

The smooth locus in infinite-level Rapoport-Zink spaces

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Abstract

Rapoport-Zink spaces are deformation spaces for p -divisible groups with additional structure. At infinite level, they become preperfectoid spaces. Let \mathcal{M}_∞ be an infinite-level Rapoport-Zink space of EL type, and let \mathcal{M}_∞° be one geometrically connected component of it. We show that \mathcal{M}_∞° contains a dense open subset which is cohomologically smooth in the sense of Scholze. This is the locus of p -divisible groups which do not have any extra endomorphisms. As a corollary, we find that the cohomologically smooth locus in the infinite-level modular curve $X(p^\infty)^\circ$ is exactly the locus of elliptic curves E with supersingular reduction, such that the formal group of E has no extra endomorphisms.

1 Introduction

Let p be a prime number. Rapoport-Zink spaces [RZ96] are deformation spaces of p -divisible groups equipped with some extra structure. This article concerns the geometry of Rapoport-Zink spaces of EL type (endomorphisms + level structure). In particular we consider the infinite-level spaces $\mathcal{M}_{\mathcal{D},\infty}$, which are preperfectoid spaces [SW13]. An example is the space $\mathcal{M}_{H,\infty}$, where $H/\overline{\mathbb{F}}_p$ is a p -divisible group of height n . The points of $\mathcal{M}_{H,\infty}$ over a nonarchimedean field K containing $W(\overline{\mathbb{F}}_p)$ are in correspondence with isogeny classes of p -divisible groups G/\mathcal{O}_K equipped with a quasi-isogeny $G \otimes_{\mathcal{O}_K} \mathcal{O}_K/p \rightarrow H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_K/p$ and an isomorphism $\mathbf{Q}_p^n \cong VG$ (where VG is the rational Tate module).

The infinite-level space $\mathcal{M}_{\mathcal{D},\infty}$ appears as the limit of finite-level spaces, each of which is a smooth rigid-analytic space. We would like to investigate the question of smoothness for the space $\mathcal{M}_{\mathcal{D},\infty}$ itself, which is quite a different matter. We need the notion of cohomological smoothness [Sch17], which makes sense for general morphisms of analytic adic spaces, and which is reviewed in Section 3. Roughly speaking, an adic space is cohomologically smooth over C (where C/\mathbf{Q}_p is complete and algebraically closed) if it satisfies local Verdier duality. In particular, if U is a quasi-compact adic space which is cohomologically smooth over $\mathrm{Spa}(C, \mathcal{O}_C)$, then the cohomology group $H^i(U, \mathbb{F}_\ell)$ is finite for all i and all primes $\ell \neq p$.

Our main theorem shows that each geometrically connected component of $\mathcal{M}_{\mathcal{D},\infty}$ has a dense open subset which is cohomologically smooth.

Theorem 1.1. *Let \mathcal{D} be a basic EL datum (cf. Section 1.1). Let C be a complete algebraically closed extension of the field of scalars of $\mathcal{M}_{\mathcal{D},\infty}$, and let $\mathcal{M}_{\mathcal{D},\infty}^\circ$ be a connected component of $\mathcal{M}_{\mathcal{D},\infty,C}$. Let $\mathcal{M}_{\mathcal{D},\infty}^{\circ,\mathrm{non-sp}} \subset \mathcal{M}_{\mathcal{D},\infty}^\circ$ be the non-special locus (cf. Section 2.5), corresponding to p -divisible groups without extra endomorphisms. Then $\mathcal{M}_{\mathcal{D},\infty}^{\circ,\mathrm{non-sp}}$ is cohomologically smooth over C .*

We remark that outside of trivial cases, $\pi_0(\mathcal{M}_{\mathcal{D},\infty,C})$ has no isolated points, which implies that no open subset of $\mathcal{M}_{\mathcal{D},\infty,C}$ can be cohomologically smooth. (Indeed, the H^0 of any quasi-compact open fails to be finitely generated.) Therefore it really is necessary to work with individual geometrically connected components of $\mathcal{M}_{\mathcal{D},\infty}$.

The geometry of Rapoport-Zink spaces is related to the geometry of Shimura varieties. As an example, consider the tower of classical modular curves $X(p^\infty)$, considered as rigid spaces over C . There is a perfectoid space $X(p^\infty)$ over C for which $X(p^\infty) \sim \varprojlim_n X(p^n)$, and a Hodge-Tate period map $\pi_{HT}: X(p^\infty) \rightarrow \mathbb{P}_C^1$ [Sch15], which is $\mathrm{GL}_2(\mathbf{Q}_p)$ -equivariant. Let $X(p^\infty)^\circ \subset X(p^\infty)$ be a connected component.

Corollary 1.2. *The following are equivalent for a C -point x of $X(p^\infty)^\circ$.*

1. *The point x corresponds to an elliptic curve E , such that the p -divisible group $E[p^\infty]$ has $\mathrm{End} E[p^\infty] = \mathbf{Z}_p$.*
2. *The stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is trivial.*
3. *There is a neighborhood of x in $X(p^\infty)^\circ$ which is cohomologically smooth over C .*

1.1 Review of Rapoport-Zink spaces at infinite level

Let k be a perfect field of characteristic p , and let H be a p -divisible group of height n and dimension d over k . We review here the definition of the infinite-level Rapoport-Zink space associated with H .

First there is the formal scheme \mathcal{M}_H over $\mathrm{Spf} W(k)$ parametrizing deformations of H up to isogeny, as in [RZ96]. For a $W(k)$ -algebra R in which p is nilpotent, $\mathcal{M}_H(R)$ is the set of isomorphism classes of pairs (G, ρ) , where G/R is a p -divisible group and $\rho: H \otimes_k R/p \rightarrow G \otimes_R R/p$ is a quasi-isogeny.

The formal scheme \mathcal{M}_H locally admits a finitely generated ideal of definition. Therefore it makes sense to pass to its adic space $\mathcal{M}_H^{\mathrm{ad}}$, which has generic fiber $(\mathcal{M}_H^{\mathrm{ad}})_\eta$, a rigid-analytic space over $\mathrm{Spa}(W(k)[1/p], W(k))$. Then $(\mathcal{M}_H^{\mathrm{ad}})_\eta$ has the following moduli interpretation: it is the sheafification of the functor assigning to a complete affinoid $(W(k)[1/p], W(k))$ -algebra (R, R^+) the set of pairs (G, ρ) , where G is a p -divisible group defined over an open and bounded subring $R_0 \subset R^+$, and $\rho: H \otimes_k R_0/p \rightarrow G \otimes_{R_0} R_0/p$ is a quasi-isogeny. There is an action of $\mathrm{Aut} H$ on $\mathcal{M}_H^{\mathrm{ad}}$ obtained by composition with ρ .

Given such a pair (G, ρ) , Grothendieck-Messing theory produces a surjection $M(H) \otimes_{W(k)} R \rightarrow \mathrm{Lie} G[1/p]$ of locally free R -modules, where $M(H)$ is the covariant Dieudonné module. There is a Grothendieck-Messing period map $\pi_{GM}: (\mathcal{M}_H^{\mathrm{ad}})_\eta \rightarrow \mathcal{F}\ell$, where $\mathcal{F}\ell$ is the rigid-analytic space parametrizing rank d locally free quotients of $M(H)[1/p]$. The morphism π_{GM} is equivariant for the action of $\mathrm{Aut} H$. It has open image $\mathcal{F}\ell^\alpha$ (the admissible locus).

We obtain a tower of rigid-analytic spaces over $(\mathcal{M}_H^{\mathrm{ad}})_\eta$ by adding level structures. For a complete affinoid $(W(k)[1/p], W(k))$ -algebra (R, R^+) , and an element of $(\mathcal{M}_H^{\mathrm{ad}})_\eta(R, R^+)$ represented locally on $\mathrm{Spa}(R, R^+)$ by a pair (G, ρ) as above, we have the Tate module $TG = \varprojlim_m G[p^m]$, considered as a adic space over $\mathrm{Spa}(R, R^+)$ with the structure of a \mathbf{Z}_p -module [SW13, (3.3)]. Finite-level spaces $\mathcal{M}_{H,m}$ are obtained by trivializing the $G[p^m]$; these are finite étale covers of $(\mathcal{M}_H^{\mathrm{ad}})_\eta$. The infinite-level space is obtained by trivializing all of TG at once, as in the following definition.

Definition 1.3 ([SW13, Definition 6.3.3]). Let $\mathcal{M}_{H,\infty}$ be the functor which sends a complete affinoid $(W(k)[1/p], W(k))$ -algebra (R, R^+) to the set of triples (G, ρ, α) , where (G, ρ) is an element of $(\mathcal{M}_H^{\mathrm{ad}})_\eta(R, R^+)$, and $\alpha: \mathbf{Z}_p^n \rightarrow TG$ is a \mathbf{Z}_p -linear map which is an isomorphism pointwise on $\mathrm{Spa}(R, R^+)$.

There is an equivalent definition in terms of *isogeny* classes of triples (G, ρ, α) , where this time $\alpha: \mathbf{Q}_p^n \rightarrow VG$ is a trivialization of the rational Tate module. Using this definition, it becomes clear that $\mathcal{M}_{H,\infty}$ admits an action of the product $\mathrm{GL}_n(\mathbf{Q}_p) \times \mathrm{Aut}^0 H$, where Aut^0 means automorphisms in the isogeny category. Then the period map $\pi_{GM}: \mathcal{M}_{H,\infty} \rightarrow \mathcal{F}\ell$ is equivariant for $\mathrm{GL}_n(\mathbf{Q}_p) \times \mathrm{Aut}^0 H$, where $\mathrm{GL}_n(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell$.

We remark that $\mathcal{M}_{H,\infty} \sim \varprojlim_m \mathcal{M}_{H,m}$ in the sense of [SW13, Definition 2.4.1].

One of the main theorems of [SW13] is the following.

Theorem 1.4. *The adic space $\mathcal{M}_{H,\infty}$ is a preperfectoid space.*

This means that for any perfectoid field K containing $W(k)$, the base change $\mathcal{M}_{H,\infty} \times_{\mathrm{Spa}(W(k)[1/p], W(k))} \mathrm{Spa}(K, \mathcal{O}_K)$ becomes perfectoid after p -adically completing.

We sketch here the proof of Theorem 1.4. Consider the “universal cover” $\tilde{H} = \varprojlim_p H$ as a sheaf of \mathbf{Q}_p -vector spaces on the category of k -algebras. This has a canonical lift to the category of $W(k)$ -algebras [SW13, Proposition 3.1.3(ii)], which we continue to call \tilde{H} . The adic generic fiber $\tilde{H}_\eta^{\mathrm{ad}}$ is a preperfectoid space, as can be checked “by hand”: it is a product of the d -dimensional preperfectoid open ball $(\mathrm{Spa} W(k)[[T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}]])_\eta$ by the constant adic space $VH^{\mathrm{ét}}$, where $H^{\mathrm{ét}}$ is the étale part of H . Given a triple (G, ρ, α) representing an element of $\mathcal{M}_{H,\infty}(R, R^+)$, the quasi-isogeny ρ induces an isomorphism $\tilde{H}_\eta^{\mathrm{ad}} \times_{\mathrm{Spa}(W(k)[1/p], W(k))} \mathrm{Spa}(R, R^+) \rightarrow \tilde{G}_\eta^{\mathrm{ad}}$; composing this with α gives a morphism $\mathbf{Q}_p^n \rightarrow \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)$. We have therefore described a morphism $\mathcal{M}_{H,\infty} \rightarrow (\tilde{H}_\eta^{\mathrm{ad}})^n$.

Theorem 1.4 follows from the fact that the morphism $\mathcal{M}_{H,\infty} \rightarrow (\tilde{H}^{\mathrm{ad}})_\eta^n$ presents $\mathcal{M}_{H,\infty}$ as an open subset of a Zariski closed subset of $(\tilde{H}^{\mathrm{ad}})_\eta^n$. We conclude this subsection by spelling out how this is done. We have a *quasi-logarithm* map $\mathrm{qlog}_H: \tilde{H}_\eta^{\mathrm{ad}} \rightarrow M(H)[1/p] \otimes_{W(k)[1/p]} \mathbb{G}_a$ [SW13, Definition 3.2.3], a \mathbf{Q}_p -linear morphism of adic spaces over $\mathrm{Spa}(W(k)[1/p], W(k))$.

Now suppose (G, ρ) is a deformation of H to (R, R^+) . The logarithm map on G fits into an exact sequence of \mathbf{Z}_p -modules:

$$0 \rightarrow TG(R, R^+) \rightarrow G_\eta^{\mathrm{ad}}(R, R^+) \rightarrow \mathrm{Lie} G.$$

After taking projective limits along multiplication-by- p , this turns into an exact sequence of \mathbf{Q}_p -vector spaces,

$$0 \rightarrow VG(R, R^+) \rightarrow \tilde{G}_\eta^{\mathrm{ad}}(R, R^+) \rightarrow \mathrm{Lie} G[1/p].$$

On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \tilde{H}_\eta(R, R^+) & \xrightarrow{\cong} & \tilde{G}_\eta(R, R^+) \\ \mathrm{qlog}_H \downarrow & & \downarrow \mathrm{log}_G \\ M(H) \otimes_{W(k)} R & \longrightarrow & \mathrm{Lie} G[1/p]. \end{array}$$

The lower horizontal map $M(H) \otimes_{W(k)} R \rightarrow \mathrm{Lie} G[1/p]$ is the quotient by the R -submodule of $M(H) \otimes_{W(k)} R$ generated by the image of $VG(R, R^+) \rightarrow \tilde{G}_\eta^{\mathrm{ad}}(R, R^+) \cong \tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \rightarrow M(H) \otimes_{W(k)} R$.

