RECONSTRUCTING DECOMPOSITION SUBGROUPS IN ARITHMETIC FUNDAMENTAL GROUPS USING REGULATORS

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Abstract. In this article we explain how to reconstruct the decomposition subgroups and norms of points on an arithmetic curve inside its fundamental group if the following data are given: the fundamental group, a part of the cyclotomic character and the family of regulators of the fields corresponding to the generic points of all étale covers of the given curve. The approach is inspired by that of Tamagawa for curves over finite fields but uses Tsfasman-Vlăduţ theorem instead of Lefschetz trace formula. To the authors' knowledge, this is a new technique in the anabelian geometry of arithmetic curves. It is conditional and depends on unknown properties of arithmetic fundamental groups.

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1. Introduction

Let K be a number field, S a finite set of primes of K containing all archimedean primes and K_S/K the maximal extension of K unramified outside S. Let $\mathcal{O}_{K,S}$ be the ring of S-integers of K. Then the étale fundamental group of $\operatorname{Spec}\mathcal{O}_{K,S}$ is equal to the Galois group $\mathrm{G}_{K,S}$ of K_S/K. We discuss in this paper, how one can deduce anabelian information on $\operatorname{Spec}\mathcal{O}_{K,S}$ from $\mathrm{G}_{K,S}$. Our final goal, which we achieve only under strong additional hypotheses, is to establish a *local correspondence on the curve* $\operatorname{Spec}\mathcal{O}_{K,S}$: i.e., given pairs (K_i, S_i) for $i = 1, 2$ and an isomorphism $\sigma\colon \mathrm{G}_{K_1,S_1} \to \mathrm{G}_{K_2,S_2}$, let for any finite subextension $K_{1,S_1}/L_1/K_1$, denote the corresponding extension of K_2 via σ by L_2. For a scheme X, let $|X|$ denote the set of closed points of X. Under certain additional requirements on σ and G_{K_i,S_i}, we will show the existence of a compatible family of bijections

$$\sigma_{*,L_1}\colon |\operatorname{Spec}\mathcal{O}_{L_1,S_1}| \smallsetminus R_1(L_1) \to |\operatorname{Spec}\mathcal{O}_{L_2,S_2}| \smallsetminus R_2(L_2)$$

on closed points, for any finite subextension $K_{1,S_1}/L_1/K_1$, where R_i is some exceptional set of primes of K_i of Dirichlet density zero. These bijections are characterized as follows: if $D_{\bar{\mathfrak{p}}}$ denote the decomposition subgroup of a prime of K_{1,S_1}, then we require $D_{\sigma_*(\bar{\mathfrak{p}})} = \sigma(D_{\bar{\mathfrak{p}}})$, where σ_* is the inverse limit of the maps σ_{*,L_1} over all $K_{1,S_1}/L_1/K_1$. More precisely, we will give a (unfortunately, not purely) group theoretic criterion for a subgroup $D \subseteq \mathrm{G}_{K,S}$ to be a decomposition subgroup of a prime in the spirit of Tamagawa's work [Ta97] in the case of curves over finite fields. Further, σ_* happens to preserve the absolute norm of primes, i.e., it also preserves the residue characteristic and the inertia degree over $\mathbb{Q}$. Thus by Chebotarev density theorem, if certain unknown properties of $\mathrm{G}_{K,S}$ are assumed to hold, then $\mathrm{G}_{K,S}$ plus the additional data (see Theorem 1.1) are enough to characterize the number field uniquely, if it is normal over $\mathbb{Q}$ and the exceptional set R is small enough. This gives an (unfortunately, conditional) approach to a question related to the Isom-form of Grothendieck's conjecture (cf. [Gr83] and [NSW08] 12.3.4,12.3.5) in the case of arithmetic curves.

Inspired by the work of Hajir-Maire [HM01], we consider only tamely ramified towers to obtain nontrivial invariants. We will work in two slightly different cases. For ℓ a rational prime and $\mathscr{L}/K$ a Galois extension, let $\mathscr{L}^{\mathrm{tm}}/K$ resp. $\mathscr{L}^{\mathrm{tm}+(\ell)}/K$ denote the maximal tame resp. the maximal tame pro-ℓ-subextension of $\mathscr{L}/K$. Let $\bullet \in \{tm, tm + (\ell)\}$. The Galois group of $K_S^{\bullet}/K$ will be denoted by $\mathrm{G}_{K,S}^{\bullet}$. Observe that $K_S^{\mathrm{tm}+(\ell)} = K_{S \smallsetminus S_\ell}^{(\ell)}$. Before stating the main result, we must introduce some further assumptions on $\mathrm{G}_{K,S}^{\bullet}$. This is due to the fact, that this group is not well-understood yet. If R is some set of primes of K and $\mathscr{L}/K$ is some extension, let $\mathscr{L}^R/K$ denote the maximal subextension of $\mathscr{L}/K$,

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which is completely split in R. Let $S_f := S \smallsetminus S_\infty$ be the set of the non-archimedean primes in S and let

$$S_{\min} := \begin{cases} \{\mathfrak{p} \in S_f : \mathrm{N}\mathfrak{p} \equiv 1 \mod \ell\} & \text{if } \bullet = \mathrm{tm} + (\ell), \\ S_f & \text{otherwise.} \end{cases}$$

We have the following assumptions on a pair K, S:

$(GRH)^\bullet$ All finite subextensions in $K_S^\bullet/K$ satisfy the generalized Riemann hypothesis.

$(FD)^\bullet$ there is a set R of primes of K of Dirichlet density 0, with $R \cap S = \varnothing$ such that for all non-archimedean primes $\mathfrak{p}$ of K with extension $\bar{\mathfrak{p}}$ to $K_S^\bullet$ one has:

$$(K_S^\bullet)_{\bar{\mathfrak{p}}} = \begin{cases} (K_\mathfrak{p})^\bullet & \text{if } \mathfrak{p} \in S_{\min} \\ (K_\mathfrak{p})^{\mathrm{nr},\bullet} & \text{if } \mathfrak{p} \notin S_{\min} \cup R \\ K_\mathfrak{p} & \text{if } \mathfrak{p} \in R, \end{cases}$$

where nr stands for the maximal unramified subextension.

Also assuming $(FD)^\bullet$ with respect to the set R, we have a further condition:

$(FKS)^\bullet$ For any finite subextension $K_S^\bullet/L/K$, for any two primes $\mathfrak{p} \neq \mathfrak{q}$ of L such that $\mathfrak{p} \notin R$, the decomposition group of $\mathfrak{p}$ in the extension $K_S^{\{\mathfrak{q}\},\bullet}/L$ is infinite.

Thus $(FD)^\bullet$ says simply that the decomposition groups at all primes outside R are the maximal possible ones and $(FKS)^\bullet$ ensures that the decomposition groups at different primes are independent in some sense. If $\bullet = \mathrm{tm} + (\ell)$, then $(FD)^\bullet$ holds for $K_S^\bullet/K$ if $\operatorname{Spec} \mathcal{O}_{K,S}$ is a $\mathrm{K}(\pi, 1)$-space for ℓ. By the work of A. Schmidt [Sch07], [Sch10] this is the case when S is big enough. Also in this pro-ℓ-case we do not need the GRH-assumption, but it is completely unclear whether $(FKS)^\bullet$ holds. At least, by [Sch10] for any given *finite* set of $\mathfrak{q}$'s one can ensure the condition by enlarging S. We think that it should be possible to weaken this condition in our approach. However, the situation (except for needing GRH) in the case $\bullet = \mathrm{tm}$ should be better, because the group is then much bigger. For a number field K let $\chi_{K,p}$ denote the p-part of its cyclotomic character, h_K its class number, Reg_K its regulator. For a finite extension L/K, let $D_{L/K}$ be its discriminant and $g_{L/K} := \log |\mathrm{N_{K/\mathbb{Q}} D_{L/K}}|^{\frac{1}{2}}$.

Theorem 1.1. *For $i = 1, 2$ let $\operatorname{Spec} \mathcal{O}_{K_i,S_i}$ be an arithmetic curve. Let*

$$\sigma\colon\ \mathrm{G}_{K_1,S_1} \to \mathrm{G}_{K_2,S_2}$$

be an isomorphism of the fundamental groups. Assume that

(1) *$S_\infty \subseteq S_i$, at least two rational primes lie under S_i and one of them, denoted p, lie under both*
(2) *K_{i,S_i} realizes locally at each $\mathfrak{p} \in S_i$ the maximal local extension*
(3) $\chi_{K_1,p} = \chi_{K_2,p} \circ \sigma$.

Let $\bullet \in \{tm, \mathrm{tm} + (\ell)\}$. Then σ induces an isomorphism

$$\sigma^\bullet\colon\ \mathrm{G}^\bullet_{K_1,S_1} \to \mathrm{G}^\bullet_{K_2,S_2}\,.$$

Assume further that

(4) *$(FD)^\bullet, (FKS)^\bullet$ hold for $K^\bullet_{i,S_i}/K_i$ with respect to a set R_i*
(5) *either $\bullet = tm + (\ell)$ and $K_1, K_2/\mathbb{Q}$ are almost normal and totally imaginary, or $(GRH)^\bullet$ holds*
(6) *for any* infinite *subextension $K^\bullet_{1,S_1}/\mathscr{L}_1/K_1$, such that $g_{L_1/K_1} \neq 0$ for $\mathscr{L}_1/L_1/K_1$ big enough, one has:*

$$\lim_{L_1} \frac{\log h_{L_1}(\mathrm{Reg}_{L_1} - \mathrm{Reg}_{L_2})}{g_{L_1/K_1}} = 0,$$

where the limit is taken over finite subextensions L_1 of $\mathscr{L}_1/K_1$ and L_2/K_2 is the field corresponding to L_1 via σ.

