Algebraic Topology I Quick Questions 2

Algebraic Topology I Tutors

January 18, 2024

Below is another list of questions which the tutors of AlgTop 1 prepared. We recommend you try to answer them first without looking at your notes, in order to check your understanding of the notions discussed in the lecture.

Problem 0.1. Let $\pi\colon S^\infty \to \mathbb{R}P^\infty$ be the familiar covering map. Then the map $E(\pi)\colon \mathbb{R} \times_{C_2} S^\infty \to \mathbb{R}P^\infty$, applying π to the S^∞-component of a representative, is the universal line bundle over paracompact spaces.

- ☐ True
- ☐ False

Problem 0.2. Spot the error in this argument: Let $\iota \in H^1(\mathbb{R}P^\infty; \mathbb{F}_2)$, $\iota_n \in H^1(\mathbb{R}P^n; \mathbb{F}_2)$ be the respective nontrivial elements. The total Stiefel–Whitney classes are $w(\gamma^1_{\mathbb{R}}) = 1 + \iota$ and $w(\gamma^{1,n+1}_{\mathbb{R}}) = 1 + \iota_n$, the inverse power series being given by $1 + \iota + \iota^2 + \cdots \in H^{\Pi}(\mathbb{R}P^\infty, \mathbb{F}_2)$ in the case of $\mathbb{R}P^\infty$, and analogously for $\mathbb{R}P^n$. From this we conclude that both $\gamma^1_{\mathbb{R}}$ and $\gamma^{1,n+1}_{\mathbb{R}}$ are not embeddable into trivial bundles.

Problem 0.3. Recall the proof that $\mathbb{R}P^{2^j}$ cannot be immersed into $\mathbb{R}^{2^{j+1}-2}$. Where do you use the immersion property?

Problem 0.4. The tangent bundle τ_{S^n} of S^n is trivial for $n \geq 1$.

- ☐ True
- ☐ False

Problem 0.5. Two real vector bundles over a paracompact base are isomorphic if and only if their Stiefel-Whitney classes agree.

- ☐ True
- ☐ False

Problem 0.6. Let $\gamma^{1,n+1}_{\mathbb{R}}$ denote the tautological bundle over $\mathbb{R}P^n$. Compute $\omega(\gamma^{1,n+1}_{\mathbb{R}})$ for n a power of 2.

Problem 0.7. The Grassmanian $\mathrm{Gr}^{\mathbb{R}}_n$ can be given the structure of a smooth manifold.

- ☐ True
- ☐ False

Problem 0.8. Let $\gamma^n_{\mathbb{C}}$ denote the tautological n-dimensional complex vector bundle over the infinite Grassmanian $\mathrm{Gr}^{\mathbb{C}}_n$, equipped with a Euclidean metric. Then the associated sphere bundle $S(\gamma^n_{\mathbb{C}}) \to \mathrm{Gr}^{\mathbb{C}}_n$ induces a surjection on cohomology $H^*(-, A)$ for every coefficient group A.

- ☐ True
- ☐ False

Problem 0.9. Let M be a smooth submanifold of $\mathbb{R}^n$. Then the tangent bundle τ_M of M is trivializable if and only if the normal bundle $\nu_{M,\mathbb{R}^n}$ is trivializable.

☐ True

☐ False

Problem 0.10. The smooth manifold $\mathbb{R}P^4$ does not allow an immersion into $\mathbb{R}^6$.

☐ True

☐ False

Problem 0.11. Let $(r_1, \ldots, r_6)$ be a sequence of non-negative numbers such that $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 + 6r_6 = 6$. Then the Stiefel-Whitney number $w_1^{r_1} \cdots w_6^{r_6}[\mathbb{R}P^6]$ is non-zero.

☐ True

☐ False

Problem 0.12. If all the Stiefel-Whitney numbers of a compact smooth manifold M are zero, then the total Stiefel-Whitney class of the tangent bundle τ_M is equal to 1.

☐ True

☐ False

Problem 0.13. $\mathbb{RP}^3 \times \mathbb{RP}^3$ is cobordant to $\mathbb{RP}^6$.

☐ True

☐ False

Problem 0.14. Euler characteristic modulo 2 is a bordism-invariant.

☐ True

☐ False

Problem 0.15. Let $I = (i_1, i_2, \ldots, i_n)$ be a sequence with associated element $Sq^I = Sq^{i_1} \cdots Sq^{i_n}$ in the Steenrod algebra. Then, if I is not admissible, we have $Sq^I = 0$.

☐ True

☐ False

Problem 0.16. The integral homology of the homotopy fibre F of the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ is zero in odd degrees and isomorphic to $\mathbb{Z}/n$ in degree $2n$, for all $n \in \mathbb{N}$.

☐ True

☐ False

Problem 0.17. Let X be a simply-connected space. If there exists an $n > 1$ such that $\pi_n(X, *)$ is infinite, then there also exists an $m > 1$ such that $H_m(X, \mathbb{Z})$ is infinite.

☐ True

☐ False

Problem 0.18. The tangent and normal bundle of the 2-torus $S^1 \times S^1$, viewed as a submanifold of $\mathbb{R}^2 \times \mathbb{R}^2$, are each trivializable.

☐ True

☐ False

Problem 0.19. Assume that ξ is a real vector bundle with paracompact base space X, satisfying $w_3(\xi) \neq 0$ and $H^2(X, \mathbb{F}_2) = 0$. Then ξ does not have a 1-dimensional subbundle.

☐ True

☐ False

Problem 0.20. Let $(F \to Y \to X, h)$ be a fibre sequence, such that three of F, Y and X are compact. Then so is the third.

☐ True

☐ False