Thus if we have a triple (G, ρ, α) representing an element of $\mathcal{M}_{H,\infty}(R, R^+)$, then we have a map $\mathbf{Q}_p^n \rightarrow \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)$, such that the cokernel of $\mathbf{Q}_p^n \rightarrow \tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \rightarrow M(H) \otimes_{W(k)} R$ is a projective R -module of rank d , namely $\mathrm{Lie} G[1/p]$. This condition on the cokernel allows us to formulate an alternate description of $\mathcal{M}_{H,\infty}$ which is independent of deformations.

Proposition 1.5. *The adic space $\mathcal{M}_{H,\infty}$ is isomorphic to the functor which assigns to a complete affinoid $(W(k)[1/p], W(k))$ -algebra (R, R^+) the set of n -tuples $(s_1, \dots, s_n) \in \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)^n$ such that the following conditions are satisfied:*

1. *The quotient of $M(H) \otimes_{W(k)} R$ by the span of the $\mathrm{qlog}(s_i)$ is a projective R -module W of rank d .*
2. *For all geometric points $\mathrm{Spa}(C, \mathcal{O}_C) \rightarrow \mathrm{Spa}(R, R^+)$, the sequence*

$$0 \rightarrow \mathbf{Q}_p^n \xrightarrow{(s_1, \dots, s_n)} \tilde{H}_\eta^{\mathrm{ad}}(C, \mathcal{O}_C) \rightarrow W \otimes_R C \rightarrow 0$$

is exact.

1.2 Infinite level Rapoport-Zink spaces of EL type

This article treats the more general class of Rapoport-Zink spaces of EL type. We review these here.

Definition 1.6. Let k be an algebraically closed field of characteristic p . A *rational EL datum* is a quadruple $\mathcal{D} = (B, V, H, \mu)$, where

- B is a semisimple \mathbf{Q}_p -algebra,
- V is a finite B -module,
- H is an object of the isogeny category of p -divisible groups over k , equipped with an action $B \rightarrow \text{End } H$,
- μ is a conjugacy class of $\overline{\mathbf{Q}_p}$ -rational cocharacters $\mathbf{G}_m \rightarrow \mathbb{G}$, where \mathbb{G}/\mathbf{Q}_p is the algebraic group $\text{GL}_B(V)$.

These are subject to the conditions:

- If $M(H)$ is the (rational) Dieudonné module of H , then there exists an isomorphism $M(H) \cong V \otimes_{\mathbf{Q}_p} W(k)[1/p]$ of $B \otimes_{\mathbf{Q}_p} W(k)[1/p]$ -modules. In particular $\dim V = \text{ht } H$.
- In the weight decomposition of $V \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}_p} \cong \bigoplus_{i \in \mathbf{Z}} V_i$ determined by μ , only weights 0 and 1 appear, and $\dim V_0 = \dim H$.

The *reflex field* E of \mathcal{D} is the field of definition of the conjugacy class μ . We remark that the weight filtration (but not necessarily the weight decomposition) of $V \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}_p}$ may be descended to E , and so we will be viewing V_0 and V_1 as $B \otimes_{\mathbf{Q}_p} E$ -modules.

The infinite-level Rapoport-Zink space $\mathcal{M}_{\mathcal{D}, \infty}$ is defined in [SW13] in terms of moduli of deformations of the p -divisible group H along with its B -action. It admits an alternate description along the lines of Proposition 1.5.

Proposition 1.7 ([SW13, Theorem 6.5.4]). *Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Let $\check{E} = E \cdot W(k)$. Then $\mathcal{M}_{\mathcal{D}, \infty}$ is isomorphic to the functor which inputs a complete affinoid $(\check{E}, \mathcal{O}_{\check{E}})$ -algebra (R, R^+) and outputs the set of B -linear maps*

$$s: V \rightarrow \check{H}_\eta^{\text{ad}}(R, R^+),$$

subject to the following conditions.

- Let W be the quotient

$$V \otimes_{\mathbf{Q}_p} R \xrightarrow{\text{ql\log}_H \circ s} M(H) \otimes_{W(k)} R \rightarrow W \rightarrow 0.$$

Then W is a finite projective R -module, which locally on R is isomorphic to $V_0 \otimes_E R$ as a $B \otimes_{\mathbf{Q}_p} R$ -module.

- For any geometric point $x = \text{Spa}(C, \mathcal{O}_C) \rightarrow \text{Spa}(R, R^+)$, the sequence of B -modules

$$0 \rightarrow V \rightarrow \check{H}(\mathcal{O}_C) \rightarrow W \otimes_R C \rightarrow 0$$

is exact.

If $\mathcal{D} = (\mathbf{Q}_p, \mathbf{Q}_p^n, H, \mu)$, where H has height n and dimension d and $\mu(t) = (t^{\oplus d}, 1^{\oplus(n-d)})$, then $E = \mathbf{Q}_p$ and $\mathcal{M}_{\mathcal{D}, \infty} = \mathcal{M}_{H, \infty}$.

The space $\mathcal{M}_{\mathcal{D}, \infty}$ admits an action by the product group $\mathbb{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where J/\mathbf{Q}_p is the algebraic group $\text{Aut}_B^\circ(H)$. A pair $(\alpha, \alpha') \in \mathbb{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$ sends s to $\alpha' \circ s \circ \alpha$.

There is once again a Grothendieck-Messing period map $\pi_{GM}: \mathcal{M}_{\mathcal{D},\infty} \rightarrow \mathcal{F}\ell_{\mu}$ onto the rigid-analytic variety whose (R, R^+) -points parametrize $B \otimes_{\mathbf{Q}_p} R$ -module quotients of $M(H) \otimes_{W(k)} R$ which are projective over R , and which are of type μ in the sense that they are (locally on R) isomorphic to $V_0 \otimes_E R$. The morphism π_{GM} sends an (R, R^+) -point of $\mathcal{M}_{\mathcal{D},\infty}$ to the quotient W of $M(H) \otimes_{W(k)} R$ as above. It is equivariant for the action of $\mathbb{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where $\mathbb{G}(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell_{\mu}$. In terms of deformations of the p -divisible group H , the period map π_{GM} sends a deformation G to $\text{Lie } G$.

There is also a Hodge-Tate period map $\pi_{HT}: \mathcal{M}_{\mathcal{D},\infty} \rightarrow \mathcal{F}\ell'_{\mu}$, where $\mathcal{F}\ell'_{\mu}(R, R^+)$ parametrizes $B \otimes_{\mathbf{Q}_p} R$ -module quotients of $V \otimes_{\mathbf{Q}_p} R$ which are projective over R , and which are (locally on R) isomorphic to $V_1 \otimes_E R$. The morphism π_{HT} sends an (R, R^+) -point of $\mathcal{M}_{\mathcal{D},\infty}$ to the image of $V \otimes_{\mathbf{Q}_p} R \rightarrow M(H) \otimes_{W(k)} R$. It is equivariant for the action of $\mathbb{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where this time $J(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell'_{\mu}(R, R^+)$. In terms of deformations of the p -divisible group H , the period map π_{HT} sends a deformation G to $(\text{Lie } G^{\vee})^{\vee}$.

2 The Fargues-Fontaine curve

2.1 Review of the curve

We briefly review here some constructions and results from [FF]. First we review the absolute curve, and then we cover the version of the curve which works in families.

Fix a perfectoid field F of characteristic p , with $F^{\circ} \subset F$ its ring of integral elements. Let $\varpi \in F^{\circ}$ be a pseudo-uniformizer for F , and let k be the residue field of F . Let $W(F^{\circ})$ be the ring of Witt vectors, which we equip with the $(p, [\varpi])$ -adic topology. Let $\mathcal{Y}_F = \text{Spa}(W(F^{\circ}), W(F^{\circ})) \setminus \{ |p[\varpi]| = 0 \}$. Then \mathcal{Y}_F is an analytic adic space over \mathbf{Q}_p . The Frobenius automorphism of F induces an automorphism ϕ of \mathcal{Y}_F . Let $B_F = H^0(\mathcal{Y}_F, \mathcal{O}_{\mathcal{Y}_F})$, a \mathbf{Q}_p -algebra endowed with an action of ϕ . Let P_F be the graded ring $P_F = \bigoplus_{n \geq 0} B_F^{\phi=p^n}$. Finally, the Fargues-Fontaine curve is $X_F = \text{Proj } P_F$. It is shown in [FF] that X_F is the union of spectra of Dedekind rings, which justifies the use of the word ‘‘curve’’ to describe X_F . Note however that there is no ‘‘structure morphism’’ $X_F \rightarrow \text{Spec } F$.

If $x \in X_F$ is a closed point, then the residue field of x is a perfectoid field F_x containing \mathbf{Q}_p which comes equipped with an inclusion $i: F \hookrightarrow F_x^{\flat}$, which presents F_x^{\flat} as a finite extension of F . Such a pair (F_x, i) is called an untilt of F . Then $x \mapsto (F_x, i)$ is a bijection between closed points of X_F and isomorphism classes of untilts of F , modulo the action of Frobenius on i . Thus if $F = E^{\flat}$ is the tilt of a given perfectoid field E/\mathbf{Q}_p , then $X_{E^{\flat}}$ has a canonical closed point ∞ , corresponding to the untilt E of E^{\flat} .

An important result in [FF] is the classification of vector bundles on X_F . (By a vector bundle on X_F we are referring to a locally free \mathcal{O}_{X_F} -module \mathcal{E} of finite rank. We will use the notation $V(\mathcal{E})$ to mean the corresponding geometric vector bundle over X_F , whose sections correspond to sections of \mathcal{E} .) Recall that an *isocrystal* over k is a finite-dimensional vector space N over $W(k)[1/p]$ together with a Frobenius semi-linear automorphism ϕ of N . Given N , we have the graded P_F -module $\bigoplus_{n \geq 0} (N \otimes_{W(k)[1/p]} B_F)^{\phi=p^n}$, which corresponds to a vector bundle $\mathcal{E}_F(N)$ on X_F . Then the Harder-Narasimhan slopes of $\mathcal{E}_F(N)$ are negative to those of N . If F is algebraically closed, then every vector bundle on X_F is isomorphic to $\mathcal{E}_F(N)$ for some N .

It is straightforward to ‘‘relativize’’ the above constructions. If $S = \text{Spa}(R, R^+)$ is an affinoid perfectoid space over k , one can construct the adic space \mathcal{Y}_S , the ring B_S , the scheme X_S , and the vector bundles $\mathcal{E}_S(N)$ as above. Frobenius-equivalence classes of untilts of S correspond to effective Cartier divisors of X_S of degree 1.

In our applications, we will start with an affinoid perfectoid space S over \mathbf{Q}_p . We will write $X_S = X_{S^{\flat}}$, and we will use ∞ to refer to the canonical Cartier divisor of X_S corresponding to the untilt S of S^{\flat} . Thus if N is an isocrystal over k , and $S = \text{Spa}(R, R^+)$ is an affinoid perfectoid space over $W(k)[1/p]$, then the fiber of $\mathcal{E}_S(N)$ over ∞ is $N \otimes_{W(k)[1/p]} R$.

2.2 Relation to p -divisible groups

Here we recall the relationships between p -divisible groups and global sections of vector bundles on the Fargues-Fontaine curve. Let us fix a perfect field k of characteristic p , and write Perf_k for the category of perfectoid spaces over k . Given a p -divisible group H over k with (covariant) isocrystal N , and an object S of Perf_k , we define the vector bundle $\mathcal{E}_S(H)$ by

$$\mathcal{E}_S(H) = \mathcal{E}_S(N) \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_S}(1).$$

Under this normalization, the Harder-Narasimhan slopes of $\mathcal{E}_S(H)$ are (pointwise on S) the same as the slopes of H .

Let us write $H^0(\mathcal{E}(H))$ for the sheafification of the functor on Perf_k which inputs a perfectoid space S over k and outputs $H^0(X_S, \mathcal{E}_S(H))$.

Proposition 2.1. *Let H be a p -divisible group over a perfect field k of characteristic p , with isocrystal N . There is an isomorphism $\tilde{H}_\eta^{\text{ad}} \cong H^0(\mathcal{E}(H))$ of sheaves on Perf_k making the diagram commute:*

$$\begin{array}{ccc} \tilde{H}_\eta^{\text{ad}} & \xrightarrow{\quad\quad\quad} & H^0(\mathcal{E}(H)) \\ & \searrow^{\text{qlog}_H} & \swarrow \\ & N \otimes_{W(k)[1/p]} \mathbb{G}_a & \end{array}$$

where the morphism $H^0(\mathcal{E}(H)) \rightarrow N \otimes_{W(k)[1/p]} \mathbb{G}_a$ sends a global section of $\mathcal{E}(H)$ to its fiber at ∞ .