Then for any $K^\bullet_{1,S_1}/L_1/K_1$ finite, $\sigma^\bullet$ induces a bijection (the local correspondence map):

$$\sigma_{*,L_1}\colon |\operatorname{Spec} \mathcal{O}_{L_1,S_1}| \smallsetminus R_1(L_1) \to |\operatorname{Spec}_{L_2,S_2}| \smallsetminus R_2(L_2),$$

characterized by $\sigma^\bullet(D_{\bar{\mathfrak{p}}}) = D_{\sigma_*(\bar{\mathfrak{p}})}$. The maps σ_{*,L_1} are compatible for varying L_1 and preserve the residue characteristic and the absolute inertia degrees of primes.

Corollary 1.2. *Under the assumptions in the theorem, if $K_1/\mathbb{Q}$ is normal and the Dirichlet density of the images of R_1, R_2 in the set of all primes of $\mathbb{Q}$ is zero, then $K_1 \cong K_2$.*

The assumptions (FD) and (FKS) were discussed above. As to the others, the assumptions (1), (2) and (3) are necessary to establish a local correspondence at the boundary (cf. [Iv14] Theorem 1.1) and to deduce information from the local Galois groups of primes in S_f. Moreover, (2) is known to hold in a large amount of cases due to [CC09] Theorem 5.1. In particular, (2) holds if (1) holds and S_f is defined over a totally real subfield. Further, (3) holds when certain finiteness hypothesis on Tate-Shafarevich groups holds (cf. [Iv14] Proposition 1.3). In the core of our method we use assumptions (6) and (FKS).

The main idea in the proof of Theorem 1.1 is a group-theoretic characterization of the decomposition subgroups of points on $\operatorname{Spec} \mathcal{O}_{K,S}$ in the spirit of Tamagawa's work [Ta97]. Instead of using étale cohomology and Lefschetz fixed point formula, which are not available in our case, we use the theorem of Tsfasman-Vlăduţ (and its further generalization by Zykin), which is a generalization of the Brauer-Siegel theorem. The application of Tsfasman-Vlăduţ theorem is the place where we need the behavior of the regulators of the fields in question. While it is completely unclear how to recover these regulators from the fundamental group $\mathrm{G}_{K,S}$, the p-adic volume $\mathrm{vol}_p(K)$ of the unit lattice (for totally real fields, it is the p-adic norm of the p-adic regulator) can be recovered if $S_p \subseteq S$ (see Section 2.4). Unfortunately, the p-adic analogue of Brauer-Siegel and hence also of Tsfasman-Vlăduţ fails (see [Wa82]), so our method would fail, if we replace Reg_K by the $\mathrm{vol}_p(K)$. This do not exclude that there is still some similar way of using $\mathrm{vol}_p(K)$ to obtain information on decomposition subgroups.

Note the following subtlety: in the application of Tsfasman-Vlăduţ, we need to restrict attention to the maximal tame subextension as the infinitely wildly ramified towers are in general asymptotically bad and hence the invariants we work with get trivial.

Finally we mention a different point of view to Theorem 1.1. Namely, assume for a number field L the number of complex and real embeddings, its class number, (the absolute value of) its discriminant and $\sharp\mu(L)$ are known (in our set up, most of them can be deduced from the position inside $\mathrm{G}_{K,S}$ of the decomposition groups at primes in S_f). Then by the class number formula the knowledge of Reg_L is equivalent to the knowledge of the residue of the zeta-function ζ_L at $s = 1$. Thus Theorem 1.1 says essentially that $\mathrm{G}_{K,S}$ (if it is big enough) plus the family of the residues of the functions ζ_L for $K_S^{\bullet}/L/K$ at $s = 1$ gives (under certain hypotheses) enough information to reconstruct K, at least when K is normal over $\mathbb{Q}$.

Notation. Let us collect the notations used throughout the paper. Let K be a number field, i.e., a finite extension of $\mathbb{Q}$. Then Σ_K is the set of all primes (archimedean or not) of K, $S_\infty := S_\infty(K)$ is the set of archimedean primes of K and $S_f := S \smallsetminus S_\infty(K)$ for any set S of primes of K. For $S, R \subseteq \Sigma_K$, K_S^R is the maximal extension of K unramified outside S and completely split in R. For a prime $\mathfrak{p}$ of K we write $\mathrm{N}\mathfrak{p}$ for its norm over $\mathbb{Q}$. Further, D_K, h_K, Reg_K are the absolute discriminant, the class number, the regulator of K and $g_K = \log |D_K|^{\frac{1}{2}}$ is the genus of K. For a finite extension L/K we set $g_{L/K} := \log |\mathrm{N_{K/\mathbb{Q}} D_{L/K}}|^{\frac{1}{2}}$.

If L/K is a Galois extension and $\bar{\mathfrak{p}}$ is a prime of L, then $D_{\bar{\mathfrak{p}},L/K} \subseteq \mathrm{G}_{L/K}$ denotes the decomposition subgroup of $\bar{\mathfrak{p}}$. If $\mathfrak{p} := \bar{\mathfrak{p}}|_K$ is the restriction of $\bar{\mathfrak{p}}$ to K, then we sometimes write $D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ or simply $D_{\bar{\mathfrak{p}}}$ instead of $D_{\bar{\mathfrak{p}},L/K}$.

A prime always mean a non-archimedean prime.

Outline of the paper. In Section 2 we recall the necessary preliminaries, do some technical computations and make a digression on how to reconstruct the p-adic volume of the unit lattice from $\mathrm{G}_{K,S}$. In Section 3 we prove two criterions for subgroups of $\mathrm{G}_{K,S}^{\bullet}$ being related to primes of K. They are the main ingredients in the proof of Theorem 1.1, which follows in Section 4.

2. Preliminaries

Here we collect some facts which we use later. First we recall results from [Iv14], which are our starting point. Then we recall the generalization of the Brauer-Siegel theorem by Tsfasman-Vlăduţ [TV02] and Zykin [Zy05], which will be the key ingredient for our main result.

2.1. **Decomposition groups of primes in S.** Let us use the following notation: if (x),(y) are some sets of invariants of K, S (like, for example, the position of the decomposition groups of primes inside $\mathrm{G}_{K,S}$), then (x) $\rightsquigarrow$ (y) resp. (x) $\leftrightsquigarrow$ (y) will have the following meaning: if the data in (x) are known, then we can deduce the data in (y) from them resp. the knowledge of (x) and (y) is equivalent. In particular, (x) $\rightsquigarrow$ (y) implies that if two pairs $(K_i, S_i), i = 1, 2$ are given, with $\mathrm{G}_{K_1,S_1} \cong \mathrm{G}_{K_2,S_2}$ and such that the data in (x) coincide for $i = 1, 2$, then also the data in (y) are coincide.

We summarize the results from [Iv14] which we need in Section 4 to prove Theorem 1.1. In contrast to the function field case, not much information can be deduced from $\mathrm{G}_{K,S}$ in general, i.e., if no assumptions on S are made. Assume that at least two rational primes are invertible in $\mathcal{O}_{K,S}$, let p denote one of them and let further $\chi_p\colon \mathrm{G}_{K,S} \to \mathbb{Z}_p^*$ denote the p-part of the cyclotomic character of K. For pairs (K, S) satisfying this assumption, the following were shown in [Iv14]:

- (i) $(\mathrm{G}_{K,S}, p) \rightsquigarrow [K : \mathbb{Q}], r_1(K), r_2(K)$
- (ii) $(\mathrm{G}_{K,S}, p, \chi_p) \rightsquigarrow (D_{\mathfrak{p}} \subseteq \mathrm{G}_{K,S})_{\mathfrak{p} \in S_f}, \sharp S_f(L), \mathrm{Cl_S(L)}$ for all finite and totally imaginary $K_S/L/K$ (also some converse statements hold; cf. [Iv14] Theorem 1.1)
- (iii) if a hypothesis on the finiteness of certain Tate-Shafarevich groups with divisible coefficients holds true, then $(\mathrm{G}_{K,S}, p) \rightsquigarrow \chi_p$ (cf. [Iv14] Proposition 1.3)
- (iv) if the Leopoldt conjecture for K and all primes holds true, then $\mathrm{G}_{K,S} \rightsquigarrow$ 'all primes ℓ such that $S_\ell \subseteq S$'.

