbation.* Preprint (2014), available from arxiv.

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- One of the most fundamental problems in geometry is the determination of the spectrum of the Laplace operator corresponding to a Riemannian metric  $g$  on  $M$ , in particular the one of the Hodge-Laplace operator  $\Delta_g^{(j)}$  which acts on differential  $j$ -forms
- If  $M$  is compact, then the spectrum  $\sigma(\Delta_g^{(j)})$  of  $\Delta_g^{(j)}$  consists of eigenvalues with a finite multiplicity and thus the situation is (analytically) very simple
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However, there is a "perturbative" way to control the *absolutely continuous* part  $\sigma_{\text{ac}}(\Delta_g^{(j)})$  of  $\sigma(\Delta_g^{(j)})$  in the noncompact case:

Assume that there is a quasi-isometric metric  $\tilde{g}$  on  $M$  such that we have some good information about the absolutely continuous part  $(\Delta_{\tilde{g}}^{(j)})_{\text{ac}}$  of  $\Delta_{\tilde{g}}^{(j)}$ . Then once we can show that the wave operators  $W_{\pm}(\Delta_g^{(j)}, \Delta_{\tilde{g}}^{(j)}, I)$  exist and are complete, they induce unitary equivalences

$$(\Delta_{\tilde{g}}^{(j)})_{\text{ac}} \sim (\Delta_g^{(j)})_{\text{ac}}, \quad \text{so that } \sigma_{\text{ac}}(\Delta_{\tilde{g}}^{(j)}) = \sigma_{\text{ac}}(\Delta_g^{(j)}).$$

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The question we address here is:

**In what sense do  $\tilde{g}$  and  $g$  have to be close to each other to ensure that  $W_{\pm}(\Delta_g, \Delta_{\tilde{g}}, I)$  exist and are complete?**

From calculating  $\Delta_{\tilde{g}} - \Delta_g$  in the (particularly important) case where one metric arises a conformal perturbation of the other, we expect the correct assumption to be a *first order* control in the deviation of  $g$  and  $\tilde{g}$ .

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- no systematic treatment (if at all) for  $k$ -forms in the literature
- For functions = 0-forms: Classically, people have considered special topologies  $M = (0, \infty) \times \mathbb{S}^{m-1}$  with warped metrics. Then the problem is typically unitarily equivalent to a scattering problem for Sturm-Liouville operators, which is a rather old (but not necessarily easy) story
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## Theorem (Müller/Salomonsen 2007, JFA 253)

Let  $g, \tilde{g}$  be complete metrics on  $M$  with  $|\sec_g|, |\sec_{\tilde{g}}| \leq L$ , such that the covariant  $C^2$ -deviation  $x \mapsto |g - \tilde{g}|_g(x)$  of  $g$  from  $\tilde{g}$  is bounded from above by some  $\beta : M \rightarrow (0, \infty)$  of moderate decay, in a way such that for appropriate constants  $a, b, c, C$  one has

$$\beta^a \in L^1(M, g), \quad |\beta^b(x) \widetilde{\text{inj}}_g(x)^c| \leq C \quad \text{for all } x,$$

where  $\widetilde{\text{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \text{inj}_g(x)\right)$ . Then  $W_{\pm}(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)})$  exist and are complete.

Although a breakthrough at that time, this result is certainly not optimal: The required control is of *second order* in the deviation of  $g$  and  $\tilde{g}$ .

Indeed, using the harmonic radius function (later...)  $x \mapsto r_g(x)$  with a certain Sobolev control, one can do much better:

## Theorem (Hempel/Weder/Post 2013, JFA 266)

Let  $g, \tilde{g}$  be complete quasi-isometric metrics on  $M$  with

$$\int_M d(g, \tilde{g})(x) h^{-(m+2)}(x) \mu_g(dx) < \infty, \quad (1)$$

where  $d(g, \tilde{g}) : M \rightarrow (0, \infty)$  is a certain function (later...) which measures a zeroth order deviation of the metrics, and where  $h : M \rightarrow (0, 1]$  is an arbitrary lower bound on

$$M \ni x \mapsto \max(\min(r_g(x), 1), \min(r_{\tilde{g}}(x), 1)) \in (0, 1].$$

Then  $W_{\pm}(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)})$  exist and are complete.

This zeroth order result should be the state of the art for *functions*.



We were interested in extending the latter result to differential forms.

Here, for some entirely algebraic reasons, we have restricted ourselves to conformal perturbations.