Proof. Let $S = \text{Spa}(R, R^+)$ be a affinoid perfectoid space over k . Then $\tilde{H}_\eta^{\text{ad}}(R, R^+) = \tilde{H}(R^\circ)$. Since R° is a perfect ring, there is an isomorphism [SW13, Theorem A] $\tilde{H}(R^\circ) \cong (B_{\text{cris}}^+(R^\circ) \otimes_{W(k)} N)^{\phi=p}$. On the other side, $H^0(X_S, \mathcal{E}_S(H)) = (B_S \otimes_{W(k)} N)^{\phi=p}$. The isomorphism between these is discussed in [LB18, Remarque 6.6].

The commutativity of the diagram in the proposition is [SW13, Proposition 5.16(ii)], at least in the case that S is a geometric point, but this suffices to prove the general case. \square

With Proposition 2.1 we can reinterpret the infinite-level Rapoport Zink spaces as moduli spaces of *modifications* of vector bundles on the Fargues-Fontaine curve. First we do this for $\mathcal{M}_{H, \infty}$. In the following, we consider $\mathcal{M}_{H, \infty}$ as a sheaf on the category of perfectoid spaces over $W(k)[1/p]$.

Proposition 2.2. *Let H be a p -divisible group of height n and dimension d over a perfect field k . Let N be the associated isocrystal over k . Then $\mathcal{M}_{H, \infty}$ is isomorphic to the functor which inputs an affinoid perfectoid space $S = \text{Spa}(R, R^+)$ over $W(k)[1/p]$ and outputs the set of exact sequences*

$$0 \rightarrow \mathcal{O}_{X_S}^n \xrightarrow{s} \mathcal{E}_S(H) \rightarrow i_{\infty*} W \rightarrow 0, \tag{2.2.1}$$

where $i_{\infty}: \text{Spec } R \rightarrow X_S$ is the inclusion, and W is a projective \mathcal{O}_S -module quotient of $N \otimes_{W(k)[1/p]} \mathcal{O}_S$ of rank d .

Proof. We briefly describe this isomorphism on the level of points over $S = \text{Spa}(R, R^+)$. Suppose that we are given a point of $\mathcal{M}_{H, \infty}(S)$, corresponding to a p -divisible group G over R° , together with a quasi-isogeny $\iota: H \otimes_k R^\circ/p \rightarrow G \otimes_{R^\circ} R^\circ/p$ and an isomorphism $\alpha: \mathbf{Q}_p^n \rightarrow VG$ of sheaves of \mathbf{Q}_p -vector spaces on S . The logarithm map on G fits into an exact sequence of sheaves of \mathbf{Z}_p -modules on S ,

$$0 \rightarrow TG \rightarrow G_\eta^{\text{ad}} \rightarrow \text{Lie } G \rightarrow 0.$$

After taking projective limits along multiplication-by- p , this turns into an exact sequence of sheaves of \mathbf{Q}_p -vector spaces on S ,

$$0 \rightarrow VG \rightarrow \tilde{G}_\eta^{\text{ad}} \rightarrow \text{Lie } G \rightarrow 0.$$

The quasi-isogeny induces an isomorphism $\tilde{H}_\eta^{\text{ad}} \times_{\text{Spa } W(k)[1/p]} S \cong \tilde{G}_\eta^{\text{ad}}$; composing this with the level structure gives an injective map $\mathbf{Q}_p^n \rightarrow \tilde{H}_\eta^{\text{ad}}(S)$, whose cokernel W is isomorphic to the projective R -module $\text{Lie } G$ of rank d . In light of Theorem 2.1, the map $\mathbf{Q}_p^n \rightarrow \tilde{H}_\eta^{\text{ad}}(S)$ corresponds to an \mathcal{O}_{X_S} -linear map $s: \mathcal{O}_{X_S}^n \rightarrow \mathcal{E}_S(H)$, which fits into the exact sequence in (2.2.1). \square

Similarly, we have a description of $\mathcal{M}_{\mathcal{D},\infty}$ in terms of modifications.

Proposition 2.3. *Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Then $\mathcal{M}_{\mathcal{D},\infty}$ is isomorphic to the functor which inputs an affinoid perfectoid space S over \check{E} and outputs the set of exact sequences of $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$ -modules*

$$0 \rightarrow V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \xrightarrow{s} \mathcal{E}_S(H) \rightarrow i_{\mathcal{D}*} W \rightarrow 0,$$

where W is a finite projective \mathcal{O}_S -module, which is locally isomorphic to $V_0 \otimes_{\mathbf{Q}_p} \mathcal{O}_S$ as a $B \otimes_{\mathbf{Q}_p} \mathcal{O}_S$ -module (using notation from Definition 1.6).

2.3 The determinant morphism, and connected components

If we are given a rational EL datum \mathcal{D} , there is a *determinant morphism* $\det: \mathcal{M}_{\mathcal{D},\infty} \rightarrow \mathcal{M}_{\det \mathcal{D},\infty}$, which we review below. For an algebraically closed perfectoid field C , the base change $\mathcal{M}_{\det \mathcal{D},\infty,C}$ is a locally profinite set of copies of $\text{Spa } C$. For a point $\tau \in \mathcal{M}_{\det \mathcal{D},\infty}(C)$, let $\mathcal{M}_{\mathcal{D},\infty}^\tau$ be the fiber of $\mathcal{M}_{\mathcal{D},\infty} \rightarrow \mathcal{M}_{\det \mathcal{D},\infty}$ over τ . We will prove in Section 4 that each $\mathcal{M}_{\mathcal{D},\infty}^{\tau,\text{non-sp}}$ is cohomologically smooth if \mathcal{D} is basic. This implies that $\pi_0(\mathcal{M}_{\mathcal{D},\infty}^{\tau,\text{non-sp}})$ is discrete, so that cohomological smoothness of $\mathcal{M}_{\mathcal{D},\infty}^{\tau,\text{non-sp}}$ is inherited by each of its connected components. This is Theorem 1.1. In certain cases (for example Lubin-Tate space) it is known that $\mathcal{M}_{\mathcal{D},\infty}^\tau$ is already connected [Che14].

We first review the determinant morphism for the space $\mathcal{M}_{H,\infty}$, where H is a p -divisible group of height n and dimension d over a perfect field k of characteristic p . Let $\check{E} = W(k)[1/p]$. For a perfectoid space S over \check{E} , we have the vector bundle $\mathcal{E}_S(H)$ and its determinant $\det \mathcal{E}_S(H)$, a line bundle of degree d . (This does not correspond to a p -divisible group “ $\det H$ ” unless $d \geq 1$.) We define $\mathcal{M}_{\det H,\infty}(S)$ to be the functor which inputs a perfectoid space $S = \text{Spa}(R, R^+)$ over \check{E} and outputs the set of morphisms $s: \mathcal{O}_{X_S} \rightarrow \det \mathcal{E}_S(H)$, such that the cokernel of s is a projective $B_{\text{dR}}^+(R)/(\xi)^d$ -module of rank 1, where (ξ) is the kernel of $B_{\text{dR}}^+(R) \rightarrow R$. Then for an algebraically closed perfectoid field C/\check{E} , the set $\mathcal{M}_{\det H,\infty}(C)$ is a \mathbf{Q}_p^\times -torsor. The morphism $\det: \mathcal{M}_{H,\infty} \rightarrow \mathcal{M}_{\det H,\infty}$ is simply $s \mapsto \det s$.

For the general case, let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Let $F = Z(B)$ be the center of B . Then F is a semisimple commutative \mathbf{Q}_p -algebra, and V is free as an F -module. Then if $\mathbb{G} = \text{Aut}_B(V)$ (as an algebraic group), then $\mathbb{G}^{\text{ab}} = \mathbb{G}/\mathbb{G}^{\text{der}} = \text{Aut}_F(\det_F V) \cong \text{Res}_{F/\mathbf{Q}_p} \mathbb{G}_m$. Let μ^{ab} be the composition of μ with $\mathbb{G} \rightarrow \mathbb{G}^{\text{ab}}$. Let $\mathcal{M}_{\det \mathcal{D},\infty}$ be the functor which inputs a perfectoid space $S = \text{Spa}(R, R^+)$ over \check{E} and outputs the set of F -linear morphisms $s: \det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \rightarrow \det_F \mathcal{E}_S(H)$, such that the cokernel of s is a $B_{\text{dR}}^+(R) \otimes_{\mathbf{Q}_p} F$ -module “of type μ^{ab} ”.

2.4 Basic Rapoport-Zink spaces

The main theorem of this article concerns basic Rapoport-Zink spaces, so we recall some facts about these here.

Let H be a p -divisible group over a perfect field k of characteristic p . The space $\mathcal{M}_{H,\infty}$ is said to be basic when the p -divisible group H (or rather, its Dieudonné module $M(H)$) is isoclinic. This is equivalent

to saying that the natural map

$$\mathrm{End}^\circ H \otimes_{\mathbf{Q}_p} W(k)[1/p] \rightarrow \mathrm{End}_{W(k)[1/p]} M(H)[1/p]$$

is an isomorphism, where on the right the endomorphisms are not required to commute with Frobenius.

More generally we have a notion of basicness for a rational EL datum (B, H, V, μ) , referring to the following equivalent conditions:

- The \mathbb{G} -isocrystal ($\mathbb{G} = \mathrm{Aut}_B V$) associated to H is basic in the sense of Kottwitz [Kot85].
- The natural map

$$\mathrm{End}_B^\circ(H) \otimes_{\mathbf{Q}_p} W(k)[1/p] \rightarrow \mathrm{End}_{B \otimes_{\mathbf{Q}_p} W(k)[1/p]} M(H)[1/p]$$

is an isomorphism.

- Considered as an algebraic group over \mathbf{Q}_p , the automorphism group $J = \mathrm{Aut}_B^\circ H$ is an inner form of \mathbb{G} .
- Let $D = \mathrm{End}_B^\circ H$. For any algebraically closed perfectoid field C containing $W(k)$, the map

$$D \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \rightarrow \mathcal{E}nd_{(B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C})} \mathcal{E}_C(H)$$

is an isomorphism.

In brief, the duality theorem from [SW13] says the following. Given a basic EL datum \mathcal{D} , there is a dual datum $\check{\mathcal{D}}$, for which the roles of the groups \mathbb{G} and J are reversed. There is a $\mathbb{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$ -equivariant isomorphism $\mathcal{M}_{\mathcal{D}, \infty} \cong \mathcal{M}_{\check{\mathcal{D}}, \infty}$ which exchanges the roles of π_{GM} and π_{HT} .

2.5 The special locus

Let $\mathcal{D} = (B, V, H, \mu)$ be a basic rational EL datum relative to a perfect field k of characteristic p , with reflex field E . Let F be the center of B . Define B -algebras D and D' by

$$\begin{aligned} D &= \mathrm{End}_F V \\ D' &= \mathrm{End}_F H \end{aligned}$$

Finally, let $\mathbb{G} = \mathrm{Aut}_B V$ and $J = \mathrm{Aut}_B H$, considered as algebraic groups over \mathbf{Q}_p . Then \mathbb{G} and J both contain $\mathrm{Res}_{F/\mathbf{Q}_p} \mathbb{G}_m$.

Let C be an algebraically closed perfectoid field containing \check{E} , and let $x \in \mathcal{M}_{\mathcal{D}, \infty}(C)$. Then x corresponds to a p -divisible group G over \mathcal{O}_C with endomorphisms by B , and also it corresponds to a $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$ -linear map $s: V \otimes_{\mathbf{Q}_p} \mathcal{O}_X \rightarrow \mathcal{E}_C(N)$ as in Proposition 2.3. Define $A_x = \mathrm{End}_F G$ (endomorphisms in the isogeny category). Then A_x is a semisimple F -algebra. It admits an alternate description in terms of s :

$$A_x \cong \left\{ (\alpha, \alpha') \in D \times D' \mid s \circ \alpha = \alpha' \circ s \right\}.$$

Lemma 2.4. *The following are equivalent:*

1. The F -algebra A_x strictly contains F .
2. The stabilizer of $\pi_{GM}(x) \in \mathcal{F}\ell_\mu(C)$ in $J(\mathbf{Q}_p)$ strictly contains F^\times .
3. The stabilizer of $\pi_{HT}(x) \in \mathcal{F}\ell'_\mu(C)$ in $\mathbb{G}(\mathbf{Q}_p)$ strictly contains F^\times .

Definition 2.5. The *special locus* in $\mathcal{M}_{\mathcal{D},\infty}$ is the subset $\mathcal{M}_{\mathcal{D},\infty}^{\text{sp}}$ defined by the condition $A_x \neq F$. The *non-special locus* $\mathcal{M}_{\mathcal{D},\infty}^{\text{non-sp}}$ is the complement of the special locus.