Thus $\mathrm{G}_{K,S}$ itself (under certain hypotheses) or $\mathrm{G}_{K,S}$ plus some part of the cyclotomic character determine intrinsically the position inside $\mathrm{G}_{K,S}$ of the decomposition groups of primes lying in S_f. Note that one does not use the (in general unknown) fact that the decomposition groups of primes in S_f are maximal possible (i.e., that K_S realizes the maximal possible local extension at this prime)[1]. Let us now consider only pairs (K, S) for which this is true. Then also the class group and the relative discriminant can be deduced:

- (iv) $(\mathrm{G}_{K,S}, (D_{\mathfrak{p}} \subseteq \mathrm{G}_{K,S})_{\mathfrak{p} \in S_f}) \rightsquigarrow \sharp\mathrm{Cl(L)}$ for any finite totally imaginary subextension $K_S/L/K$; $\mathrm{N_{L/\mathbb{Q}}D_{L'/L}}$, if $K_S/L'/L/K$ are finite subextensions such that L'/L is tame.

2.2. **Asymptotically exact towers.** We will make use of the generalized Brauer-Siegel theorem proven by Tsfasman-Vlăduţ [TV02] and extended further by Zykin [Zy05], so let us recall their results. Tsfasman and Vlăduţ proved their results in a rather big generality, for asymptotically exact *families* of number fields. We need only the special case of infinite number fields (equivalently, of towers of number fields). Let $\mathscr{K}$ be an infinite number field, i.e., an algebraic extension of $\mathbb{Q}$ of infinite degree. We can choose a tower

$$\mathbb{Q} \subsetneq K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq \mathscr{K}$$

of subfields, finite over $\mathbb{Q}$, such that $\mathscr{K} = \bigcup_{n=1}^{\infty} K_n$. Such towers are clearly not unique. All results in this section (and also later on) depend only on $\mathscr{K}$, not on the choice of the tower. For a number field K, set

- $\Phi_\alpha(K) := \{\mathfrak{p} \in \Sigma_K : \mathrm{N}\mathfrak{p} = \alpha\}$ if α is a rational prime power
- $\Phi_\alpha(K)$ is the set of real resp. complex primes of K, if $\alpha = \mathbb{R}$ resp. $\mathbb{C}$.

For $\mathscr{K}, K_n$ as above and α either a power of a rational prime or $\mathbb{R}$ or $\mathbb{C}$, define

$$\phi_\alpha(\mathscr{K}) := \lim_{n \to \infty} \frac{\sharp\Phi_\alpha(K_n)}{g_{K_n}}.$$

Then [TV02] Lemma 2.4 shows that this limit exists for each α (in the terminology of [TV02] this means that $\{K_n\}_n$ is an asymptotically exact family). Moreover, [TV02] Lemma 2.5 shows that these limits depend only on $\mathscr{K}$ and α and not on the choice of the tower. Further, $\mathscr{K}$ is called *asymptotically good*, if there exists an α such that $\phi_\alpha(\mathscr{K}) \neq 0$, and *asymptotically bad* otherwise. Also define

[1]Note that although this is not known in general, [Ch07] [CC09] shows that this is true if S is defined over a totally real subfield and $S \supseteq S_{p_1} \cup S_{p_2} \cup S_\infty$ for at least two rational primes $p_1 \neq p_2$.

$$\mathrm{BS}(\mathscr{K}) := \lim_{n\to\infty} \frac{\log(h_{K_n}\operatorname{Reg}_{K_n})}{g_{K_n}}.$$

(We will see in a moment that this notation makes sense). A finite extension L/K is called *almost normal*, if there are subextensions $L = K_n \supseteq K_{n-1} \supseteq \cdots \supseteq K_0 = K$, such that K_{i+1}/K_i is normal. An infinite extension $\mathscr{K}/K$ is called almost normal, if it is the limit of a tower of finite almost normal extensions. The following generalization of the classical Brauer-Siegel theorem is shown by Tsfasman-Vlăduţ in the asymptotically good case and also under GRH and then by Zykin in the remaining asymptotically bad unconditional case.

Theorem 2.1 ([TV02] GRH Corollary D, Corollary F; [Zy05] Corollary 3)**.** *Let $\mathscr{K}$ be an infinite number field and let $\{K_n\}_n$ be a tower of subextensions of $\mathscr{K}$, such that $K_n \subsetneq K_{n+1}$ for all n and $\mathscr{K} = \bigcup_{i=0}^{\infty} K_n$. Assume that at least one of the following holds:*

- *each K_n is almost normal over $\mathbb{Q}$, or*
- *GRH holds for each K_n*

Then $\mathrm{BS}(\mathscr{K})$ exists, is independent of the choice of $(K_n)_n$ and

$$\mathrm{BS}(\mathscr{K}) = 1 + \sum_q \phi_q(\mathscr{K}) \log\frac{q}{q-1} - \phi_{\mathbb{R}}(\mathscr{K})\log 2 - \phi_{\mathbb{C}}(\mathscr{K})\log 2\pi,$$

where the sum is taken over all rational prime powers.

Also observe that by [TV02] Theorem H (and GRH Theorem G) $\mathrm{BS}(\mathscr{K})$ is always finite. The term on the right side in Theorem 2.1 is closely related with the value at 1 of the zeta function of $\mathscr{K}$ (which is defined in [TV02]). For an infinite number field $\mathscr{K}$, satisfying one of the conditions from the theorem, we define:

$$\lambda(\mathscr{K}) := \mathrm{BS}(\mathscr{K}) - 1 + \phi_{\mathbb{R}}(\mathscr{K})\log 2 + \phi_{\mathbb{C}}(\mathscr{K})\log 2\pi = \sum_q \phi_q(\mathscr{K})\log\frac{q}{q-1}. \tag{2.1}$$

Moreover, assume we are given an infinite Galois extension $\mathscr{K}/K$ of number fields, with K finite over $\mathbb{Q}$. Let G be the corresponding profinite Galois group. If $G \supseteq U \rhd H$ are two subgroups such that U is open in G and H is normal, closed and not open in U, let $\mathscr{L} := \mathscr{K}^H, L := \mathscr{K}^U$ be the corresponding fixed fields. Then we will write

$$\lambda(U;H) := \lambda(\mathscr{L}) \qquad \text{and} \qquad \phi_\alpha(U;H) := \phi_\alpha(\mathscr{L}).$$

Then $\lambda(U;H), \phi_\alpha(U;H)$ depend only on H, not on U.

2.3. **Some computations.** The following two easy Lemmas below are closely related to [HM01] Lemma 5, Definition 6 and the invariant μ, which we introduce below is related to the root discriminant $\mathrm{rd}_K := |D_{K/\mathbb{Q}}|^{\frac{1}{[K:\mathbb{Q}]}}$.

Lemma 2.2. *Let L/K be a finite Galois extension, which is unramified outside a set $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ of primes of K, and let $\mathfrak{p}_i\mathcal{O}_L = \prod_{j=1}^{g_i} \mathfrak{P}_{ij}^{e_i}$, such that $[L:K] = e_i f_i g_i$, where f_i is the inertia degree of $\mathfrak{P}_{ij}/\mathfrak{p}_i$ for some (any) j. Then*

$$N_{K/\mathbb{Q}} D_{L/K} = \prod_{i=1}^r \mathrm{N}\mathfrak{p}_i^{[L:K](1-\frac{1}{e_i}+\frac{\beta_i}{e_i})},$$

with some $\beta_i \geqslant 0$ and $\beta_i = 0 \Leftrightarrow L/K$ is tamely ramified in $\mathfrak{p}_i$. Moreover, one has

$$\frac{g_L}{[L:K]} = g_K + \sum_{i=1}^r (1 - \frac{1}{e_i} + \frac{\beta_i}{e_i})\mathrm{N}\mathfrak{p}_i$$

Proof. Let $\delta_{L/K}$ be the different of L/K. Then the $\mathfrak{P}_{ij}$-valuation of $\delta_{L/K}$ is $e_i - 1 + \sum_{u=1}^{\infty}(\sharp D_{\mathfrak{P}_{ij}/\mathfrak{p}_i,u} - 1)$, where $D_{\mathfrak{P}_{ij}/\mathfrak{p}_i,u}$ denote the higher ramification subgroups in the lower numbering (cf. e.g. [Se79] Chap. IV Proposition 4). Let $\beta_i := \sum_{u=1}^{\infty}(\sharp D_{\mathfrak{P}_{ij}/\mathfrak{p}_i,u} - 1)$. As $D_{L/K} = \mathrm{N_{L/K}}\delta_{\mathrm{L/K}}$, we get

$$\mathrm{N_{K/\mathbb{Q}}D_{L/K}} = \mathrm{N_{K/\mathbb{Q}}N_{L/K}}\delta_{\mathrm{L/K}} = \mathrm{N_{L/\mathbb{Q}}}\delta_{\mathrm{L/K}} = \prod_{\mathrm{i=1}}^{\mathrm{r}} \mathrm{N}\mathfrak{p}_{\mathrm{i}}^{\mathrm{g_i f_i (e_i - 1 + \beta_i)}} = \prod_{\mathrm{i=1}}^{\mathrm{r}} \mathrm{N}\mathfrak{p}_{\mathrm{i}}^{\mathrm{[L:K](1 - \frac{1}{e_i} + \frac{\beta_i}{e_i})}}.$$

It is clear that $\beta_i = 0 \Leftrightarrow L/K$ is tamely ramified in $\mathfrak{p}_i$. Now taking the logarithm of the formula $D_{L/\mathbb{Q}} = D_{K/\mathbb{Q}}^{[L:K]}\mathrm{N_{K/\mathbb{Q}}D_{L/K}}$ gives the last statement of the lemma. □

Lemma 2.3. *Let K be a number field and $\mathscr{K}/K$ an infinite Galois extension. Let $(K_n)_{n=0}^{\infty}$ be a tower corresponding to $\mathscr{K}/K$, consisting of Galois subextensions with $K = K_0$. Then*

$$\mu(\mathscr{K}/K) := \lim_{n\to\infty} \frac{[K_n : K]}{g_{K_n}} \geqslant 0$$

exists, is finite and depends only on $\mathscr{K}/K$, not on the choice of the tower. If moreover, $\mathscr{K}/K$ is unramified outside a finite set S of primes of K and only tamely ramified in S, then $\mu(\mathscr{K}/K) > 0$.

