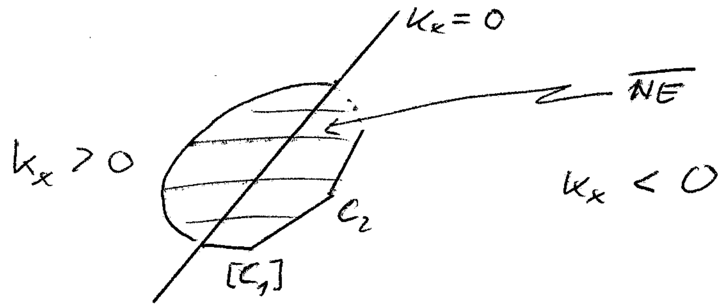
MMP: Reduce any variety (e.g. smooth projective) to one with $K_X$ nef.

<u>Cone theorem (smooth case)</u>

Let $X$ be a smooth projective variety. Then there exist rational curves $C_i$ with $0 > (K_X . C_i) \geq -(\dim(X)+1)$ s.t. $\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$, where the $\mathbb{R}_{\geq}[C_i]$ are extremal rays of $\overline{NE(X)}$.

(Moreover, accumulation only happens near $K_X = 0$, i.e. $\forall \varepsilon > 0$ and any ample divisor $H$: $\overline{NE(X)}_{K_X+\varepsilon H \leq 0}$ contains only finitely many $[C_i]$.)




What is missing: • Can the extremal rays $\mathbb{R}_{\geq 0}[C_i]$ be contracted?

• What about singular varieties?

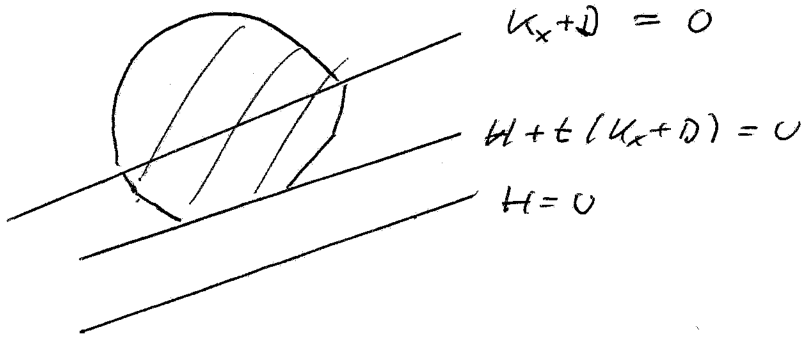
__Rationality theorem__: Suppose $X$ is projective and $(X, D)$ is klt. with $K_X + D$ not nef

Pick $a \in \mathbb{Z}_{>0}$ s.t. $a(K_X + D)$ is Cartier.

Then for any ample (Cartier) divisor $H$

$$r := \max\{t \in \mathbb{R} \mid H + t(K_X + D) \text{ nef}\}$$

is __rational__ of the form $\frac{u}{v}$ with $0 < v \leq a \cdot (\dim(X) + 1)$.




__Basepoint free theorem__: Suppose $X$ projective, $(X, D)$ weakly klt and $H$ is a nef Cartier divisor s.t. $aH - (K_X + D)$ is ample for some $a > 0$. Then $mH$ is basepoint free for $m \gg 0$.

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Both theorems combined yield: For the extremal rays in $\overline{NE(X)}_{K_X + D < 0}$ one finds divisors $H + t(K_X + D)$ which are nef and vanish along the ray. Since $t$ is rational $H' := \frac{1}{t} H + (K_X + D)$ is $\mathbb{Q}$-Cartier, which is nef an $H' - (K_X + D)$ is ample. Then $mH'$ is base point free and will therefore yield a morphism $X \to \mathbb{P}^N$ contracting the extremal ray.

Proof of the BPF-theorem:

Choose resolution $f: Y \to X$ with except. divisors $E_i$ s.t.

- $D_Y = \tilde{D} + \sum E_i$ is nc, $Y$ smooth
- $K_Y + D_Y = f^*(K_X + D) + \sum b_i E_i$ with $b_i > 0$
- $f^*(aH - (K_X + D)) - \sum \delta_i E_i$ is ample for certain $0 < \delta_i \ll 1$.

1. Rewrite: $f^*(aH - (K_X + D)) - \sum \delta_i E_i$ ample

$= a f^* H - (K_Y + D_Y) + \sum (b_i - \delta_i) E_i$

$= a f^* H + A - K_Y$, where $A = -\tilde{D} + \sum (b_i - \delta_i - 1) E_i$

Clearly: $A$ is nc and using $d_i < 1$ $(D = \sum d_i D_i)$ and $b_i > 0$ one finds $-d_i > -1$, $b_i - \delta_i - 1 > -1$, i.e. $\lceil A \rceil \geq 0$ (effective).

Non-vanishing theorem: $Y$ smooth projective, $H_Y$ nef, $A$ s.t. $\{A\}$ nc and $\lceil A \rceil \geq 0$, $aH_Y + A - K_Y$ nef and big (e.g. ample) for some $a > 0$. Then

$$H^0(Y, m H_Y + \lceil A \rceil) \neq 0 \qquad m \gg 0$$

This clearly applies to our situation and yields:

$$H^0(Y, m f^* H + \lceil A \rceil) \neq 0 \quad \text{for } m \gg 0$$

$(H_Y := f^* H)$

Since $\lceil A \rceil = \sum a_i E_i$ with $a_i \geq 0$ ( use $0 \leq d_i < 1$ ),

one has $f_* \mathcal{O}(m f^* H + \lceil A \rceil) = \mathcal{O}(m H)$.