The special locus is built out of “smaller” Rapoport-Zink spaces, in the following sense. Let A be a semisimple \mathbf{Q}_p -algebra containing B , equipped with two B -embeddings $A \rightarrow D$ and $A \rightarrow D'$, so that A acts on V and H . Also assume that a cocharacter in the conjugacy class μ factors through a cocharacter $\mu_0: \mathbb{G}_m \rightarrow \text{Aut}_A V$. Let $\mathcal{D}_0 = (A, V, H, \mu_0)$. Then there is an evident morphism $\mathcal{M}_{\mathcal{D}_0,\infty} \rightarrow \mathcal{M}_{\mathcal{D},\infty}$. The special locus $\mathcal{M}_{\mathcal{D},\infty}^{\text{sp}}$ is the union of the images of all the $\mathcal{M}_{\mathcal{D}_0,\infty}$, as A ranges through all semisimple \mathbf{Q}_p -subalgebras of $D \times D'$ containing B as a proper subalgebra.

3 Cohomological smoothness

Let Perf be the category of perfectoid spaces in characteristic p , with its pro-étale topology [Sch17, Definition 8.1]. For a prime $\ell \neq p$, there is a notion of ℓ -cohomological smoothness [Sch17, Definition 23.8]. We only need the notion for morphisms $f: Y' \rightarrow Y$ between sheaves on Perf which are separated and representable in locally spatial diamonds. If such an f is ℓ -cohomologically smooth, and Λ is an ℓ -power torsion ring, then the relative dualizing complex $Rf^! \Lambda$ is an invertible object in $D_{\text{ét}}(Y', \Lambda)$ (thus, it is v -locally isomorphic to $\Lambda[n]$ for some $n \in \mathbb{Z}$), and the natural transformation $Rf^! \Lambda \otimes f^* \rightarrow Rf^!$ of functors $D_{\text{ét}}(Y, \Lambda) \rightarrow D_{\text{ét}}(Y', \Lambda)$ is an equivalence [Sch17, Proposition 23.12]. In particular, if f is projection onto a point, and $Rf^! \Lambda \cong \Lambda[n]$, one derives a statement of Poincaré duality for Y' :

$$R\text{Hom}(R\Gamma_c(Y', \Lambda), \Lambda) \cong R\Gamma(Y', \Lambda)[n].$$

We will say that f is cohomologically smooth if it is ℓ -cohomologically smooth for all $\ell \neq p$. As an example, if $f: Y' \rightarrow Y$ is a separated smooth morphism of rigid-analytic spaces over \mathbf{Q}_p , then the associated morphism of diamonds $f^\diamond: (Y')^\diamond \rightarrow Y^\diamond$ is cohomologically smooth [Sch17, Proposition 24.3]. There are other examples where f does not arise from a finite-type map of adic spaces. For instance, if $\tilde{B}_C = \text{Spa } C \langle T^{1/p^\infty} \rangle$ is the perfectoid closed ball over an algebraically closed perfectoid field C , then \tilde{B}_C is cohomologically smooth over C .

If Y is a perfectoid space over an algebraically closed perfectoid field C , it seems quite difficult to detect whether Y is cohomologically smooth over C . We will review in Section 3.2 a “Jacobian criterion” from [FS] which applies to certain kinds of Y . But first we give a classical analogue of this criterion in the context of schemes.

3.1 The Jacobian criterion: classical setting

Proposition 3.1. *Let X be a smooth projective curve over an algebraically closed field k . Let $Z \rightarrow X$ be a smooth morphism. Define \mathcal{M}_Z to be the functor which inputs a k -scheme T and outputs the set of sections of $Z \rightarrow X$ over X_T , that is, the set of morphisms s making*

$$\begin{array}{ccc} & & Z \\ & \nearrow s & \downarrow \\ X \times_k T & \longrightarrow & X \end{array}$$

commute, subject to the condition that, fiberwise on T , the vector bundle $s^ \text{Tan}_{Z/X}$ has vanishing H^1 fiberwise on T . Then $\mathcal{M}_Z \rightarrow \text{Spec } k$ is formally smooth.*

Here $\mathrm{Tan}_{Z/X}$ is the tangent bundle, equal to the \mathcal{O}_Z -linear dual of the sheaf of differentials $\Omega_{Z/X}$, which is locally free of finite rank. Let $\pi: X \times_k T \rightarrow T$ be the projection. For $t \in T$, let X_t be the fiber of π over t , and let $s_t: X_t \rightarrow Z$ be the restriction of s to X_t . By proper base change, the fiber of $R^1\pi_*s^*\mathrm{Tan}_{Z/X}$ at $t \in T$ is $H^1(X_t, s_t^*\mathrm{Tan}_{Z/X})$. The condition about the vanishing of H^1 in the proposition is equivalent to $H^1(X_t, s_t^*\mathrm{Tan}_{Z/X}) = 0$ for each $t \in T$. By Nakayama's lemma, this condition is equivalent to $R^1\pi_*s^*\mathrm{Tan}_{Z/X} = 0$.

Proof. Suppose we are given a commutative diagram

$$\begin{array}{ccc} T_0 & \longrightarrow & \mathcal{M}_Z \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{Spec} k, \end{array} \quad (3.1.1)$$

where $T_0 \rightarrow T$ is a first-order thickening of affine schemes; thus T_0 is the vanishing locus of a square-zero ideal sheaf $I \subset \mathcal{O}_T$. Note that I becomes an \mathcal{O}_{T_0} -module.

The morphism $T_0 \rightarrow \mathcal{M}_Z$ in (3.1.1) corresponds to a section of $Z \rightarrow X$ over T_0 . Thus there is a solid diagram

$$\begin{array}{ccc} X \times_k T_0 & \xrightarrow{s_0} & Z \\ \downarrow & \nearrow s & \downarrow \\ X \times_k T & \longrightarrow & X. \end{array} \quad (3.1.2)$$

We claim that there exists a dotted arrow making the diagram commute. Since $Z \rightarrow X$ is smooth, it is formally smooth, and therefore this arrow exists Zariski-locally on X . Let $\pi: X \times_k T \rightarrow T$ and $\pi_0: X \times_k T_0 \rightarrow T_0$ be the projections. Then $X \times_k T_0$ is the vanishing locus of the ideal sheaf $\pi^*I \subset \mathcal{O}_{X \times_k T}$. Note that sheaves of sets on $X \times_k T$ are equivalent to sheaves of sets on $X \times_k T_0$; under this equivalence, π^*I and π_0^*I correspond. By [Sta14, Remark 36.9.6], the set of such morphisms form a (Zariski) sheaf of sets on $X \times_k T$, which when viewed as a sheaf on $X \times_k T_0$ is a torsor for

$$\mathcal{H}om_{\mathcal{O}_{X \times_k T_0}}(s_0^*\Omega_{Z/X}, \pi_0^*I) \cong s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I.$$

This torsor corresponds to class in

$$H^1(X \times_k T_0, s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I).$$

This H^1 is the limit of a spectral sequence with terms

$$H^p(T_0, R^q\pi_{0*}(s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I)).$$

But since T_0 is affine and $R^q\pi_{0*}(s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I)$ is quasi-coherent, the above terms vanish for all $p > 0$, and therefore

$$H^1(X \times_k T_0, s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I) \cong H^0(T_0, R^1\pi_{0*}(s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I)).$$

Since $s_0^*\mathrm{Tan}_{Z/X}$ is locally free, we have $s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I \cong s_0^*\mathrm{Tan}_{Z/X} \otimes^{\mathbb{L}}\pi_{0*}I$, and we may apply the projection formula [Sta14, Lemma 35.21.1] to obtain

$$R\pi_{0*}(s_0^*\mathrm{Tan}_{Z/X} \otimes \pi_0^*I) \cong R\pi_{0*}s_0^*\mathrm{Tan}_{Z/X} \otimes^{\mathbb{L}}I.$$

Now we apply the hypothesis about vanishing of H^1 , which implies that $R\pi_{0*}s_0^*\mathrm{Tan}_{Z/X}$ is quasi-isomorphic

to the locally free sheaf $\pi_{0*}s_0^* \text{Tan}_{Z/X}$ in degree 0. Therefore the complex displayed above has $H^1 = 0$.

Thus our torsor is trivial, and so a morphism $s: X \times_k T \rightarrow Z$ exists filling in (3.1.2). The final thing to check is that s corresponds to a morphism $T \rightarrow \mathcal{M}_Z$, i.e., that it satisfies the fiberwise $H^1 = 0$ condition. But this is automatic, since T_0 and T have the same schematic points. \square

In the setup of Proposition 3.1, let $s: X \times_k \mathcal{M}_Z \rightarrow Z$ be the universal section. That is, the pullback of s along a morphism $T \rightarrow \mathcal{M}_Z$ is the section $X \times_k T \rightarrow Z$ to which this morphism corresponds. Let $\pi: X \times_k \mathcal{M}_Z \rightarrow \mathcal{M}_Z$ be the projection.

Now assume that the hypothesis of Proposition 3.1 is satisfied, so that $\mathcal{M}_Z \rightarrow \text{Spec } k$ is formally smooth. There is an isomorphism

$$\pi_* s^* \text{Tan}_{Z/X} \cong \text{Tan}_{\mathcal{M}_Z/\text{Spec } k}.$$

Indeed, the proof of Proposition 3.1 shows that $\pi_* s^* \text{Tan}_{Z/X}$ has the same universal property with respect to first order deformations as $\text{Tan}_{\mathcal{M}_Z/\text{Spec } k}$.

The following example is of similar spirit as our main application of the perfectoid Jacobian criterion below.

Example 3.2. Let $X = \mathbb{P}^1$ over the algebraically closed field k . For $d \in \mathbb{Z}$, let $V_d = \underline{\text{Spec}}_X \text{Sym}_{\mathcal{O}_X}(\mathcal{O}(-d))$ be the geometric vector bundle over X whose global sections are $\Gamma(X, \mathcal{O}(d))$. Fix integers $n, d, \delta > 0$ and let P be a homogeneous polynomial over k of degree δ in n variables. Then P defines a morphism $P: \prod_{i=1}^n V_d \rightarrow V_{d\delta}$, by sending sections $(s_i)_{i=1}^n$ of V_d to the section $P(s_1, \dots, s_n)$ of $V_{d\delta}$. Fix a global section $f: X \rightarrow V_{d\delta}$ to the projection morphism and consider the pull-back of P along f :

$$\begin{array}{ccccc} Z \hookrightarrow & P^{-1}(f) & \longrightarrow & X & \\ & \downarrow & & \downarrow f & \searrow \text{id}_X \\ & \prod_{i=1}^n V_d & \xrightarrow{P} & V_{d\delta} & \longrightarrow X \end{array}$$

Moreover, let Z be the smooth locus of $P^{-1}(f)$ over X , i.e., the open subscheme of $P^{-1}(f)$, defined by the condition $\text{rk} \left(\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_n} \right) = 1$. We wish to apply Proposition 3.1 to Z/X . Let \mathcal{M}'_Z denote the space of global sections of Z over X , that is for a k -scheme T , $\mathcal{M}'_Z(T)$ is the set of morphisms $s: X \times_k T \rightarrow Z$ as in the proposition (without any further conditions). A k -point $g \in \mathcal{M}'_Z(k)$ corresponds to a section $g: X \rightarrow \prod_{i=1}^n V_d$, satisfying $P \circ g = f$. In general, for a (geometric) vector bundle V on X with corresponding locally free \mathcal{O}_X -module \mathcal{E} , the pull-back of the tangent space $\text{Tan}_{V/X}$ along a section $s: X \rightarrow V$ is canonically isomorphic to \mathcal{E} . Hence in our situation (using that $Z \subseteq P^{-1}(f)$ is open) the tangent space $g^* \text{Tan}_{Z/X}$ can be computed from the short exact sequence,

$$0 \rightarrow g^* \text{Tan}_{Z/X} \rightarrow \bigoplus_{i=1}^n \mathcal{O}(d) \xrightarrow{D_g P} \mathcal{O}(d\delta) \rightarrow 0,$$

where $D_g P$ is the derivative of P at g . It is the \mathcal{O}_X -linear map given by $(t_i)_{i=1}^n \mapsto \sum_{i=1}^n \frac{\partial P}{\partial x_i}(g) t_i$ (note that $\frac{\partial P}{\partial x_i}(g)$ are global sections of $\mathcal{O}(d(\delta - 1))$). Note that $D_g P$ is surjective: by Nakayama, it suffices to check this fiberwise, where it is true by the $\text{rk} = 1$ condition defining Z .

The space \mathcal{M}'_Z is the subfunctor of \mathcal{M}'_Z consisting of all g such that (fiberwise) $g^* \text{Tan}_{Z/X} = \ker(D_g P)$ has vanishing H^1 . Writing $\ker(D_g P) = \bigoplus_{i=1}^r \mathcal{O}(m_i)$ ($m_i \in \mathbb{Z}$), this is equivalent to $m_i \geq -1$. By the Proposition 3.1 we conclude that \mathcal{M}'_Z is formally smooth over k .