Proof. This follows from the last statement in Lemma 2.2. □

If $\mathscr{K}/\mathscr{L}/L/K$ and $H \lhd U \subseteq G$ are as at the end of Section 2.2, then we also write $\mu(U;H) := \mu(\mathscr{L}/L)$. Note that if $H \subseteq V \subseteq U$ with last inclusion open, then we have

$$\mu(U;H) = [U:V]\mu(V;H).$$

As we can only read off $N_{K/\mathbb{Q}}D_{L/K}$ from $\mathrm{G}_{K,S}$, but not $D_{K/\mathbb{Q}}$, we introduce the following variant of our invariants. Let K be a fixed base field. For L/K finite, let

$$g_{L/K} := \log|\mathrm{N_{K/\mathbb{Q}}D_{L/K}}|^{\frac{1}{2}}.$$

Then $g_L = [L:K]g_K + g_{L/K}$. For $\mathscr{K}/\mathscr{L}/L/K, U, H$ as above choose a tower $\mathscr{L} \supsetneq \cdots \supsetneq L_2 \supsetneq L_1 \supsetneq L_0 = L$ and assume that $g_{L_n/K} > 0$ for $n \gg 0$ (equivalently, $\mathscr{L}/K$ is ramified). We define (α varies through rational prime powers, $\mathbb{R}$ and $\mathbb{C}$)

$$\begin{aligned}
\phi_{\alpha,\mathrm{rel}}(U;H) := \phi_{\alpha,\mathrm{rel}}(\mathscr{L}/L) &:= \lim_n \frac{\sharp\Phi_\alpha(L_n)}{g_{L_n/K}}\\
\mu_{\mathrm{rel}}(U;H) := \mu_{\mathrm{rel}}(\mathscr{L}/L) &:= \lim_n \frac{[L_n:L]}{g_{L_n/K}}\\
\lambda_{\mathrm{rel}}(U;H) := \lambda_{\mathrm{rel}}(\mathscr{L}/L) &:= \lim_n \frac{\log(h_{L_n}\mathrm{Reg}_{L_n})}{g_{L_n/K}} - 1 + \phi_{\mathbb{R},\mathrm{rel}}(U;H)\log 2 + \phi_{\mathbb{C},\mathrm{rel}}(U;H)\log 2\pi.
\end{aligned}$$

Lemma 2.4. *With the above notations assume $g_{L_n/K} > 0$ for $n \gg 0$. We have*

(i) *$\mu_{\mathrm{rel}}(U;H), \phi_{q,\mathrm{rel}}(U;H)$ and $\lambda_{\mathrm{rel}}(U;H)$ exist and are independent of the choice of the subtower (L_n).*

(ii) *One has $\mu_{\mathrm{rel}}(U;H) < \infty$ and $\mu(U;H) = \mu_{\mathrm{rel}}(U;H)(1 + [L:K]g_K\mu_{\mathrm{rel}}(U;H))^{-1}$. In particular, $\mu(U;H) = 0 \Leftrightarrow \mu_{\mathrm{rel}}(U;H) = 0$.*

(ii) *Assume $\mu(U;H), \mu(U;1)$ are non-zero. Then the following holds:*

$$\frac{\lambda(U;H)}{\mu(U;H)} - \frac{\lambda(U;1)}{\mu(U;1)} = \frac{\lambda_{\mathrm{rel}}(U;H)}{\mu_{\mathrm{rel}}(U;H)} - \frac{\lambda_{\mathrm{rel}}(U;1)}{\mu_{\mathrm{rel}}(U;1)}.$$

Proof. To simplify notation in the proof we omit U, H and write μ_{rel} for $\mu_{\mathrm{rel}}(U;H)$, μ for $\mu(U;H)$, etc. We have

$$\lim_n \frac{g_{L_n}}{[L_n:L]} = \lim_n \frac{g_{L_n/K}}{[L_n:L]} + g_K[L:K] = \lim_n \frac{g_{L_n/L}}{[L_n:L]} + g_{L/K} + [L:K]g_K.$$

As the limit involving g_{L_n}'s exists and is independent of the choice of the tower, the same is true for both other limits. The limit involving $g_{L_n/K}$'s is positive (either $g_{L/K} > 0$, or by assumption, $g_{L_n/L} > 0$ for $n \gg 0$ and then by Lemma 2.3, $\lim_n \frac{g_{L_n/L}}{[L_n:L]} > 0$). Hence μ_{rel} exists, is finite, independent of the choice of the tower, $\mu = 0 \Leftrightarrow \mu_{\mathrm{rel}} = 0$ and if both are unequal zero, then $\mu^{-1} = \mu_{\mathrm{rel}}^{-1} + [L:K]g_K$. From this part (ii) of the lemma follows. Further, we compute:

$$\phi_q(1+[L:K]g_K\mu_{\mathrm{rel}}) = \lim_n \frac{\sharp\Phi_q(L_n)}{g_{L_n}}(1+[L:K]g_K \lim_n \frac{[L_n:L]}{g_{L_n/K}}) = \lim_n \frac{\sharp\Phi_q(L_n)}{g_{L_n/K}} = \phi_{q,\mathrm{rel}}$$

As the limits in the second term exist and are independent of the choice of the tower, the same holds true for the limit defining $\phi_{q,\mathrm{rel}}$. An analogous computation shows that λ_{rel} exists and is independent of the choice of the tower. This shows part (i) of the lemma. Then one easily computes:

$$\lambda = (\lambda_{\mathrm{rel}}+1)(1+[L:K]g_K\mu_{\mathrm{rel}})^{-1} - 1.$$

Putting this and the above in equation (2.1) (which is the Tsfasman-Vlăduţ theorem) we obtain

$$\lambda_{\mathrm{rel}} - [L:K]g_K\mu_{\mathrm{rel}} = \sum_q \phi_{q,\mathrm{rel}} \log\frac{q}{q-1}. \tag{2.2}$$

Further, the above implies that if $\mu \neq 0$, then

$$\frac{\phi_q}{\mu} = \frac{\phi_{q,\mathrm{rel}}}{\mu_{\mathrm{rel}}}.$$

This and equation (2.2) imply part (iii) of the lemma. □

2.4. **The p-adic volume of the unit lattice.** We do not use the results of this section in the rest of this paper. Let p be a (say, odd) prime, K a number field, $S \supseteq S_\infty \cup S_p$ a finite set of primes of K. In contrast to the usual regulator, it is possible to reconstruct the p-adic volume $\mathrm{vol}_p(K)$ of the unit lattice of K from the fundamental group $\mathrm{G}_{K,S}$ (+ some more information). As we do not use $\mathrm{vol}_p(K)$ outside this section, we omit its definition here and refer to [NSW08] 10.3.3. We use same notations as in Section 2.1.

Proposition 2.5. *Let K be a number field, $S \supseteq S_\infty$ a finite set of primes of K, such that at least two rational primes lie in $\mathcal{O}^*_{K,S}$ and let p be one of them. Assume $p > 2$ or K totally imaginary. Assume the Leopoldt conjecture holds for K and p. Then*

$$(\mathrm{G}_{K,S}, p, \chi_p) \rightsquigarrow \mathrm{vol}_p(K).$$

Proof. By [Iv14] Theorem 1.1, the given information is enough to reconstruct the position of decomposition subgroups of primes in S_f inside $\mathrm{G}_{K,S}$ and hence also in its pro-p-quotient $\mathrm{G}^{(p)}_{K,S}$. For a prime $\bar{\mathfrak{p}} \in (S_f \smallsetminus S_p)(K^{(p)}_S)$, we have a map with open image $D_{\bar{\mathfrak{p}},K^{(p)}_S/K} \to \mathbb{Z}_p$, which is induced by the cyclotomic character $\chi'_p\colon \mathrm{G}^{(p)}_{K,S} \to \mathbb{Z}_p$. The inertia subgroups $I_{\bar{\mathfrak{p}},K^{(p)}_S/K} \subseteq D_{\bar{\mathfrak{p}},K^{(p)}_S/K}$ are the kernels of this map. Therefore we can reconstruct the quotient $\mathrm{G}^{(p)}_{K,S_p}$ of $\mathrm{G}^{(p)}_{K,S}$ together with the decomposition subgroups at S_p by dividing out the normal subgroup generated by $I_{\bar{\mathfrak{p}},K^{(p)}_S/K}$ for $\bar{\mathfrak{p}} \in S_f \smallsetminus S_p$. Hence we can also reconstruct the following exact sequence from class field theory (cf. [NSW08] 8.3.21)