Hence $H^0(X, mH) \neq 0$ for $m \gg 0$.

2. Consider (reduced) base locus $B(m) := Bs\,|mH|$

For $m_0$ fixed one has $B(m_0^{n_2}) \subset B(m_0^{n_1})$ if $n_2 \geq n_1$

$\rightsquigarrow$ stabilizes for $n \gg 0$

Suppose one can prove $B(m_0^n) = B(m_1^n) = \emptyset$ for $n \gg 0$
and coprime $m_0, m_1$. Since any large $m$ can
be written as $m = a_0 m_0^n + a_1 m_1^n$, $a_0, a_1 \geq 0$
one has $B(m) \subset B(a_0 m_0^n) \cap B(a_1 m_1^n) = \emptyset$.

Thus, it suffices to prove $B(m_0^n) = \emptyset$ $n \gg 0$.

3. Choose the resolution $f : Y \to X$ such that
moreover $f^{-1} \mathcal{I}_B \subset \mathcal{O}_Y$ invertible. ("Principalization"),
where $B := B(m_0^n)$ $n \gg 0$ (it stabilizes)

From now on the $E_i$ could be exceptional or
codim-1 components of $B$.

$$\rightsquigarrow m f^* H = L + \sum a_i E_i \qquad m = m_0^n$$

movable part,
i.e. $|L|$ is bpf

$Bs\,(m f^* H)$

More precisely,
$Bs\,(m f^* H) = \bigcup\limits_{a_i \neq 0} E_i$

(Recall that $H^0(Y, f^* mH) = H^0(X, mH)$
and therefore $Bs(f^* mH) = f^{-1} B$.)

Idea: Try to ~~get~~ contradiction by finding
$E_1$ with $a_1 > 0$ s.t. $f(E_1) \not\subset B(m_0^n)$ $n \gg 0$.

4. For $b \geq cm + a$ define

$$N(b,c) := b f^* H - K_Y - D_Y + \sum (b_i - \delta_i - c a_i) E_i$$

put $b_i = 1$ for codim. 1 cpt of $B$

$$= \underbrace{f^*(b - cm - a)H}_{\text{nef}} + c\underbrace{(f^* mH - \sum a_i E_i)}_{\substack{=L \\ \text{basepoint free}}} + \underbrace{f^* aH - K_Y - D_Y + \sum (b_i - \delta_i) E_i}_{\text{ample}}$$

nef + ample = ample
bpf + ample = ample
$\Rightarrow$ $N(b,c)$ is ample

Using that $\{N(b,c)\}$ is nc the Kawamata-Viehweg vanishing in the form 2b) applies:

$$H^1(Y, \underbrace{\ulcorner N(b,c) + K_Y \urcorner}_{(*)}) = 0$$

$$\ulcorner N(b,c) + K_Y \urcorner = \ulcorner b f^* H + \sum (b_i - \delta_i - c a_i - 1) E_i \urcorner$$

$$= b f^* H + \sum \ulcorner b_i - \delta_i - c a_i - 1 \urcorner E_i$$

Choose $c$ (and possibly $\delta_i$) s.t. for one $i$ (say $i=1$),

$$b_1 - \delta_1 - c a_1 - 1 = -1 \quad \text{and}$$

$$b_i - \delta_i - c a_i - 1 > -1 \quad \text{for } i > 1$$

(Note that $E_1$ is not nec. exceptional, but $a_1 > 0$)

Then we may write

$$\Sigma (b_i - \delta_i - c a_i - 1) E_i = A - E_1 \quad \text{with } A \text{ having}$$

no $E_1$ component. Moreover $\ulcorner A \urcorner$ is effective and exceptional.

Hence $\circledast = b f^* H + \ulcorner A \urcorner - E_1$.

Now use $0 \to \mathcal{O}(-E_1) \to \mathcal{O} \to \mathcal{O}_{E_1} \to 0$

$\rightsquigarrow$

$$H^0(Y, b f^* H + \ulcorner A \urcorner) \to H^0(E_1, (b f^* H + \ulcorner A \urcorner)|_{E_1}) \to \underbrace{H^1(Y, b f^* H + \ulcorner A \urcorner - E_1)}_{=0}$$

$\Rightarrow$ "$\twoheadrightarrow$"

Claim: $H^0(E_1, (b f^* H + \ulcorner A \urcorner)|_{E_1}) \neq 0$.

Then $H^0(Y, b f^* H + \ulcorner A \urcorner)$ contains a section $s$ with $s|_{E_1} \neq 0$

Writing $s = \pi^* t$ : $H^0(X, bH) \cong H^0(Y, b f^* H + \ulcorner A \urcorner)$ ($\ulcorner A \urcorner$ except)

$t \longmapsto \pi^* t$

yields $t \in H^0(X, bH)$ with $t|_{f(E_1)} \neq 0$

$\Rightarrow f(E_1) \not\subset Bs(b)$. Use this for $b = m_0$ "

5. Proof of Claim

Want to apply non-vanishing on $E_1$:

$K_{E_1} = (K_Y + E_1)|_{E_1}$

$N(b,c)|_{E_1}$ ample

$= (b f^*H + A - E_1 - K_Y)|_{E_1}$

$= (b f^*H + A)|_{E_1} - K_{E_1}$ $\Rightarrow$ non-vanishing on $E_1$ applies

nef $\quad$ nc and $\ulcorner A \urcorner \geq 0$