Consider now a numerical example. Let $n = 3$, $d = 1$ and $\delta = 4$ and let $g \in \mathcal{M}'_Z(k)$. Then $D_g P \in \text{Hom}_{\mathcal{O}_X}(\mathcal{O}(1)^{\oplus 3}, \mathcal{O}(4)) = \Gamma(X, \mathcal{O}(3)^{\oplus 3})$, a 12-dimensional k -vector space, and moreover, $D_g P$ lies in the

open subspace of surjective maps. We have the short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow g^* \mathrm{Tan}_{Z/X} \rightarrow \mathcal{O}(1)^{\oplus 3} \xrightarrow{D_g P} \mathcal{O}(4) \rightarrow 0 \quad (3.1.3)$$

This shows that $g^* \mathrm{Tan}_{Z/X}$ has rank 2 and degree -1 . Moreover, being a subbundle of $\mathcal{O}(1)^{\oplus 3}$ it only can have slopes ≤ 1 . There are only two options, either $g^* \mathrm{Tan}_{Z/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}$ or $g^* \mathrm{Tan}_{Z/X} \cong \mathcal{O}(-2) \oplus \mathcal{O}(1)$. The point g lies in \mathcal{M}_Z if and only if the first option occurs g . Which option occurs can be seen from the long exact cohomology sequence associated to (3.1.3):

$$0 \rightarrow \Gamma(X, g^* \mathrm{Tan}_{Z/X}) \rightarrow \underbrace{\Gamma(X, \mathcal{O}(1))^{\oplus 3}}_{6\text{-dim'l}} \xrightarrow{\Gamma(D_g P)} \underbrace{\Gamma(X, \mathcal{O}(4))}_{5\text{-dim'l}} \rightarrow H^1(X, g^* \mathrm{Tan}_{Z/X}) \rightarrow 0,$$

It is clear that $\Gamma(X, g^* \mathrm{Tan}_{Z/X})$ is 1-dimensional if and only if $g^* \mathrm{Tan}_{Z/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}$ and 2-dimensional otherwise. The first option is generic, i.e., \mathcal{M}_Z is an open subscheme of \mathcal{M}'_Z .

3.2 The Jacobian criterion: perfectoid setting

We present here the perfectoid version of Proposition 3.1.

Theorem 3.3 (Fargues-Scholze [FS]). *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space in characteristic p . Let $Z \rightarrow X_S$ be a smooth morphism of schemes. Let \mathcal{M}_Z be the functor which inputs a perfectoid space $T \rightarrow S$ and outputs the set of sections of $Z \rightarrow X_S$ over T , that is, the set of morphisms s making*

$$\begin{array}{ccc} & & Z \\ & \nearrow s & \downarrow \\ X_T & \longrightarrow & X_S \end{array}$$

commute, subject to the condition that, fiberwise on T , all Harder-Narasimhan slopes of the vector bundle $s^ \mathrm{Tan}_{Z/X_S}$ are positive. Then $\mathcal{M}_Z \rightarrow S$ is a cohomologically smooth morphism of locally spatial diamonds.*

Example 3.4. Let $S = \eta = \mathrm{Spa}(C, \mathcal{O}_C)$, where C is an algebraically closed perfectoid field of characteristic p , and let $Z = \mathbb{V}(\mathcal{E}_S(H)) \rightarrow X_S$ be the geometric vector bundle attached to $\mathcal{E}_S(H)$, where H is a p -divisible group over the residue field of C . Then $\mathcal{M}_Z = H^0(\mathcal{E}_S(H))$ is isomorphic to $\tilde{H}_\eta^{\mathrm{ad}}$ by Theorem 2.1. Let $s: X_{\mathcal{M}_Z} \rightarrow Z$ be the universal morphism; then $s^* \mathrm{Tan}_{Z/X_S}$ is the constant Banach-Colmez space associated to H . This has vanishing H^1 if and only if H has no étale part. This is true if and only if \mathcal{M}_Z is isomorphic to a perfectoid open ball. The perfectoid open ball is cohomologically smooth, in accord with Theorem 3.3. In contrast, if the étale quotient $H^{\mathrm{ét}}$ has height $d > 0$, then $\pi_0(\tilde{H}_\eta^{\mathrm{ad}}) \cong \mathbf{Q}_p^d$ implies that $\tilde{H}_\eta^{\mathrm{ad}}$ is not cohomologically smooth.

In the setup of Theorem 3.3, suppose that $x = \mathrm{Spa}(C, \mathcal{O}_C) \rightarrow S$ is a geometric point, and that $x \rightarrow \mathcal{M}_Z$ is an S -morphism, corresponding to a section $s: X_C \rightarrow Z$. Then $s^* \mathrm{Tan}_{Z/X_S}$ is a vector bundle on X_C . In light of the discussion in the previous section, we are tempted to interpret $H^0(X_C, s^* \mathrm{Tan}_{Z/X_S})$ as the “tangent space of $\mathcal{M}_Z \rightarrow S$ at x ”. At points x where $s^* \mathrm{Tan}_{Z/X_S}$ has only positive Harder-Narasimhan slopes, this tangent space is a perfectoid open ball.

4 Proof of the main theorem

4.1 Dilatations and modifications

As preparation for the proof of Theorem 1.1, we review the notion of a dilatation of a scheme at a locally closed subscheme [BLR90, §3.2].

Throughout this subsection, we fix some data. Let X be a curve, meaning that X is a scheme which is locally the spectrum of a Dedekind ring. Let $\infty \in X$ be a closed point with residue field C . Let $i_\infty: \text{Spec } C \rightarrow X$ be the embedding, and let $\xi \in \mathcal{O}_{X,\infty}$ be a local uniformizer at ∞ .

Proposition 4.1. *Let $V \rightarrow X$ be a morphism of finite type, and let $Y \subset V_\infty$ be a locally closed subscheme of the fiber of V at ∞ .*

There exists a morphism of X -schemes $V' \rightarrow V$ which is universal for the following property: $V' \rightarrow X$ is flat at ∞ , and $V'_\infty \rightarrow V_\infty$ factors through $Y \subset V_\infty$.

The X -scheme V' is the *dilatation* of V at Y . We review here its construction.

First suppose that $Y \subset V_\infty$ is closed. Let $\mathcal{I} \subset \mathcal{O}_V$ be the ideal sheaf which cuts out Y . Let $B \rightarrow V$ be the blow-up of V along Y . Then $\mathcal{I} \cdot \mathcal{O}_B$ is a locally principal ideal sheaf. The dilatation V' of V at Y is the open subscheme of B obtained by imposing the condition that the ideal $(\mathcal{I} \cdot \mathcal{O}_B)_x \subset \mathcal{O}_{B,x}$ is generated by ξ at all $x \in B$ lying over ∞ .

We give here an explicit local description of the dilatation V' . Let $\text{Spec } A$ be an affine neighborhood of ∞ , such that $\xi \in A$, and let $\text{Spec } R \subset V$ be an open subset lying over $\text{Spec } A$. Let $I = (f_1, \dots, f_n)$ be the restriction of \mathcal{I} to $\text{Spec } R$, so that I cuts out $Y \cap \text{Spec } A$. Then the restriction of $V' \rightarrow V$ to $\text{Spec } R$ is $\text{Spec } R'$, where

$$R' = R \left[\frac{f_1}{\xi}, \dots, \frac{f_n}{\xi} \right] / (\xi\text{-torsion}).$$

Now suppose $Y \subset V_\infty$ is only locally closed, so that Y is open in its closure \bar{Y} . Then the dilatation of V at Y is the dilatation of $V \setminus (\bar{Y} \setminus Y)$ at Y .

Note that a dilatation $V' \rightarrow V$ is an isomorphism away from ∞ , and that it is affine.

Example 4.2. Let

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow i_{\infty*} W \rightarrow 0$$

be an exact sequence of \mathcal{O}_X -modules, where \mathcal{E} (and thus \mathcal{E}') is locally free, and W is a C -vector space. (This is an elementary modification of the vector bundle \mathcal{E} .) Let $K = \ker(\mathcal{E}_\infty \rightarrow W)$.

Let $\mathbb{V}(\mathcal{E}) \rightarrow X$ be the geometric vector bundle corresponding to \mathcal{E} . Similarly, we have $\mathbb{V}(\mathcal{E}') \rightarrow X$, and an X -morphism $\mathbb{V}(\mathcal{E}') \rightarrow \mathbb{V}(\mathcal{E})$. Let $\mathbb{V}(K) \subset \mathbb{V}(\mathcal{E})_\infty$ be the affine space associated to $K \subset \mathcal{E}_\infty$. We claim that $\mathbb{V}(\mathcal{E}')$ is isomorphic to the dilatation $\mathbb{V}(\mathcal{E})'$ of $\mathbb{V}(\mathcal{E})$ at $\mathbb{V}(K)$. Indeed, by the universal property of dilatations, there is a morphism $\mathbb{V}(\mathcal{E}') \rightarrow \mathbb{V}(\mathcal{E})'$, which is an isomorphism away from ∞ .

To see that $\mathbb{V}(\mathcal{E}') \rightarrow \mathbb{V}(\mathcal{E})'$ is an isomorphism, it suffices to work over $\mathcal{O}_{X,\infty}$. Over this base, we may give a basis f_1, \dots, f_n of global sections of \mathcal{E} , with f_1, \dots, f_k lifting a basis for $K \subset \mathcal{E}_\infty$. Then the localization of $\mathbb{V}(\mathcal{E}') \rightarrow \mathbb{V}(\mathcal{E})'$ at ∞ is isomorphic to

$$\text{Spec } \mathcal{O}_{X,\infty} \left[\frac{f_1}{\xi}, \dots, \frac{f_k}{\xi}, f_{k+1}, \dots, f_n \right] \rightarrow \text{Spec } \mathcal{O}_{X,\infty} [f_1, \dots, f_n].$$

This agrees with the localization of $\mathbb{V}(\mathcal{E}') \rightarrow \mathbb{V}(\mathcal{E})$ at ∞ .

Lemma 4.3. *Let $V \rightarrow X$ be a smooth morphism, let $Y \subset V_\infty$ be a smooth locally closed subscheme, and let $\pi: V' \rightarrow V$ be the dilatation of V at Y . Then $V' \rightarrow X$ is smooth, and $\text{Tan}_{V'/X}$ lies in an exact sequence of*

$\mathcal{O}_{V'}$ -modules

$$0 \rightarrow \mathrm{Tan}_{V'/X} \rightarrow \pi^* \mathrm{Tan}_{V/X} \rightarrow \pi^* j_* N_{Y/V_\infty} \rightarrow 0, \quad (4.1.1)$$

where N_{Y/V_∞} is the normal bundle of $Y \subset V_\infty$, and $j: Y \rightarrow V$ is the inclusion.

Finally, let $T \rightarrow X$ be a morphism which is flat at ∞ , and let $s: T \rightarrow V$ be a morphism of X -schemes, such that s_∞ factors through Y . By the universal property of dilatations, s factors through a morphism $s': T \rightarrow V'$. Then we have an exact sequence of \mathcal{O}_V -modules

$$0 \rightarrow (s')^* \mathrm{Tan}_{V'/X} \rightarrow s^* \mathrm{Tan}_{V/X} \rightarrow i_{T_\infty} s_\infty^* N_{Y/V_\infty} \rightarrow 0. \quad (4.1.2)$$

Proof. One reduces to the case that Y is closed in V_∞ . The smoothness of $V' \rightarrow X$ is [BLR90, §3.2, Proposition 3]. We turn to the exact sequence (4.1.1). The morphism $\mathrm{Tan}_{V'/X} \rightarrow \pi^* \mathrm{Tan}_{V/X}$ comes from functoriality of the tangent bundle. To construct the morphism $\pi^* \mathrm{Tan}_{V/X} \rightarrow \pi^* j_* N_{Y/V_\infty}$, we consider the diagram

$$\begin{array}{ccc} V'_\infty & \xrightarrow{\pi'_\infty} & Y \\ \downarrow \pi_\infty & \searrow & \downarrow i_Y \\ V'_\infty & & V_\infty \\ \downarrow i_{V'} & & \downarrow i_V \\ V' & \xrightarrow{\pi} & V \end{array} \quad \begin{array}{c} \curvearrowright \\ j \end{array}$$

in which the outer rectangle is cartesian. For its part, the normal bundle N_{Y/V_∞} sits in an exact sequence of \mathcal{O}_Y -modules

$$0 \rightarrow \mathrm{Tan}_{Y/C} \rightarrow i_Y^* \mathrm{Tan}_{V_\infty/C} \rightarrow N_{Y/V_\infty} \rightarrow 0.$$

The composite

$$\begin{aligned} i_{V'}^* \pi^* \mathrm{Tan}_{V/X} &= \pi_\infty^* i_{V'}^* \mathrm{Tan}_{V/X} \\ &\cong \pi_\infty^* \mathrm{Tan}_{V_\infty/C} \\ &= (\pi'_\infty)^* i_Y^* \mathrm{Tan}_{V/C} \\ &\rightarrow (\pi'_\infty)^* N_{Y/V_\infty} \end{aligned}$$

induces by adjunction a morphism

$$\pi^* \mathrm{Tan}_{V/X} \rightarrow i_{V'}^* (\pi'_\infty)^* N_{Y/V_\infty} \cong \pi^* j_* N_{Y/V_\infty},$$

where the last step is justified because j is a closed immersion.