$$0 \to \overline{\mathcal{O}^*_{K,S_p}} \to \prod_{\mathfrak{p}\in S_p} K^{*,(p)}_{\mathfrak{p}} \to \mathrm{G}^{\mathrm{ab},(p)}_{K,S_p} \to \mathrm{Cl_{S_p}(K)^{(p)}} \to 0$$

where the upper index (p) denotes the pro-p completion and $\overline{\mathcal{O}^*_{K,S_p}}$ denotes the closure of the image of $\mathcal{O}^*_{K,S_p}$ in $\prod_{\mathfrak{p}\in S_p} K^{*,(p)}_{\mathfrak{p}}$. Note that the decomposition groups $D_{\bar{\mathfrak{p}},K_S/K}$ are the full local groups by [CC09] and hence by class fields theory $K^{*,(p)}_{\mathfrak{p}} \cong \mathrm{G}^{\mathrm{ab},(p)}_{K_{\mathfrak{p}}} \cong D^{\mathrm{ab},(p)}_{\bar{\mathfrak{p}},K_S/K}$. Let $U_{\mathfrak{p}} \subset K^*_{\mathfrak{p}}$ be the units of $\mathcal{O}_{K_{\mathfrak{p}}}$. The assumed Leopoldt conjecture and [NSW08] 10.3.13 with $S = S_p \cup S_\infty$, $T = \varnothing$ shows the exactness of the rows of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\mathcal{O}^*_K} & \longrightarrow & \prod_{\mathfrak{p}\in S_p} U^{(p)}_{\mathfrak{p}} & \longrightarrow & \mathrm{G}^{\mathrm{ab},(p)}_{K,S_p} & \longrightarrow & \mathrm{Cl}^{(\mathrm{p})}_{\mathrm{K}} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{O}^*_{K,S_p}} & \hookrightarrow & \prod_{\mathfrak{p}\in S_p} K^{*,(p)}_{\mathfrak{p}} & \longrightarrow & \mathrm{G}^{\mathrm{ab},(p)}_{K,S_p} & \longrightarrow & \mathrm{Cl}^{(\mathrm{p})}_{\mathrm{K,S_p}} & \longrightarrow & 0
\end{array}$$

where $\overline{\mathcal{O}_K^*}$ is the closure of the image of $\mathcal{O}_K^*$ inside $\prod_{\mathfrak{p}\in S_p} U_{\mathfrak{p}}^{(p)}$. We can reconstruct the second vertical map from the given data ($U_{\mathfrak{p}}^{(p)}$ correspond to the inertia subgroup via the reciprocity isomorphism $K_{\mathfrak{p}}^{*,(p)} \to \mathrm{G}_{K_{\mathfrak{p}}}^{\mathrm{ab},(p)}$). An easy diagram chase shows $\overline{\mathcal{O}_K^*} = \overline{\mathcal{O}_{K,S_p}^*} \cap \prod_{\mathfrak{p}\in S_p} U_{\mathfrak{p}}^{(p)}$. Hence we can reconstruct the upper left horizontal map in the above diagram. Apply $-\otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ to it; now the proposition follows from [NSW08] 10.3.8 as $r_1(K), r_2(K)$ can be reconstructed from the given data and as the natural map $\mathcal{O}_K^* \otimes_{\mathbb{Z}} \mathbb{Z}_p \twoheadrightarrow \overline{\mathcal{O}_K^*}$ is an isomorphism by Leopoldt's conjecture. □

3. Small norm criterions

Throughout this section we fix a number field K with a set of primes $S \supseteq S_\infty$ and let $\bullet \in \{\mathrm{tm}, \mathrm{tm}+(\ell)\}$. Let R be a further set of primes of K. Recall that prime always means non-archimedean prime.

3.1. Normal case.

Definition 3.1. Let $U \subseteq \mathrm{G}_{K,S}^\bullet$ be an open subgroup and $H \lhd U$ a closed normal subgroup, which is not open. Let $L = (K_S^\bullet)^U$ and $\mathfrak{p}$ a non-archimedean prime of L. We say that $\mathfrak{p}$ **lies under** H, if for some (any) extension $\bar{\mathfrak{p}}$ of $\mathfrak{p}$ to $K_S^\bullet$, the inclusion $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ is open (equivalently, $[\mathfrak{p}_H : \mathfrak{p}] < \infty$, where $\mathfrak{p}_H := \bar{\mathfrak{p}}|_{(K_S^\bullet)^H}$). We also define:

$$z_R(U;H) := \text{the number of primes in } L \text{ lying under } H \text{ and outside } R.$$

Note that the definition is independent on the choice of $\bar{\mathfrak{p}}$ as H is normal in U.

Proposition 3.2 (Criterion 1)**.** *Let $H \lhd U$, L be as in Definition 3.1. Assume that either $L/\mathbb{Q}$ is almost normal, or that $(GRH)^\bullet$ holds. Assume that $(FD)^\bullet, (FKS)^\bullet$ hold for (K,S) with respect to the set R. We have:*

(i) $z_R(U;H) = 0 \Leftrightarrow \frac{\lambda(U;H)}{\mu(U;H)} - \frac{\lambda(U;1)}{\mu(U;1)} = 0.$

(ii) $z_R(U;H) > 1 \Leftrightarrow \exists H_1, H_2 \subseteq H$ *which are closed and normal in U such that $z_R(U;H_i) > 0$ for $i = 1,2$ and $z_R(U;H_1 \cap H_2) = 0$.*

Proof. (i): By our assumptions we can apply Tsfasman-Vlăduţ to any infinite subfield of $K_S^\bullet/L$ which is (almost) normal over L. Let $(L_n)_n$ be some tower corresponding to $K_S^\bullet/L$ with $L_0 = L$ (consisting of almost normal subextensions). By $(FD)^\bullet$, for each q there is an n_0, such that all primes lying outside $R(L_{n_0})$ have norm $> q$. For a set T of primes of K, let $T^{\mathrm{N=q}}$ denote the subset of primes having the norm q. Then for each $n \geqslant n_0$ we have $\Phi_q(L_n) = R(L_n)^{\mathrm{N=q}}$ and as R is completely split in $K_S^\bullet/L$, we get:

$$\phi_q(U;1) = \lim_n \frac{\sharp R(L_n)^{\mathrm{N=q}}}{g_{L_n}} = \lim_n \frac{\sharp R(L)^{\mathrm{N=q}} \cdot [L_n : L]}{g_{L_n}} = \sharp R(L)^{\mathrm{N=q}} \cdot \mu(U;1) \tag{3.1}$$

and hence

$$\lambda(U;1) = \sum_q \phi_q(U;1) \log \frac{q}{q-1} = \mu(U;1) \sum_q \sharp R(L)^{\mathrm{N=q}} \cdot \log \frac{q}{q-1}.$$

Similarly, the primes in $R(\cdot)^{\mathrm{N=q}}$ contribute to $\phi_q(U;H)$. Hence we have

$$\phi_q(U;H) \geqslant \sharp R(L)^{\mathrm{N=q}} \cdot \mu(U;H).$$

As $\lambda(U;H) = \sum_q \phi_q(U;H) \log \frac{q}{q-1}$ and $\phi_q(U;H) \geqslant 0$, $\log \frac{q}{q-1} > 0$ for each q we get

$$\frac{\lambda(U;H)}{\mu(U;H)} - \frac{\lambda(U;1)}{\mu(U;1)} = 0 \Leftrightarrow \forall q \colon \phi_q(U;H) = \sharp R(L)^{\mathrm{N=q}} \cdot \mu(U;H). \tag{3.2}$$

Thus we have to show the equivalence of the right hand side of (3.2) with $z_R(U;H) = 0$. Assume first $z_R(U;H) = 0$, i.e., for any prime $\mathfrak{p}$ of L outside R and any extension $\bar{\mathfrak{p}}$ to $K_S^\bullet$, the intersection $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ is not open. Let $\mathfrak{p} \notin R$ be a prime of L and let $\mathfrak{p}_H := \bar{\mathfrak{p}}|_{(K_S^\bullet)^H}$. We claim that then the inertia degree of $\mathfrak{p}_H/\mathfrak{p}$ is infinite. The claim is obvious for primes $\mathfrak{p} \notin S_{\min}$, as $K_S^\bullet/L$ is then unramified at $\mathfrak{p}$ and hence $\sharp(D_{\bar{\mathfrak{p}}/\mathfrak{p}}/D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H) = \sharp D_{\mathfrak{p}_H/\mathfrak{p}}$ is the inertia degree. For a prime $\mathfrak{p} \in S_{\min}$, we have by our assumption $(FD)^\bullet$ in the case $\bullet = \mathrm{tm}$: $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cong \hat{\mathbb{Z}} \ltimes \hat{\mathbb{Z}}^{(p')}$, where p is the residue characteristic of $\mathfrak{p}$ (here and later (p') means the prime-to-p part) and in the case $\bullet = \mathrm{tm} + (\ell)$, we have $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cong \mathbb{Z}_\ell \ltimes \mathbb{Z}_\ell$, in both cases the semi-direct product defined by sending 1 to multiplication by the norm of $\mathfrak{p}$. By Lemma 3.3 below resp. [Iv14] Lemma 2.2 and the normality of H, it follows from $(D_{\bar{\mathfrak{p}}/\mathfrak{p}} : D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H) = \infty$ that also $(D^{\mathrm{nr}}_{\bar{\mathfrak{p}}/\mathfrak{p}} : \overline{H}) = \infty$, where $D^{\mathrm{nr}}_{\bar{\mathfrak{p}}/\mathfrak{p}}$ is the unramified quotient of $D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ and $\overline{H}$ the image of $D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ in it. This shows our claim.