Given a Riemannian metric  $g$  we have

- $\nabla_g$ : the Levi-Civita connection
- $Q_g : \wedge^2 TM \rightarrow \wedge^2 TM$  the s.a. curvature endomorphism
- for a smooth 1-form  $\alpha$ ,  $\text{int}_g(\alpha) = \text{ext}^{\dagger g} : \wedge T^*M \rightarrow \wedge T^*M$  is interior multiplication with  $\alpha$
- the codifferential  $\delta_g := d^{\dagger g} : \Omega_{C^\infty}(M) \rightarrow \Omega_{C^\infty}(M)$
- the Dirac type operator  $D_g := d + \delta_g : \Omega_{C^\infty}(M) \rightarrow \Omega_{C^\infty}(M)$
- the Hodge-Laplacian  $\Delta_g := D_g^2 : \Omega_{C^\infty}(M) \rightarrow \Omega_{C^\infty}(M)$
- the Friedrichs realization  $H_g$  of  $\Delta_g$  in  $\Omega_{L^2}(M, g)$
- the resolvents  $R_{g,\lambda} := (H_g + \lambda)^{-1}$ ,  $\lambda > 0$ .
- everything filters through the form degree; notation:  
 $\Omega_{L^2}(M) = \bigoplus_{j=0}^m \Omega_{L^2}^{(j)}(M, g)$ ,  $H_g = \bigoplus_{j=0}^m H_g^{(j)}$  etc.

Our main technical tool will be harmonic coordinates with Sobolev control:

### Definition (Cheeger/Anderson)

Let  $p \in (m, \infty)$ ,  $q \in (1, \infty)$ ,  $x \in M$ . Then the  $W_g^{1,p}$ -harmonic radius at  $x$  with Euclidean distortion  $q$ ,  $r_g(x, p, q) \in (0, \infty]$ , is defined to be supremum of all  $r > 0$  such that there is a  $\Delta_g^{(0)}$ -harmonic chart  $\Phi : B_g(x, r) \rightarrow U \subset \mathbb{R}^m$  which, with respect to the  $\Phi$ -coordinates, satisfies the estimates

$$q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq q(\delta_{ij}) \text{ as symmetric bilinear forms,} \quad (2a)$$

$$r^{1-\frac{m}{p}} \left( \int_U |\partial_k g_{ij}(y)|^p dy \right)^{1/p} \leq q - 1 \text{ for all } i, j, k \in \{1, \dots, m\}. \quad (2b)$$

It is not obvious at all that  $r_g(x, p, q) > 0$ . Anyway, one has the following elementary fact:

### Proposition (B/G/M)

*For any fixed  $p, q$ , the function  $x \mapsto \min(1, r_g(x, p, q))$  is 1-Lipschitz w.r.t.  $g$ .*

The fact that indeed  $r_g(x, p, q) > 0$  as claimed in the definition follows from applying the following results near  $x$ :

## Proposition (Cheeger/Anderson 90%; B/G/M)

Assume that  $\text{Ric}_g(x) \geq -\frac{1}{\beta^2}$  and  $\text{inj}_g(x) \geq \tilde{h}(x)$ , where  $\beta > 0$  is a constant and  $\tilde{h} : M \rightarrow (0, \infty)$  is a continuous. Then:

- a) If  $\tilde{h}$  is  $g$ -Lipschitz, then for any  $p, q$  there is  $C = C(m, p, q) > 0$  such that for all  $x \in M$  one has

$$\min(r_g(x, p, q), 1) \geq C \min\left(1, \frac{\tilde{h}(x)}{1 + \|\text{d}\tilde{h}\|_{\infty, g}}, \beta\right).$$

- b) If there is a  $x_0 \in M$ , and  $c_1 > 0$ ,  $c_2 \geq 0$  such that  $\tilde{h} \geq c_1 e^{-c_2 d_g(\cdot, x_0)}$ , then for any  $p, q$  there is  $C = C(m, p, q) > 0$  such that for all  $x \in M$  one has

$$\min(r_g(x, p, q), 1) \geq C \min\left(1, \frac{c_1}{e^{c_2}} e^{-c_2 d_g(x, x_0)}, \beta\right).$$

The importance of Sobolev harmonic coordinates: By embedding theorems, we get a Hölder control on  $g_{ij}$ . To make an effective use of this observation in the form-case, we add:

### Definition

For any  $K > 0$  and any function  $h : M \rightarrow (0, 1]$ , let

$$\mathcal{M}_{K,h}(M) := \left\{ \tilde{g} \mid \tilde{g} \text{ is a complete metric on } M \text{ with } Q_{\tilde{g}} \geq -K \right. \\ \left. \text{and } \min(1, r_{\tilde{g}}(\cdot, p, q)) \geq h \text{ for some } p \in (m, \infty), q \in (1, \sqrt{2}) \right\}.$$

Carleman type resolvent estimate:

### Theorem (B/G/M)

Assume that  $g \in \mathcal{M}_{K,h}(M)$  for some pair  $(K, h)$ . Then for all  $n \in \mathbb{N}$  with  $n \geq m/4 + 2$  there is a  $C = C(m, n) > 0$ , such that for all  $\lambda > K \max_{j=0, \dots, m} j(m-j) + 1$ , the operator  $R_{g,\lambda}^n$  is an integral operator, with a Borel integral kernel

$$M \times M \ni (x, y) \longmapsto R_{g,\lambda}^n(x, y) \in \text{Hom}(\wedge T_y^* M, \wedge T_x^* M)$$

which satisfies the estimate

$$\int_M |R_{g,\lambda}^n(x, y)|_{\mathcal{G}^2}^2 \mu_g(dy) \leq Ch(x)^{-m} \text{ for all } x \in M.$$

The proof is rather complicated. The key observations are:

- $V_g^{(j)} := \Delta_g^{(j)} - \nabla_{g,j}^\dagger \nabla_{g,j}$  is zeroth order and s.a. by Weitzenböck's formula
- The Gallot-Meyer estimate states that under  $Q_g \geq -K$  one has  $V_g^{(j)} \geq -K \cdot j(m-j)$
- Now one can use probabilistic domination results for covariant Schrödinger semigroups  $e^{-t(\nabla^\dagger \nabla + V)}$  (e.g. my paper in JFA 262) to control  $R_{g,\lambda}^{(j),n}$  by  $R_{g,1}^{(0),n}$ . The latter can be controlled by  $\min(1, r_g(\cdot, p, q))$



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- Given a smooth function  $\psi : M \rightarrow \mathbb{R}$  let  $g_\psi$  denote the conformally equivalent metric  $g_\psi := e^{2\psi}g$ . Then  $g$  and  $g_\psi$  are quasi-isometric if and only if  $\psi$  is bounded and then we have the canonical identification operator  $I = I_{g, g_\psi} : \Omega_{L^2}(M, g) \rightarrow \Omega_{L^2}(M, g_\psi)$ .
- Given a Borel function  $h : M \rightarrow (0, \infty)$  and a smooth function  $\psi : M \rightarrow \mathbb{R}$  define

$$d(g, \psi)(x) := \max\{\sinh(2|\psi(x)|), |d\psi(x)|_g\}, \quad x \in M,$$

$$d_h(g, \psi) := \int_M d(g, \psi)(x) h(x)^{-(m+2)} \mu_g(dx) \in [0, \infty].$$

We call  $\psi$  a *h-scattering perturbation* of  $g$ , if one has  $d_h(g, \psi) < \infty$ .

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Now we can formulate our main result for forms:

### Theorem (B/G/M)

Let  $\psi : M \rightarrow \mathbb{R}$  be smooth with  $\psi$ ,  $|\mathrm{d}\psi|_g$  bounded, and assume that  $g, g_\psi \in \mathcal{M}_{K,h}(M)$  for some pair  $(K, h)$ , in a way such that  $\psi$  is a  $h$ -scattering perturbation of  $g$ . Then the wave operators

$$W_\pm(H_{g_\psi}, H_g, I) = \underset{t \rightarrow \pm\infty}{s\text{-}\lim} e^{itH_{g_\psi}} I e^{-itH_g} P_{\text{ac}}(H_g)$$

exist and are complete, and everything filters (a posteriori...  $\rightsquigarrow$  total forms and Dirac type operators!) through the form degree.

## Corollary

*Assume that  $g$  is complete with  $Q_g \geq -K$  for some  $K > 0$  and that  $\tilde{g}$  is a metric on  $M$  which is conformally equivalent to  $g$  and which coincides with  $g$  at infinity. Then the assumptions of our main result are satisfied.*

Indeed, since  $\psi$  is compactly supported by assumption, we can take

$h(x) := \min(1, r_g(x, p, q), r_{g_\psi}(x, p, q))$  for all  $p > m$ ,  $1 < q < \sqrt{2}$ ,

which is a positive continuous function, to make  $\psi$  a  $h$ -scattering perturbation of  $g$ .