We check that (4.1.1) is exact using our explicit description of V' . The sequence is clearly exact away from the preimage of Y in V' , since on the complement of this locus, the morphism π is an isomorphism, and $\pi^* j_* = 0$. Therefore we let $y \in Y$ and check exactness after localization at y . Let $\mathcal{I} \subset \mathcal{O}_V$ be the ideal sheaf which cuts out Y , and let $I \subset \mathcal{O}_{V,y}$ be the localization of \mathcal{I} at y . Then $\mathcal{O}_{V_\infty,y} = \mathcal{O}_{V,y}/\xi$. Since $Y \subset V_\infty$ are both smooth at y , we can find a system of local coordinates $\bar{f}_1, \dots, \bar{f}_n \in \mathcal{O}_{V_\infty,y}$ (meaning that the differentials $d\bar{f}_i$ form a basis for $\Omega_{V_\infty/C,y}^1$), such that $\bar{f}_{k+1}, \dots, \bar{f}_n$ generate I/ξ . If $\partial/\partial \bar{f}_i$ are the dual basis, then the stalk of N_{Y/V_∞} at y is the free $\mathcal{O}_{Y,y}$ -module with basis $\partial/\partial \bar{f}_{k+1}, \dots, \partial/\partial \bar{f}_n$.

Choose lifts $f_i \in \mathcal{O}_{V,y}$ of the \bar{f}_i . Then I is generated by ξ, f_k, \dots, f_n . The localization of $V' \rightarrow V$ over y is $\mathrm{Spec} \mathcal{O}_{V',y}$, where $\mathcal{O}_{V',y} = \mathcal{O}_{V,y}[g_{k+1}, \dots, g_n]/(\xi\text{-torsion})$, where $\xi g_i = f_i$ for $i = k+1, \dots, n$. Then the stalk of $\mathrm{Tan}_{V'/X}$ at y is the free $\mathcal{O}_{V',y}$ -module with basis $\partial/\partial f_1, \dots, \partial/\partial f_k, \partial/\partial g_{k+1}, \dots, \partial/\partial g_n$, whereas the

stalk of $\pi^* \text{Tan}_{V/X}$ at y is the free $\mathcal{O}_{V',y}$ -module with basis $\partial/\partial f_1, \dots, \partial/\partial f_n$. The quotient between these stalks is evidently the free module over $\mathcal{O}_{V',y}/\xi$ with basis $\partial/\partial f_{k+1}, \dots, \partial/\partial f_n$, and this agrees with the stalk of $\pi^* j_* N_{Y/V_\infty}$.

Given a morphism of X -schemes $s: T \rightarrow V$ as in the lemma, we apply $(s')^*$ to (4.1.1); this is exact because s' is flat. The term on the right is $s^* j_* N_{Y/V_\infty} \cong i_{T_\infty} s_\infty^* N_{Y/V_\infty}$ (once again, this is valid because j is a closed immersion). \square

4.2 The space $\mathcal{M}_{H,\infty}$ as global sections of a scheme over X_C

We will prove Theorem 1.1 for the Rapoport-Zink spaces of the form $\mathcal{M}_{H,\infty}$ before proceeding to the general case. Let H be a p -divisible group of height n and dimension d over a perfect field k . In this context, $\check{E} = W(k)[1/p]$. Let $\mathcal{E} = \mathcal{E}_C(H)$. Throughout, we will be interpreting $\mathcal{M}_{H,\infty}$ as a functor on $\text{Perf}_{\check{E}}$ as in Proposition 2.3.

We have a determinant morphism $\det: \mathcal{M}_{H,\infty} \rightarrow \mathcal{M}_{\det H,\infty}$. Let $\tau \in \mathcal{M}_{\det H,\infty}(C)$ be a geometric point of $\mathcal{M}_{\det H,\infty}$. This point corresponds to a section τ of $\mathbb{V}(\det \mathcal{E}) \rightarrow X_C$, which we also call τ . Let $\mathcal{M}_{H,\infty}^\tau$ be the fiber of \det over τ .

Our first order of business is to express $\mathcal{M}_{H,\infty}^\tau$ as the space of global sections of a smooth morphism $Z \rightarrow X_C$, defined as follows. We have the geometric vector bundle $\mathbb{V}(\mathcal{E}^n) \rightarrow X$, whose global sections parametrize morphisms $s: \mathcal{O}_{X_C}^n \rightarrow \mathcal{E}$. Let U_{n-d} be the locally closed subscheme of the fiber of $\mathbb{V}(\mathcal{E}^n)$ over ∞ , which parametrizes all morphisms of rank $n-d$. We consider the dilatation $\mathbb{V}(\mathcal{E}^n)^{\text{rk}_\infty = n-d} \rightarrow \mathbb{V}(\mathcal{E}^n)$ of $\mathbb{V}(\mathcal{E}^n)$ along U_{n-d} . For any flat X_C -scheme T , $\mathbb{V}(\mathcal{E}^n)^{\text{rk}_\infty = n-d}(T)$ is the set of all $s: \mathcal{O}_T^n \rightarrow \mathcal{E}_T$ such that $\text{cok}(s) \otimes C$ is projective $\mathcal{O}_T \otimes C$ -module of rank d . Define Z as the Cartesian product:

$$\begin{array}{ccc} Z & \longrightarrow & \mathbb{V}(\mathcal{E}^n)^{\text{rk}_\infty = n-d} \\ \downarrow & & \downarrow \det \\ X_C & \xrightarrow{\tau} & \mathbb{V}(\det \mathcal{E}). \end{array} \quad (4.2.1)$$

Lemma 4.4. *Let \mathcal{M}_Z be the functor which inputs a perfectoid space T/C and outputs the set of sections of $Z \rightarrow X_C$ over X_T . Then \mathcal{M}_Z is isomorphic to $\mathcal{M}_{H,\infty}^\tau$.*

Proof. Let $T = \text{Spa}(R, R^+)$ be an affinoid perfectoid space over C . The morphism $X_T \rightarrow X_C$ is flat. (This can be checked locally: $B_{\text{dR}}^+(R)$ is torsion-free over the discrete valuation ring $B_{\text{dR}}^+(C)$, and so it is flat.) By the description in (4.2.1), an X_T -point of \mathcal{M}_Z corresponds to a morphism $F: \mathcal{O}_{X_T}^n \rightarrow \mathcal{E}_T(H)$ which has the properties:

- (1) The cokernel of F_∞ is a projective R -module quotient of $\mathcal{E}_T(H)_\infty$ of rank d .
- (2) The determinant of F equals τ .

On the other hand, by Proposition 2.2, $\mathcal{M}_{H,\infty}(T)$ is the set of morphisms $F: \mathcal{O}_{X_T}^n \rightarrow \mathcal{E}_T(H)$ satisfying

- (1') The cokernel of F is $i_{\infty} W$, for a projective R -module quotient W of $\mathcal{E}_T(H)_\infty$ of rank d .
- (2) The determinant of F equals τ .

We claim the two sets of conditions are equivalent for a morphism $F: \mathcal{O}_{X_T}^n \rightarrow \mathcal{E}_T(H)$. Clearly (1') implies (1), so that (1') and (2) together imply (1) and (2) together. Conversely, suppose (1) and (2) hold. Since τ represents a point of $\mathcal{M}_{\det H,\infty}$, it is an isomorphism outside of ∞ , and therefore so is F . This means that $\text{cok } F$ is supported at ∞ . Thus $\text{cok } F$ is a $B_{\text{dR}}^+(R)$ -module. For degree reasons, the length of $(\text{cok } F) \otimes_{B_{\text{dR}}^+(R)} B_{\text{dR}}^+(C')$

has length d for every geometric point $\mathrm{Spa}(C', (C')^+) \rightarrow T$. Whereas condition (1) says that $(\mathrm{cok} F) \otimes_{B_{\mathrm{dR}}^+(R)} R$ is a projective R -module of rank d . This shows that $(\mathrm{cok} F)$ is already a projective R -module of rank d , which is condition (1'). \square

Lemma 4.5. *The morphism $Z \rightarrow X_C$ is smooth.*

Proof. Let $\omega' \in X_C$ be a closed point, with residue field C' . It suffices to show that the stalk of Z at ω' is smooth over $\mathrm{Spec} B_{\mathrm{dR}}^+(C')$.

If $\omega' \neq \omega$, then this stalk is isomorphic to the variety $(\mathbb{A}^{n^2})^{\det=\tau}$ consisting of $n \times n$ matrices with fixed determinant τ . Since τ is invertible in $B_{\mathrm{dR}}^+(C')$, this variety is smooth.

Now suppose $\omega' = \omega$. Let ξ be a generator for the kernel of $B_{\mathrm{dR}}^+(C) \rightarrow C$. Then the stalk of Z at ω is isomorphic to the flat $B_{\mathrm{dR}}^+(C)$ -scheme Y , whose T -points for a flat $B_{\mathrm{dR}}^+(C)$ -scheme T are $n \times n$ matrices with coefficients in $\Gamma(T, \mathcal{O}_T)$, which are rank $n - d$ modulo ξ , and which have fixed determinant τ (which must equal $u\xi^d$ for a unit $u \in B_{\mathrm{dR}}^+(C)$). Consider the open subset $Y_0 \subset Y$ consisting of matrices M where the first $(n - d)$ columns have rank $(n - d)$. Then the final d columns of M are congruent modulo ξ to a linear combination of the first $(n - d)$ columns. After row reduction operations only depending on those first $(n - d)$ columns, M becomes

$$\left(\begin{array}{c|c} I_{n-d} & P \\ \hline 0 & \xi Q \end{array} \right),$$

with $\det Q = w$ for a unit $w \in B_{\mathrm{dR}}^+(C)$ which only depends on the first $(n - d)$ columns of M . We therefore have a fibration $Y_0 \rightarrow \mathbb{A}^{n(n-d)}$, namely projection onto the first $(n - d)$ columns, whose fibers are $\mathbb{A}^{d(n-d)} \times (\mathbb{A}^{d^2})^{\det=w}$, which is smooth. Therefore Y_0 is smooth. The variety Y is covered by opens isomorphic to Y_0 , and so it is smooth. \square

We intend to apply Theorem 3.3 to the morphism $Z \rightarrow X$, and so we need some preparations regarding the relative tangent space of $\mathbb{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty=n-d} \rightarrow X_C$.

4.3 A linear algebra lemma

Let $f: V' \rightarrow V$ be a rank r linear map between n -dimensional vector spaces over a field C . Thus there is an exact sequence

$$0 \rightarrow W' \rightarrow V' \xrightarrow{f} V \xrightarrow{q} W \rightarrow 0.$$

with $\dim W = \dim W' = n - r$.

Consider the minor map $\Lambda: \mathrm{Hom}(V', V) \rightarrow \mathrm{Hom}(\bigwedge^{r+1} V', \bigwedge^{r+1} V)$ given by $F \mapsto \bigwedge^{r+1} F$. This is a polynomial map, whose derivative at f is a linear map

$$D_f \Lambda: \mathrm{Hom}(V', V) \rightarrow \mathrm{Hom} \left(\bigwedge^{r+1} V', \bigwedge^{r+1} V \right).$$

Explicitly, this map is

$$D_f \Lambda(F)(v_1 \wedge \cdots \wedge v_{r+1}) = \sum_{i=1}^{r+1} f(v_1) \wedge \cdots \wedge f(v_{i-1}) \wedge F(v_i) \wedge \cdots \wedge f(v_{r+1}).$$

Lemma 4.6. *Let*

$$K = \ker (\mathrm{Hom}(V', V) \rightarrow \mathrm{Hom}(W', W))$$

be the kernel of the map $F \mapsto q \circ (F|_{W'})$. Then $\ker D_f \Lambda = K$.

Proof. Suppose $F \in K$. Since f has rank r , the exterior power $\bigwedge^{r+1} V'$ is spanned over C by elements of the form $v_1 \wedge \cdots \wedge v_{r+1}$, where $v_{r+1} \in \ker f = W'$. Since $f(v_{r+1}) = 0$, the sum in (4.3) reduces to

$$D_f \Lambda(F)(v_1 \wedge \cdots \wedge v_{r+1}) = f(v_1) \wedge \cdots \wedge f(v_r) \wedge F(v_{r+1}).$$

Since $F \in K$ and $v_{r+1} \in W'$ we have $F(v_{r+1}) \in \ker q = f(V')$, which means that $D_f \Lambda(F) \in \bigwedge^{r+1} f(V') = 0$. Thus $F \in \ker D_f \Lambda$.

Now suppose $F \in \ker D_f \Lambda$. Let $w \in W'$. We wish to show that $F(w) \in f(V')$. Let $v_1, \dots, v_r \in V'$ be vectors for which $f(v_1), \dots, f(v_r)$ is a basis for $f(V')$. Since $F \in \ker D_f \Lambda$, we have $D_f \Lambda(F)(v_1 \wedge \cdots \wedge v_r \wedge w) = 0$. On the other hand,

$$D_f \Lambda(F)(v_1 \wedge \cdots \wedge v_r \wedge w) = f(v_1) \wedge \cdots \wedge f(v_r) \wedge F(w),$$

because all other terms in the sum in (4.3) are 0, owing to $f(w) = 0$. Since the wedge product above is 0, and the $f(v_i)$ are a basis for $f(V)$, we must have $F(w) \in f(V')$. Thus $F \in K$. \square

We interpret Lemma 4.6 as the calculation of a certain normal bundle. Let $Y = \mathbb{V}(\text{Hom}(V', V))$ be the affine space over C representing morphisms $V' \rightarrow V$ over a C -scheme, and let $j: Y^{\text{rk}=r} \rightarrow Y$ be the locally closed subscheme representing morphisms which are everywhere of rank r . Thus, $Y^{\text{rk}=r}$ is an open subset of the fiber over 0 of (the geometric version of) the minor map Λ . It is well known that $Y^{\text{rk}=r}/C$ is smooth of codimension $(n-r)^2$ in Y/C , and so the normal bundle $N_{Y^{\text{rk}=r}/Y}$ is locally free of this rank.