Fix now a rational prime power $q = p^r$. By our claim we can choose $H \subset V \subseteq U$ open such that each prime $\mathfrak{p} \notin R$ of L with residue characteristic p has norm $\mathrm{N}\mathfrak{p} > \mathrm{q}$. Replacing V by $\bigcap_{g \in U} gVg^{-1}$ we can assume it is normal in U. Let M/L be the extension corresponding to $V \subseteq U$. As in (3.1) we get:

$$\begin{aligned}\phi_q(U;H) &= \phi_q(V;H) = \sharp R(M)^{\mathrm{N}=\mathrm{q}} \cdot \mu(V;H) = \sharp R(L)^{\mathrm{N}=\mathrm{q}} \cdot [U:V]\mu(V;H) \\ &= \sharp R(L)^{\mathrm{N}=\mathrm{q}} \cdot \mu(U;H).\end{aligned}$$

Conversely, assume there is a prime $\mathfrak{p} \notin R$ in L lying under H. Let $\bar{\mathfrak{p}}$ be an extension of $\mathfrak{p}$ to $K_S^\bullet$. Then $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ is open. Let $H \lhd V \lhd U$ be open and normal in U, such that $V \cap D_{\bar{\mathfrak{p}}/\mathfrak{p}} = H \cap D_{\bar{\mathfrak{p}}/\mathfrak{p}}$. Such V exists: indeed, the image $\overline{D_{\bar{\mathfrak{p}}}}$ of $D_{\bar{\mathfrak{p}}}$ under $\pi\colon U \twoheadrightarrow U/H$ is finite, so we can choose some open and normal $\overline{V} \subseteq U/H$ such that $\overline{V} \cap \overline{D_{\bar{\mathfrak{p}}}} = \{1\}$. Then $V := \pi^{-1}(\overline{V}) \subseteq U$ is open and normal with $V \cap D_{\bar{\mathfrak{p}}} = H \cap D_{\bar{\mathfrak{p}}}$. Let $M := (K_S^\bullet)^V$. Denote by $\mathfrak{p}_M$ the restriction of $\bar{\mathfrak{p}}$ to M and by $\mathfrak{p}_H$ the restriction of $\bar{\mathfrak{p}}$ to $(K_S^\bullet)^H$. As $D_{\mathfrak{p}_H/\mathfrak{p}_M} = D_{\bar{\mathfrak{p}}/\mathfrak{p}_M}/D_{\bar{\mathfrak{p}}/\mathfrak{p}_H} = D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap V / D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H = 1$ and as H normal in V, the prime $\mathfrak{p}_M$ is completely decomposed in $(K_S^\bullet)^H/M$. Let now $q := \mathrm{N}\mathfrak{p}_{\mathrm{M}}$. As in (3.1) we get:

$$\phi_q(U;H) = \phi_q(V;H) \geqslant (1 + \sharp R(M)^{\mathrm{N}=\mathrm{q}})\mu(V;H) = \mu(V;H) + \sharp R(L)^{\mathrm{N}=\mathrm{q}}\mu(U;H).$$

By Lemma 2.3, $\mu(V;H) > 0$. This finishes the proof of (i).

(ii): Assume first $z_R(U;H) > 1$. Then there are two different primes $\mathfrak{p}_1, \mathfrak{p}_2 \notin R$ in L, both lying under H. For $i = 1,2$ let $\bar{\mathfrak{p}}_i$ be an extension of $\mathfrak{p}_i$ to $K_S^\bullet$ and let $H_i := \langle\langle D_{\bar{\mathfrak{p}}_i/\mathfrak{p}_i} \cap H \rangle\rangle_U$ be the closed normal subgroup of U generated by $D_{\bar{\mathfrak{p}}_i/\mathfrak{p}_i} \cap H$. Then obviously, $\mathfrak{p}_i$ lies under H_i, hence $z_R(U;H_i) > 0$. Further, for any prime $\mathfrak{p} \neq \mathfrak{p}_1$ of L outside R with some extension $\bar{\mathfrak{p}}$ to $K_S^\bullet$, we have:

$$D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H_i \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap \langle\langle D_{\bar{\mathfrak{p}}/\mathfrak{p}_i} \rangle\rangle_U \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}},$$

and the second inclusion is not open by $(FKS)^\bullet$. Hence (doing the same for $\mathfrak{p}_2$ instead of $\mathfrak{p}_1$) no prime outside R lies under $H_1 \cap H_2$.

Conversely, assume there are two closed subgroups $H_1, H_2 \subseteq H$, normal in U, such that $z_R(U;H_i) > 0$ for $i = 1,2$ and $z_R(U;H_1 \cap H_2) = 0$. Let $\mathfrak{p}_i \notin R$ be a prime of L lying under H_i. Then clearly, $\mathfrak{p}_i$ also lie under H. If we would have $\mathfrak{p}_1 = \mathfrak{p}_2 =: \mathfrak{p}$, then $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H_i \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ would be an open inclusion for $i = 1,2$. But then also $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap H_1 \cap H_2 \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ would be open, i.e. $\mathfrak{p}$ would lie under $H_1 \cap H_2$, which contradicts $z_R(U;H_1 \cap H_2) = 0$. Thus $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and hence $z_R(U;H_1 \cap H_2) > 1$.

□

Lemma 3.3. *Let $D = \hat{\mathbb{Z}} \ltimes \hat{\mathbb{Z}}^{(p')}$, the operation of the first factor on the second given by sending* 1 *to multiplication by some power of the prime p. Let $N \lhd D$ be a closed normal subgroup. If the composition $N \hookrightarrow D \twoheadrightarrow \hat{\mathbb{Z}}$, where the last map is the projection on the first factor, has open image, then N is open in D.*

Proof. Replacing D by the preimage of the image of N under $D \twoheadrightarrow \hat{\mathbb{Z}}$, we can assume that the composition $N \to \hat{\mathbb{Z}}$ is surjective. Find generators of D as a profinite group such that $D = \langle \sigma, \tau \colon \sigma\tau\sigma^{-1} = \tau^q \rangle$ and $D \twoheadrightarrow \hat{\mathbb{Z}}$ is given by $\sigma \mapsto 1$, $\tau \mapsto 0$. There is an $a \in \hat{\mathbb{Z}}^{(p')}$ such that $\tau^a\sigma \in N$. Hence D/N is generated by the image $\bar{\tau}$ of τ in D/N, i.e., D/N is procyclic and hence abelian. Hence N contains the commutator of D, which is equal to $\langle \tau^{q-1} \rangle$. Now it is immediate to check that N is open in D. □

3.2. **General criterion.** We keep the notations from Section 3.1. If G is a profinite group and H a subgroup, then we denote by $\langle\langle H \rangle\rangle_G$ the smallest closed normal subgroup of G containing H.

Definition 3.4. Assume that $(FD)^\bullet, (FKS)^\bullet$ hold for $K_S^\bullet/K$ with respect to the set R. Let $Z \subseteq \mathrm{G}_{K,S}^\bullet$ be a closed subgroup. We say that Z has **decomposition behavior** if the following conditions are satisfied for all open subgroups $U \subseteq \mathrm{G}_{K,S}^\bullet$ containing Z:

(1) $\langle\langle Z\rangle\rangle_U \subseteq U$ is not open
(2) $z_R(U; \langle\langle Z\rangle\rangle_U) = z_{R\cup S_{\min}}(U; \langle\langle Z\rangle\rangle_U) = 1$
(3) there is a constant $C > 0$ and an open subgroup $Z \subset U_0 \subseteq \mathrm{G}_{K,S}$, such that if $Z \subset U \subseteq U_0$, then
$$\frac{\lambda(U; \langle\langle Z\rangle\rangle_U)}{\mu(U; \langle\langle Z\rangle\rangle_U)} - \frac{\lambda(U;1)}{\mu(U;1)} = C.$$

Theorem 3.5 (Criterion 2)**.** *Assume that $(FD)^\bullet, (FKS)^\bullet$ hold for $K_S^\bullet/K$ with respect to a set R. Assume either $\bullet = \mathrm{tm} + (\ell)$ and $K/\mathbb{Q}$ is almost normal or $(GRH)^\bullet$ holds. Let $\langle x\rangle \subseteq \mathrm{G}_{K,S}^\bullet$ be a (closed) procyclic subgroup. Then $\langle x\rangle$ has decomposition behavior if and only if there is a prime $\bar{\mathfrak{p}}$ of $K_S^\bullet$ with restriction $\mathfrak{p} \notin R$ to K, such that $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap \langle x\rangle \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ is open. In this case $\langle x\rangle \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$. Moreover, $\bar{\mathfrak{p}}$ is unique and lies outside $R \cup S_{\min}$.*

The proof will show that the theorem remains true if we replace 'procyclic' by 'abelian' The next corollary follows from the theorem and Lemma 3.7.