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## Corollary

Assume that  $\psi : M \rightarrow \mathbb{R}$  is smooth and bounded,  $g$  is complete such that  $|\sec_g|, |\sec_{g_\psi}| \leq L$  for some  $L > 0$ , that there is some  $\beta : [0, \infty) \rightarrow (0, \infty)$  exponentially bounded from below, and a point  $x_0 \in M$  such that with  $\beta(x) := \beta(1 + d_g(x, x_0))$  one has:

- (i) There are constants  $b \in (0, 1)$  with  $\beta^b \in L^1(M, g)$ , and  $C_1 > 0$  such that for all  $x \in M$ ,

$$\widetilde{\text{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \text{inj}_g(x)\right) \geq C_1 \cdot \beta(x)^{\frac{1-b}{m+2}}. \quad (3)$$

- (ii) For some constant  $C > 0$  one has

$${}^1|g - g_\psi| := |g - g_\psi|_g + |\nabla_g - \nabla_{g_\psi}|_g \leq C \cdot \beta \quad (4)$$

Then the assumptions of our main result are satisfied.



The latter result can be considered as a generalization in the conformal case of the initial Müller/Salomonsen result to forms. Note however that, being a *first order result*, it is better even on functions.

Some steps in the proof of our main result...

An essential tool is to use a decomposition formula (the algebra of which forced us to restrict ourselves to the conformal case) efficiently with harmonic coordinates:

## Proposition

Assume that  $\psi$  and  $|\mathrm{d}\psi|_g$  are bounded, let  $\lambda > 0$ ,  $n \geq 1$  and let  $g$  (and thus  $g_\psi$ ) be complete. Then the bounded operator

$$R_{g_\psi, \lambda}^n (H_{g_\psi} I - I H_g) R_{g, \lambda}^n : \Omega_{L^2}(M, g) \longrightarrow \Omega_{L^2}(M, g_\psi)$$

can be decomposed as

$$\begin{aligned} R_{g_\psi, \lambda}^n (H_{g_\psi} I - I H_g) R_{g, \lambda}^n = \\ R_{g_\psi, \lambda}^n \left( D_{g_\psi} \cdot 2 \sinh(2\psi) I D_g + D_{g_\psi} I (1 - e^{-2\psi}) \mathrm{d} - \mathrm{d} \circ (1 - e^{2\psi}) I D_g \right. \\ \left. + D_{g_\psi} \operatorname{int}_{g_\psi}(\mathrm{d}\psi) \tau I - \tau \operatorname{int}_g(\mathrm{d}\psi) D_g \right) R_{g, \lambda}^n. \quad (5) \end{aligned}$$

Here  $\tau := \bigoplus_{j=0}^m (m - 2j) 1_{\wedge^j T^* M} : \wedge T^* M \longrightarrow \wedge T^* M$ .

- Now we combine the latter decomposition formula with our Carleman type resolvent estimate and the commutator relations  $[A, R_{g,\lambda}^n] = 0$ , where  $A \in \{D_g, d, \delta_g\}$ , to get that  $R_{g_\psi,\lambda}^n (H_{g_\psi} I - IH_g) R_{g,\lambda}^n$  is trace class for large  $n \rightsquigarrow$  the assumptions of Belopol'skii-Birman's theorem are satisfied
- The decomposition formula heavily requires that the underlying Hamiltonians are of the form  $L^*L$ . That is why we work with *total forms and the Dirac type operator*  $D_g$  and  $H_g = D_g^2 = D_g^*D_g$  instead of on a fixed form degree. On functions, all of this is very simple as  $\Delta^{(0)} = d^\dagger \varepsilon d$  where the differential  $d$  does not depend on the metric (and this leads to a zeroth order condition in this case).

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- I am almost sure that one can drop the restriction to conformal perturbations. But you have to solve the following entirely algebraic problem: Give yourself two metrics  $g$  and  $\tilde{g}$  on  $M$ . You can always write the one as a multiplicative perturbation of the other. But: How do you calculate  $\delta_{\tilde{g}} = F_{\tilde{g},g}(\delta_g)$ ? In the conformal case, there are somewhat accessible perturbative formulae.
- Further studies for functions: Local Dirichlet forms (not clear at all)? Weighted infinite graphs (this should at least admit a clear formulation in terms of the edge and vertex weight functions)?

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Introduction

Existing "scalar" results for functions = 0-forms

Our main result for differential forms

Key steps in the proof of our main result

Outlook

Thank you for listening!