We have a universal morphism of $\mathcal{O}_{Y^{\text{rk}=r}}$ -modules $F: \mathcal{O}_{Y^{\text{rk}=r}} \otimes_C V' \rightarrow \mathcal{O}_{Y^{\text{rk}=r}} \otimes_C V$. Let $\mathcal{W}' = \ker F$ and $\mathcal{W} = \text{cok } F$, so that \mathcal{W}' and \mathcal{W} are locally free $\mathcal{O}_{Y^{\text{rk}=r}}$ -modules of rank $n-r$. We also have the $\mathcal{O}_{Y^{\text{rk}=r}}$ -linear morphism $D\Lambda: \mathcal{O}_{Y^{\text{rk}=r}} \otimes_C \text{Hom}(V', V) \rightarrow \mathcal{O}_{Y^{\text{rk}=r}} \otimes_C \text{Hom}(\Lambda^{r+1} V', \Lambda^{r+1} V)$, whose kernel is precisely $\text{Tan}_{Y^{\text{rk}=r}/C}$. The geometric interpretation of Lemma 4.6 is a commutative diagram with short exact rows:

$$\begin{array}{ccccc} \ker D\Lambda & \longrightarrow & \mathcal{O}_{Y^{\text{rk}=r}} \otimes_C \text{Hom}(V', V) & \longrightarrow & \mathcal{H}om(\mathcal{W}', \mathcal{W}) \\ \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ \text{Tan}_{Y^{\text{rk}=r}/C} & \longrightarrow & j^* \text{Tan}_{Y/C} & \longrightarrow & N_{Y^{\text{rk}=r}/Y}. \end{array} \quad (4.3.1)$$

4.4 Moduli of morphisms of vector bundles with fixed rank at ∞

We return to the setup of §4.1. We have a curve X and a closed point $\infty \in X$, with inclusion map i_∞ and residue field C .

Let \mathcal{E} and \mathcal{E}' be rank n vector bundles over X , with fibers $V = \mathcal{E}_\infty$ and $V' = \mathcal{E}'_\infty$. We have the geometric vector bundle $\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E})) \rightarrow X$. If $f: T \rightarrow X$ is a morphism, then T -points of $\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))$ classify \mathcal{O}_T -linear maps $f^* \mathcal{E}' \rightarrow f^* \mathcal{E}$.

Let $\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\text{rk}_\infty=r}$ be the dilatation of $\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))$ at the locally closed subscheme $\mathbb{V}(\text{Hom}(V', V))^{\text{rk}=r}$ of the fiber $\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))_\infty = \mathbb{V}(\text{Hom}(V', V))$. This has the following property, for a flat morphism $f: T \rightarrow X$: the X -morphisms $s: T \rightarrow \mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\text{rk}_\infty=r}$ parametrize those \mathcal{O}_T -linear maps $F: f^* \mathcal{E}' \rightarrow f^* \mathcal{E}$, for which the fiber $F_\infty: f_\infty^* V' \rightarrow f_\infty^* V$ has rank r everywhere on T_∞ .

Given a morphism s as above, corresponding to a morphism $F: f^* \mathcal{E}' \rightarrow f^* \mathcal{E}$, we let \mathcal{W}' and \mathcal{W} denote the kernel and cokernel of F_∞ . Then \mathcal{W}' and \mathcal{W} are locally free \mathcal{O}_{T_∞} -modules of rank r . Let $i_{T_\infty}: T_\infty \rightarrow T$ denote the pullback of i_∞ through f .

We intend to use Lemma 4.3 to compute $s^* \text{Tan}_{\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\text{rk}_\infty=r}/X}$. The tangent bundle $\text{Tan}_{\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))/X}$ is isomorphic to the pullback $f^* \mathcal{H}om(\mathcal{E}', \mathcal{E})$. Also, we have identified the normal bundle $N_{\mathbb{V}(\text{Hom}(V', V))^{\text{rk}=r}/\mathbb{V}(\text{Hom}(V', V))}$

in (4.3.1). So when we apply the lemma to this situation, we obtain an exact sequence of \mathcal{O}_T -modules

$$0 \rightarrow s^* \mathrm{Tan}_{\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_{\mathcal{O}}=r}/X} \rightarrow f^* \mathcal{H}om(\mathcal{E}', \mathcal{E}) \rightarrow i_{T_{\mathcal{O}}}^* \mathcal{H}om(\mathcal{W}', \mathcal{W}) \rightarrow 0, \quad (4.4.1)$$

where the third arrow is adjoint to the map

$$i_{T_{\mathcal{O}}}^* f^* \mathcal{H}om(\mathcal{E}', \mathcal{E}) = \mathrm{Hom}(f_{\mathcal{O}}^* V', f_{\mathcal{O}}^* V) \rightarrow \mathcal{H}om(\mathcal{W}', \mathcal{W}),$$

which sends $F \in \mathcal{H}om(f_{\mathcal{O}}^* V', f_{\mathcal{O}}^* V)$ to the composite

$$\mathcal{W}' \rightarrow f_{\mathcal{O}}^* V' \xrightarrow{F_{\mathcal{O}}} f_{\mathcal{O}}^* V \rightarrow \mathcal{W}.$$

The short exact sequence in (4.4.1) identifies the \mathcal{O}_T -module $s^* \mathrm{Tan}_{\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_{\mathcal{O}}=r}/X}$ as a modification of $f^* \mathcal{H}om(\mathcal{E}', \mathcal{E})$ at the divisor $T_{\mathcal{O}}$. We can say a little more in the case that F itself is a modification. Let us assume that F fits into an exact sequence

$$0 \rightarrow \mathcal{E}' \xrightarrow{F} \mathcal{E} \xrightarrow{\alpha} i_{T_{\mathcal{O}}}^* \mathcal{W} \rightarrow 0.$$

Dualizing gives another exact sequence

$$0 \rightarrow \mathcal{E}^{\vee} \xrightarrow{F^{\vee}} (\mathcal{E}')^{\vee} \xrightarrow{\alpha'} i_{T_{\mathcal{O}}}^*(\mathcal{W}')^{\vee} \rightarrow 0.$$

Then

$$\begin{aligned} s^* \mathrm{Tan}_{\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_{\mathcal{O}}=r}/X} &= \ker [\mathcal{H}om(\mathcal{E}', \mathcal{E}) \rightarrow i_{T_{\mathcal{O}}}^* \mathcal{H}om(\mathcal{W}', \mathcal{W})] \\ &\cong \ker(\alpha \otimes \alpha') \end{aligned}$$

The kernel of $\alpha \otimes \alpha'$ can be computed in terms of $\ker \alpha = \mathcal{E}'$ and $\ker \alpha' = \mathcal{E}^{\vee}$, see Lemma 4.7 below. It sits in a diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{H}om(\mathcal{E}, \mathcal{E}') & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{E}', \mathcal{E}') & \longrightarrow & s^* \mathrm{Tan}_{\mathbb{V}(\mathcal{H}om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_{\mathcal{O}}=r}/X} \longrightarrow 0. \\ & & \downarrow & & & & \\ & & \mathrm{Tor}_1(i_{\mathcal{O}*} \mathcal{W}', i_{\mathcal{O}*} \mathcal{W}) & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array} \quad (4.4.2)$$

Lemma 4.7. *Let \mathcal{A} be an abelian \otimes -category. Let*

$$0 \rightarrow K \xrightarrow{i} A \xrightarrow{f} B \rightarrow 0$$

$$0 \rightarrow K' \xrightarrow{i'} A' \xrightarrow{f'} B' \rightarrow 0$$

be two exact sequences in \mathcal{A} , with A, A', K, K' projective. The homology of the complex

$$K \otimes K' \xrightarrow{(i \otimes 1_{K'}, 1_K \otimes i')} (A \otimes K') \oplus (K \otimes A) \xrightarrow{1_A \otimes i' - i \otimes 1_{A'}} A \otimes A'$$

is given by $H_2 = 0$, $H_1 \cong \text{Tor}_1(B, B')$, and $H_0 \cong B \otimes B'$. Thus, $K'' = \ker(f \otimes f': A \otimes A' \rightarrow B \otimes B')$ appears in a diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & K \otimes K' & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & F & \longrightarrow & (A \otimes K') \oplus (K \otimes A) & \longrightarrow & K'' \longrightarrow 0 \\ & & \downarrow & & & & \\ & & \text{Tor}_1(B, B') & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where both sequences are exact.

Proof. Let C_\bullet be the complex $K \rightarrow A$, and let C'_\bullet be the complex $K' \rightarrow A'$. Since C'_\bullet is a projective resolution of B' , we have a Tor spectral sequence [Sta14, Tag 061Z]

$$E_{i,j}^2: \text{Tor}_j(H_i(C_\bullet), B') \Longrightarrow H_{i+j}(C_\bullet \otimes C'_\bullet).$$

We have $E_{0,0}^2 = B \otimes B'$ and $E_{0,1}^2 = \text{Tor}_1(B, B')$, and $E_{i,j}^2 = 0$ for all other (i, j) . Therefore $H_0(C_\bullet \otimes C'_\bullet) \cong B \otimes B'$ and $H_1(C_\bullet \otimes C'_\bullet) \cong \text{Tor}_1(B, B')$, which is the lemma. \square

4.5 A tangent space calculation

We return to the setup of §4.2. Thus we have fixed a p -divisible group H over a perfect field k , and an algebraically closed perfectoid field C containing $W(k)[1/p]$. But now we specialize to the case that H is isoclinic. Therefore $D = \text{End } H$ (up to isogeny) is a central simple \mathbf{Q}_p -algebra. Let $\mathcal{E} = \mathcal{E}_C(H)$; we have $\mathcal{H}em(\mathcal{E}, \mathcal{E}) \cong D \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$.

Recall the scheme $Z \rightarrow X_C$, defined as a fiber product in (4.2.1). Let $s: X_C \rightarrow Z$ be a section. This corresponds to a morphism $F: \mathcal{O}_{X_C}^n \rightarrow \mathcal{E}$. Let W' and W be the cokernel of F_∞ ; these are C -vector spaces.

We are interested in the vector bundle $s^* \text{Tan}_{Z/X_C}$. This is the kernel of the derivative of the determinant map:

$$s^* \text{Tan}_{Z/X_C} = \ker(D_s \det: s^* \text{Tan}_{\mathbb{V}(\mathcal{E}^n)_{\text{rk}_\infty = n-d}/X_C} \rightarrow \det \mathcal{E}).$$

We apply (4.4.2) to give a description of $s^* \text{Tan}_{\mathbb{V}(\mathcal{E}^n)^{\text{rk}_{\infty} = n-d}/X_C}$. We get a diagram of \mathcal{O}_{X_C} -modules

$$\begin{array}{ccccccc}
& & 0 & & & & (4.5.1) \\
& & \downarrow & & & & \\
& & (\mathcal{E}^\vee)^n & & & & \\
& & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & (M_n(\mathbf{Q}_p) \times D) \otimes \mathcal{O}_{X_C} & \longrightarrow & s^* \text{Tan}_{\mathbb{V}(\mathcal{E}^n)^{\text{rk}_{\infty} = n-d}/X_C} \longrightarrow 0. \\
& & \downarrow & & & & \\
& & \text{Tor}_1(i_{\infty*} W', i_{\infty*} W) & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

On the other hand, the horizontal exact sequence fits into a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & (M_n(\mathbf{Q}_p) \times D) \otimes \mathcal{O}_{X_C} & \longrightarrow & s^* \text{Tan}_{\mathbb{V}(\mathcal{E}^n)^{\text{rk}_{\infty} \leq n-d}/X_C} \longrightarrow 0 & (4.5.2) \\
& & & & \downarrow \text{tr} & & \downarrow D_s \det & \\
& & & & \mathcal{O}_{X_C} & \xrightarrow{\tau} & \det \mathcal{E} &
\end{array}$$

The arrow labeled tr is induced from the \mathbf{Q}_p -linear map $M_n(\mathbf{Q}_p) \times D \rightarrow \mathbf{Q}_p$ carrying (α', α) to $\text{tr}(\alpha') - \text{tr}(\alpha)$ (reduced trace on D). The commutativity of the lower right square boils down to the identity, valid for sections $s_1, \dots, s_n \in H^0(X_C, \mathcal{E})$ and $\alpha \in D$:

$$((\alpha s_1) \wedge s_2 \wedge \dots \wedge s_n) + \dots + (s_1 \wedge \dots \wedge (\alpha s_n)) = (\text{tr } \alpha)(s_1 \wedge \dots \wedge s_n).$$

(There is a similar identity for $\alpha' \in M_n(\mathbf{Q}_p)$.) Because the arrow labeled τ is injective, we can combine (4.5.1) and (4.5.2) to arrive at a description of $s^* \text{Tan}_{Z/X_C}$:

$$\begin{array}{ccccccc}
& & 0 & & & & (4.5.3) \\
& & \downarrow & & & & \\
& & (\mathcal{E}^\vee)^n & & & & \\
& & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & (M_n(\mathbf{Q}_p) \times D)^{\text{tr}=0} \otimes \mathcal{O}_X & \longrightarrow & s^* \text{Tan}_{Z/X_C} \longrightarrow 0. \\
& & \downarrow & & & & \\
& & \text{Tor}_1(i_{\infty*} W', i_{\infty*} W) & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

We pass to duals to obtain

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
0 & \longrightarrow & (s^* \text{Tan}_{Z/X_C})^\vee & \longrightarrow & ((M_n(\mathbf{Q}_p) \times D)/\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & 0 \\
& & & & \searrow \text{dotted} & & \downarrow & & \\
& & & & & & \mathcal{E}^n & & \\
& & & & & & \downarrow & & \\
& & & & & & \text{Tor}_1(i_{\mathcal{O}*}((W')^\vee, i_{\mathcal{O}*}W^\vee)) & & \\
& & & & & & \downarrow & & \\
& & & & & & 0 & & \\
& & & & & & & & (4.5.4)
\end{array}$$

The dotted arrow is induced from the map $(M_n(\mathbf{Q}_p) \times D) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \rightarrow \mathcal{E}^n$ sending $(\alpha', \alpha) \otimes 1$ to $\alpha \circ F - F \circ \alpha'$.