Corollary 3.6. *With the assumptions as in Theorem 3.5, the decomposition subgroups inside $\mathrm{G}_{K,S}^\bullet$ of primes lying outside $R \cup S_{\min}$ are exactly the maximal procyclic subgroups of $\mathrm{G}_{K,S}^\bullet$ of decomposition behavior.*

Proof of Theorem 3.5. Uniqueness of $\bar{\mathfrak{p}}$ follows from $(FKS)^\bullet$. Observe that as a pro-ℓ-subgroup is nilpotent, any open subgroup of it is almost normal. Using either this or the $(GRH)^\bullet$-assumption, we can apply Theorem 2.1 to any subtower of $K_S^\bullet/K$.

Assume first there is a prime $\bar{\mathfrak{p}}$ of $K_S^\bullet$ with restriction $\mathfrak{p} \notin R$ to K such that $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap \langle x\rangle \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ is open. Then $\mathfrak{p} \notin S_{\min}$, otherwise $D_{\bar{\mathfrak{p}}/\mathfrak{p}}$ would contain an open subgroup generated by one element, which is impossible by $(FD)^\bullet$. As $\langle x\rangle$ is abelian, Lemma 3.7 below implies $\langle x\rangle \subseteq D_{\bar{\mathfrak{p}}/\mathfrak{p}}$.

For any $\langle x\rangle \subseteq U \subseteq \mathrm{G}_{K,S}^\bullet$ open with fixed field L, let $\mathfrak{p}_L := \bar{\mathfrak{p}}|_L$. We have: $\langle\langle x\rangle\rangle_U \subseteq \langle\langle D_{\bar{\mathfrak{p}}/\mathfrak{p}_L}\rangle\rangle_U \subseteq U$ and the last inclusion is not open by $(FKS)^\bullet$. This shows part (1) of Definition 3.4. Further, $\mathfrak{p}_L$ lies under $\langle\langle x\rangle\rangle_U$. If a further prime $\mathfrak{q} \neq \mathfrak{p}_L$ of L with $\mathfrak{q} \notin R$ would lie under $\langle\langle x\rangle\rangle_U$, then the composition $D_{\bar{\mathfrak{q}}/\mathfrak{q}} \cap \langle\langle x\rangle\rangle_U \subseteq D_{\bar{\mathfrak{q}}/\mathfrak{q}} \cap \langle\langle D_{\bar{\mathfrak{p}}/\mathfrak{p}_L}\rangle\rangle_U \subseteq D_{\bar{\mathfrak{q}}/\mathfrak{q}}$ would be open. Hence also the second inclusion would be open, and this contradicts $(FKS)^\bullet$. This shows (2).

To show (3), we compute the numbers $\lambda(U; \langle\langle x\rangle\rangle_U)$. Let $\langle x\rangle \subseteq U_0 \subseteq \mathrm{G}_{K,S}^\bullet$ be open with fixed field L_0, such that $U_0 \cap D_{\bar{\mathfrak{p}}/\mathfrak{p}} = \langle x\rangle$. Let $q := \mathrm{N}\mathfrak{p}_{\mathrm{L}_0}$. For any $\langle x\rangle \subseteq U \subseteq U_0$ open with fixed field L, we have $D_{\mathfrak{p}_L/\mathfrak{p}_{L_0}} = D_{\bar{\mathfrak{p}}/\mathfrak{p}_{L_0}}/D_{\bar{\mathfrak{p}}/\mathfrak{p}_L} = D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap U_0/D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap U = 1$, i.e. $\mathrm{N}\mathfrak{p}_{\mathrm{L}} = \mathrm{q}$ too. Consider now the tower $(K_S^\bullet)^{\langle\langle x\rangle\rangle_U}/L$ corresponding to the inclusion $\langle\langle x\rangle\rangle_U \subseteq U$. Similar as in the proof of Proposition 3.2, $(FKS)^\bullet$ and a computation show:

$$\frac{\lambda(U; \langle\langle x\rangle\rangle_U)}{\mu(U; \langle\langle x\rangle\rangle_U)} - \frac{\lambda(U;1)}{\mu(U;1)} = \log\frac{q}{q-1}, \tag{3.3}$$

which implies (3). This finishes the proof of one direction in the theorem.

To prove the other direction, let $\langle x\rangle \subseteq G_{K,S}^\bullet$ be of decomposition behavior. Let $\langle x\rangle \subseteq U \subseteq \mathrm{G}_{K,S}^\bullet$ be an open subgroup with fixed field L. By assumptions, there is a unique prime $\mathfrak{p}_L \notin R \cup S_{\min}$ of L lying under $\langle\langle x\rangle\rangle_U$. This uniqueness implies that if $\langle x\rangle \subseteq U' \subseteq U \subseteq \mathrm{G}_{K,S}^\bullet$ with fixed fields L', L, then $\mathfrak{p}_{L'}|_L = \mathfrak{p}_L$. Thus the sets $\{\mathfrak{p}_L\}$ with one element form a projective system, which limit is again a one element set, i.e., we obtain a unique prime $\bar{\mathfrak{p}}$ of $K_S^\bullet$ lying outside $R \cup S_{\min}$ with $\bar{\mathfrak{p}}|_L = \mathfrak{p}_L$ for each L.

Let now $\langle x\rangle \subseteq U \subseteq \mathrm{G}_{K,S}^\bullet$ be open. By condition (1) of Definition 3.4, $\langle\langle x\rangle\rangle_U \subset U$ is not open and we can compute $\lambda(U; \langle\langle x\rangle\rangle_U)$. Let therefore $\langle\langle x\rangle\rangle_U \subseteq V_0 \subseteq U$ be open and normal with fixed field M_0 such that $D_{\bar{\mathfrak{p}}/\mathfrak{p}_L} \cap \langle\langle x\rangle\rangle_U = D_{\bar{\mathfrak{p}}/\mathfrak{p}_L} \cap V_0$. Let $\mathfrak{p}_{M_0,1}, \dots, \mathfrak{p}_{M_0,r}$ be the primes in M_0 lying over $\mathfrak{p}_L$. Let further f_U be the inertia degree of $\mathfrak{p}_{M_0,i}$ over $\mathfrak{p}_L$. Then

$$\begin{aligned} f_U &= (D_{\bar{\mathfrak{p}}/\mathfrak{p}_L} : D_{\bar{\mathfrak{p}}/\mathfrak{p}_L} \cap \langle\langle x\rangle\rangle_U) \\ rf_U &= [M_0 : L] \end{aligned}$$

(in particular, f_U does not depend on the choice of M_0). Further, $\mathfrak{p}_{M_0,i}$ are completely decomposed in the tower $(K_S^\bullet)^{\langle\langle x\rangle\rangle_U}/M_0$. Set $q_{0,U} := \mathrm{N}\mathfrak{p}_\mathrm{L}$ and $q_U := q_{0,U}^{f_U} = \mathrm{N}\mathfrak{p}_{\mathrm{M_0,i}}$. Then we have:

$$\begin{aligned}\phi_{q_U}(U;\langle\langle x\rangle\rangle_U) &= \phi_{q_U}(V_0;\langle\langle x\rangle\rangle_U) = \lim_M \frac{\sharp R(M)^{\mathrm{N}=\mathrm{q_U}} + r[M:M_0]}{g_M} \\ &= \mu(U;\langle\langle x\rangle\rangle_U)\sharp R(L)^{\mathrm{N}=\mathrm{q_U}} + \frac{1}{f_U}\mu(U;\langle\langle x\rangle\rangle_U),\end{aligned}$$

where M runs through the tower $(K_S^\bullet)^{\langle\langle x\rangle\rangle_U}/M_0$. As by condition (2) of Definition 3.4 no further prime of L outside $R \cup S_{\min}$ lies under $\langle\langle x\rangle\rangle_U$, we get:

$$\frac{\lambda(U;\langle\langle x\rangle\rangle_U)}{\mu(U;\langle\langle x\rangle\rangle_U)} - \frac{\lambda(U;1)}{\mu(U;1)} = \frac{1}{f_U}\log\frac{q_U}{q_U-1} = \log q_{0,U} - \frac{1}{f_U}\log(q_{0,U}^{f_U}-1). \tag{3.4}$$

Now let $\langle x\rangle \subset U_0 \subseteq \mathrm{G}_{K,S}^\bullet$ be open as in condition (3) of Definition 3.4 and let $\langle x\rangle \subset U' \subseteq U \subseteq U_0$ be two open subgroups with fixed fields L', L. We compare the expressions (3.4) for U and U'. Let $c := [\mathfrak{p}_{L'} : \mathfrak{p}_L] = (D_{\bar{\mathfrak{p}}/\mathfrak{p}_L} : D_{\bar{\mathfrak{p}}/\mathfrak{p}_{L'}})$. Then $q_{0,U'} = q_{0,U}^c$ and hence

$$\frac{\lambda(U';\langle\langle x\rangle\rangle_{U'})}{\mu(U';\langle\langle x\rangle\rangle_{U'})} - \frac{\lambda(U';1)}{\mu(U';1)} = \log q_{0,U'} - \frac{1}{f_{U'}}\log(q_{0,U'}^{f_{U'}}-1) = c\log q_{0,U} - \frac{1}{f_{U'}}\log(q_{0,U}^{cf_{U'}}-1)$$

By condition (3), these quantities for U and U' must coincide, i.e., we get (for simplicity let $t := q_{0,U}$):

$$(c-1)\log t = \log\left(\frac{(t^{cf'_U}-1)^{1/f_{U'}}}{(t^{f_U}-1)^{1/f_U}}\right).$$

Applying exp and taking $f_U f_{U'}$-th power we obtain

$$t^{(c-1)f_U f_{U'}}(t^{f_U}-1)^{f_{U'}} = (t^{cf_{U'}}-1)^{f_U}.$$

All numbers in this equation are positive integers and if $c \neq 1$, then the left side would be divisible by t (which is a prime power > 1), whereas the right side would not. This contradiction shows $c = 1$ and therefore, the intersection $D_{\bar{\mathfrak{p}}/\mathfrak{p}} \cap U$ is independent of the open subgroup $\langle x\rangle \subseteq U \subseteq U_0$. Hence

$$D_{\bar{\mathfrak{p}}/\mathfrak{p}_K} \cap \langle x\rangle = D_{\bar{\mathfrak{p}}/\mathfrak{p}_K} \cap \bigcap_{\langle x\rangle\subseteq U} U = D_{\bar{\mathfrak{p}}/\mathfrak{p}_K} \cap U_0$$

is open in $D_{\bar{\mathfrak{p}}/\mathfrak{p}_K}$. This finishes the proof. □

Lemma 3.7. *Assume $(FD)^\bullet, (FKS)^\bullet$ hold for $K_S^\bullet/K$ with respect to the set R. Let $\mathfrak{p} \notin R$ be a prime of K with extension $\bar{\mathfrak{p}}$ to $K_S^\bullet$. Let $H \subseteq D_{\bar{\mathfrak{p}}}$ be an open subgroup. Then $\mathrm{N}_{\mathrm{G}_{\mathrm{K,S}}^\bullet}(\mathrm{H}) \subseteq \mathrm{D}_{\bar{\mathfrak{p}}}$, where $\mathrm{N}_\mathrm{G}(\mathrm{H})$ denote the normalizer of H in G.*

Proof. Let $x \in \mathrm{N}_{\mathrm{G}_{\mathrm{K,S}}^\bullet}(\mathrm{H})$. Then

$$H = xHx^{-1} \subseteq xD_{\bar{\mathfrak{p}}}x^{-1} = D_{x\bar{\mathfrak{p}}},$$

i.e. $D_{x\bar{\mathfrak{p}}} \cap D_{\bar{\mathfrak{p}}} \subseteq D_{\bar{\mathfrak{p}}}$ is open. By $(FKS)^\bullet$, $x \in \bar{\mathfrak{p}}$. □

4. Anabelian geometry

Proof of Theorem 1.1. For each finite $K_{1,S_1}/L_1/K_1$, let L_2 be the corresponding field via σ. The assumptions imply by [Iv14] Theorem 1.1 that σ maps decomposition groups of primes in $S_1 \smallsetminus S_\infty$ to decomposition groups of primes in $S_2 \smallsetminus S_\infty$ (cf. Section 2.1). As in [Iv14] Corollary 1.4, for any $K_{1,S_1}/L_1/K_1$, we have an induced local correspondence at the boundary

$$\sigma_{*,L_1} : (S_1 \smallsetminus S_\infty)(L_1) \to (S_2 \smallsetminus S_\infty)(L_2),$$

which is an isomorphism preserving residue characteristics, absolute inertia degrees and ramification indices. As by assumption, the decomposition groups of primes in $S_{i,f}$ are the full local groups, σ also preserves the wild inertia subgroups and hence induces an isomorphism

$$\sigma^\bullet\colon\ \mathrm{G}_{K_1,S_1}^\bullet \to \mathrm{G}_{K_2,S_2}^\bullet.$$

Choose $K^\bullet_{1,S_1}/M_1/K_1$ finite such that M_1 is totally imaginary and such that $g_{M_1/K_1} > 0$. Then M_2 is also totally imaginary ([Iv14] Proposition 4.2) and $\mu_{\rm rel}(U;H) < \infty$ for any $H \lhd U \subseteq G^\bullet_{M_1,S_1}$ (here and later we take $\mu_{\rm rel}$ with respect to the base field K_i). Also Lemmas 2.3 and 2.4 show that $\mu_{\rm rel}(U;H) \neq 0 \neq \mu(U;H)$. Now [Iv14] Proposition 4.2 implies that the local correspondence at the boundary preserves the absolute inertia and ramification degrees, hence we have $g_{L_1/K_1} = g_{L_2/K_2}$ for each $K^\bullet_{1,S_1}/L_1/M_1$ and that $h_{L_1} = h_{L_2}$. This and the assumption in the theorem on the behavior of the regulators imply that for any $H_1 \subset U_1 \subseteq {\rm G}^\bullet_{M_1,S_1}$ with first inclusion closed and non-open and the second inclusion open, we have:

$$\begin{aligned}\lambda_{\rm rel}(U_1;H_1) &= \lambda_{\rm rel}(\sigma(U_1);\sigma(H_1))\\ \mu_{\rm rel}(U_1;H_1) &= \mu_{\rm rel}(\sigma(U_1);\sigma(H_1)),\end{aligned}$$

with $\mu_{\rm rel}, \lambda_{\rm rel}$ as in Section 2.3. Applying Lemma 2.4(iii) we obtain

$$\frac{\lambda(U_1;H_1)}{\mu(U_1;H_1)} - \frac{\lambda(U_1;1)}{\mu(U_1;1)} = \frac{\lambda(\sigma(U_1);\sigma(H_1))}{\mu(\sigma(U_1);\sigma(H_1))} - \frac{\lambda(\sigma(U_1);1)}{\mu(\sigma(U_1);1)}. \tag{4.1}$$

This and Proposition 3.2 imply that $\sigma^\bullet$ preserves decomposition behavior of subgroups (note that the knowledge of z_R implies the knowledge of $z_{R\cup S_{\min}}$ by the local correspondence at the boundary). Thus by Corollary 3.6, $\sigma^\bullet$ maps the decomposition subgroups of primes outside $R_1 \cup S_{1,\min}$ in ${\rm G}^\bullet_{K_1,S_1}$ to decomposition subgroups of primes outside $R_2 \cup S_{2,\min}$ in ${\rm G}^\bullet_{K_2,S_2}$. Let $\Sigma_{K^\bullet_{i,S_i}}$ denote the set of all primes of $K^\bullet_{i,S_i}$ (it can be seen as the inverse limit of the finite level sets). Therefore we obtain (using $(FKS)^\bullet$) a bijective map

$$\sigma_*\colon \Sigma_{K^\bullet_{1,S_1}} \smallsetminus (R_1 \cup S_{1,\min} \cup S_\infty) \to \Sigma_{K^\bullet_{2,S_2}} \smallsetminus (R_2 \cup S_{2,\min} \cup S_\infty), \tag{4.2}$$

determined by $\sigma^\bullet(D_{\bar{\mathfrak p}}) = D_{\sigma_*(\bar{\mathfrak p})}$, which is equivariant for the action of ${\rm G}^\bullet_{K_1,S_1}$ on these sets (the action comes from the conjugation action on subgroups and on the right side it is defined via σ). Thus we obtain the maps σ_{*,L_1} from the theorem by factoring out the action of open subgroups of ${\rm G}_{K_1,S_1}$. Now let $\mathfrak p \in \Sigma_{L_1} \smallsetminus (R_1 \cup S_{1,\min} \cup S_\infty)$, where L_1 correspond to the open subgroup $U_1 \subseteq {\rm G}_{K_1,S_1}$. Then equations (3.3) and (4.1) show

$$\log \frac{{\rm N}\mathfrak p}{{\rm N}\mathfrak p - 1} = \log \frac{{\rm N}\sigma(\mathfrak p)}{{\rm N}\sigma(\mathfrak p) - 1} =: A,$$

i.e., ${\rm N}\mathfrak p = \frac{{\rm e}^{\rm A}}{{\rm e}^{\rm A}-1} = {\rm N}\sigma_{*,{\rm L}_1}(\mathfrak p)$.

□

Now the same argument as in [Ne69] Theorem 2 shows Corollary 1.2. We repeat it for the convenience of the reader.

Proof of Corollary 1.2. Let $P^{\geqslant 1}(K_i/\mathbb Q)$ denote the set of primes of $\mathbb Q$ unramified in $K_i/\mathbb Q$ and having at least one factor of degree 1 in $K_i/\mathbb Q$. Let ${\rm cs}(K_i/\mathbb Q)$ be the set of primes of $\mathbb Q$ completely split in $K_i/\mathbb Q$. For sets of primes of a number field K, let $S \simeq T$ mean that S and T are equal up to a subset of Dirichlet density zero. Then as the local correspondence preserves absolute norms of primes we have:

$$P^{\geqslant 1}(K_2/\mathbb Q) \simeq P^{\geqslant 1}(K_1/\mathbb Q) = {\rm cs}(K_1/\mathbb Q) \simeq {\rm cs}(K_2/\mathbb Q).$$

Then Chebotarev density theorem implies that $K_2/\mathbb Q$ is also normal. As ${\rm cs}(K_1/\mathbb Q) \simeq {\rm cs}(K_2/\mathbb Q)$, a second application of Chebotarev implies $K_1 \cong K_2$. □

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