Theorem 4.8. *The following are equivalent for a morphism $F: \mathcal{O}_{X_C}^n \rightarrow \mathcal{E}$ corresponding to a point x of $\mathcal{M}_{H,\mathcal{O}}^\tau$:*

1. *The vector bundle $s^* \text{Tan}_{Z/X_C}$ has a Harder-Narasimhan slope which is ≤ 0 .*
2. *The point x lies in the special locus $\mathcal{M}_{H,\mathcal{O}}^{\tau,\text{sp}}$.*

Proof. Condition (1) is true if and only if $H^0(X_C, s^* \text{Tan}_{Z/X_C}^\vee) \neq 0$. We now take H^0 of (4.5.4), noting that $H^0(X_C, \mathcal{F}^\vee) \rightarrow H^0(X_C, \mathcal{E}^n)$ is injective. We find that

$$\begin{aligned}
H^0(X_C, s^* \text{Tan}_{Z/X_C}^\vee) &\cong \left\{ (\alpha', \alpha) \in M_n(\mathbf{Q}_p) \times D \mid \alpha \circ F = F \circ \alpha' \right\} / \mathbf{Q}_p. \\
&= A_x / \mathbf{Q}_p.
\end{aligned}$$

This is nonzero exactly when x lies in the special locus. □

Combining Theorem 4.8 with the criterion for cohomological smoothness in Theorem 3.3 proves Theorem 1.1 for the space $\mathcal{M}_{H,\mathcal{O}}$.

Naturally we wonder whether it is possible to give a complete description of $s^* \text{Tan}_{Z/X_C}$, as this is the “tangent space” of $\mathcal{M}_{H,\mathcal{O}}^\tau$ at the point x . Note that $s^* \text{Tan}_{Z/X_C}$ can only have nonnegative slopes, since it is a quotient of a trivial bundle. Therefore Theorem 4.8 says that 0 appears as a slope of $s^* \text{Tan}_{Z/X_C}$ if and only if s corresponds to a special point of $\mathcal{M}_{H,\mathcal{O}}^\tau$.

Example 4.9. Consider the case that H has dimension 1 and height n , so that $\mathcal{M}_{H,\mathcal{O}}$ is an infinite-level Lubin-Tate space. Suppose that $x \in \mathcal{M}_{H,\mathcal{O}}(C)$ corresponds to a section $s: X_C \rightarrow Z$. Then $s^* \text{Tan}_{Z/X_C}$ is a vector bundle of rank $n^2 - 1$ and degree $n - 1$, with slopes lying in $[0, 1/n]$; this already limits the possibilities for the slopes to a finite list.

If $n = 2$ there are only two possibilities for the slopes appearing in $s^* \text{Tan}_{Z/X_C}$: $\{1/3\}$ and $\{0, 1/2\}$. These correspond exactly to the nonspecial and special loci, respectively.

If $n = 3$, there are a priori five possibilities for the slopes appearing in $s^* \text{Tan}_{Z/X_C}$: $\{1/4, 1/4\}$, $\{1/3, 1/5\}$, $\{1/3, 1/3, 0, 0\}$, $\{2/7, 0\}$, and $\{1/3, 1/4, 0\}$. But in fact the final two cases cannot occur: if 0 appears as a slope, then x lies in the special locus, so that $A_x \neq \mathbf{Q}_p$. But as A_x is isomorphic to a subalgebra of $\text{End}^\circ H$, the division algebra of invariant $1/3$, it must be the case that $\dim_{\mathbf{Q}_p} A_x = 3$, which forces 0 to appear as a

slope with multiplicity $\dim_{\mathbf{Q}_p} A_x/\mathbf{Q}_p = 2$. On the nonspecial locus, we suspect that the generic (semistable) case $\{1/4, 1/4\}$ always occurs, as otherwise there would be some unexpected stratification of $\mathcal{M}_{H,\infty}^{\text{non-sp}}$. But currently we do not know how to rule out the case $\{1/3, 1/5\}$.

4.6 The general case

Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum over k , with reflex field E . Let $F = Z(B)$. As in Section 2.5, let $D = \text{End}_F V$ and $D' = \text{End}_F H$, so that D and D' are both B -algebras.

Let C be a perfectoid field containing \check{E} , and let $\tau \in \mathcal{M}_{\det \mathcal{D}, \infty}^\tau(C)$. Let $\mathcal{M}_{\mathcal{D}, \infty}^\tau$ be the fiber of the determinant map over τ . We will sketch the proof that $\mathcal{M}_{\mathcal{D}, \infty}^\tau \rightarrow \text{Spa } C$ is cohomologically smooth. It is along the same lines as the proof for $\mathcal{M}_{H, \infty}$, but with some extra linear algebra added.

The space $\mathcal{M}_{\mathcal{D}, \infty}^\tau$ may be expressed as the space of global sections of a smooth morphism $Z \rightarrow X_C$, defined as follows. We have the geometric vector bundle $\mathbb{V}(\mathcal{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_X, \mathcal{E}_C(H)))$. In its fiber over ∞ , we have the locally closed subscheme whose R -points for a C -algebra R are morphisms, whose cokernel is as a $B \otimes_{\mathbf{Q}_p} R$ -module isomorphic to $V_0 \otimes_{\check{E}} R$, where V_0 is the weight 0 subspace of $V \otimes_{\mathbf{Q}_p} \check{E}$ determined by μ . We then have the dilatation $\mathbb{V}(\mathcal{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)))^\mu$ of $\mathbb{V}(\mathcal{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_X, \mathcal{E}_C(H)))$ at this locally closed subscheme. Its points over $S = \text{Spa}(R, R^+)$ parametrize B -linear morphisms $s: V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \rightarrow \mathcal{E}_S(H)$, such that (locally on S) the cokernel of the fiber s_∞ is isomorphic as a $(B \otimes_{\mathbf{Q}_p} R)$ -module to $V_0 \otimes_{\check{E}} R$. Finally, the morphism $Z \rightarrow X_C$ is defined by the cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & \mathbb{V}(\mathcal{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)))^\mu \\ \downarrow & & \downarrow \text{det} \\ X_C & \xrightarrow{\tau} & \mathbb{V}(\mathcal{H}om_F(\det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \det_F \mathcal{E}_C(H))). \end{array}$$

Let $x \in \mathcal{M}_{\mathcal{D}, \infty}^\tau(C)$ correspond to a B -linear morphism $s: V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \rightarrow \mathcal{E}_C(H)$ and a section of $Z \rightarrow X_C$ which we also call s . Define $B \otimes_{\mathbf{Q}_p} C$ -modules W' and W by

$$0 \rightarrow W' \rightarrow V \otimes_{\mathbf{Q}_p} C \xrightarrow{s_x} \mathcal{E}_C(H)_x \rightarrow W \rightarrow 0.$$

The analogue of (4.5.4) is a diagram which computes the dual of $s^* \text{Tan}_{Z/X_C}$:

$$\begin{array}{ccccccc} & & & & 0 & & (4.6.1) \\ & & & & \downarrow & & \\ 0 & \longrightarrow & (s^* \text{Tan}_{Z/X_C})^\vee & \longrightarrow & ((D' \times D)/F) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} & \longrightarrow & \mathcal{F}^\vee \longrightarrow 0 \\ & & & \searrow & \downarrow & & \\ & & & & \mathcal{H}om(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)) & & \\ & & & & \downarrow & & \\ & & & & \text{Tor}_1^F(i_{x*}((W')^\vee, i_{x*} W^\vee)) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

This time, the dotted arrow is induced from the map $(D' \times D) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \rightarrow \mathcal{H}om(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H))$ sending $(\alpha', \alpha) \otimes 1$ to $\alpha \circ s - s \circ \alpha'$. Taking H^0 in (4.6.1) shows that $H^0(X_C, s^* \text{Tan}_{Z/X_C}^\vee) = A_x/F$, and this

is nonzero exactly when x lies in the special locus.

4.7 Proof of Corollary 1.2

We conclude with a discussion of the infinite-level modular curve $X(p^\infty)$. Recall from [Sch15] the following facts about the Hodge-Tate period map $\pi_{HT}: X(p^\infty) \rightarrow \mathbb{P}^1$. The ordinary locus in $X(p^\infty)$ is sent to $\mathbb{P}^1(\mathbf{Q}_p)$. The supersingular locus is isomorphic to finitely many copies of $\mathcal{M}_{H,\infty}$, where H is a connected p -divisible group of height 2 and dimension 1 over the residue field of C ; the restriction of π_{HT} to this locus agrees with the π_{HT} we had already defined on each $\mathcal{M}_{H,\infty}$.

We claim that the following are equivalent for a C -point x of $X(p^\infty)^\circ$:

1. The point x corresponds to an elliptic curve E/\mathcal{O}_C , such that the p -divisible group $E[p^\infty]$ has $\text{End } E[p^\infty] = \mathbf{Z}_p$.
2. The stabilizer of $\pi_{HT}(x)$ in $\text{PGL}_2(\mathbf{Q}_p)$ is trivial.
3. There is a neighborhood of x in $X(p^\infty)^\circ$ which is cohomologically smooth over C .

First we discuss the equivalence of (1) and (2). If E is ordinary, then $E[p^\infty] \cong \mathbf{Q}_p/\mathbf{Z}_p \times \mu_{p^\infty}$ certainly has endomorphism ring larger than \mathbf{Z}_p , so that (1) is false. Meanwhile, the stabilizer of $\pi_{HT}(x) \in \text{PGL}_2(\mathbf{Q}_p)$ is a Borel subgroup, so that (2) is false as well. The equivalence between (1) and (2) in the supersingular case is a special case of the equivalence discussed in Section 2.5.

Theorem 1.1 tells us that $\mathcal{M}_{H,\infty}^{\circ,\text{non-sp}}$ is cohomologically smooth, which implies that shows that (2) implies (3). We therefore are left with showing that if (2) is false for a point $x \in X(p^\infty)^\circ$, then no neighborhood of x is cohomologically smooth.

First suppose that x lies in the ordinary locus. This locus is fibered over $\mathbb{P}^1(\mathbf{Q}_p)$. Suppose U is a sufficiently small neighborhood of x . Then U is contained in the ordinary locus, and so $\pi_0(U)$ is nondiscrete. This implies that $H^0(U, \mathbb{F}_\ell)$ is infinite, and so U cannot be cohomologically smooth.

Now suppose that x lies in the supersingular locus, and that $\pi_{HT}(x)$ has nontrivial stabilizer in $\text{PGL}_2(\mathbf{Q}_p)$. We can identify x with a point in $\mathcal{M}_{H,\infty}^{\circ,\text{sp}}(C)$. We intend to show that every neighborhood of x in $\mathcal{M}_{H,\infty}^\circ$ fails to be cohomologically smooth.

Not knowing a direct method, we appeal to the calculations in [Wei16], which constructed semistable formal models for each $\mathcal{M}_{H,m}^\circ$. The main result we need is Theorem 5.1.2, which uses the term ‘‘CM points’’ for what we have called special points. There exists a decreasing basis of neighborhoods $Z_{x,0} \supset Z_{x,1} \supset \dots$ of x in $\mathcal{M}_{H,\infty}^\circ$. For each affinoid $Z = \text{Spa}(R, R^+)$, let $\bar{Z} = \text{Spec } R^+ \otimes_{\mathcal{O}_C} \kappa$, where κ is the residue field of C . For each $m \geq 0$, there exists a nonconstant morphism $\bar{Z}_{x,m} \rightarrow C_{x,m}$, where $C_{x,m}$ is an explicit nonsingular affine curve over κ . This morphism is equivariant for the action of the stabilizer of $Z_{x,m}$ in $\text{SL}_2(\mathbf{Q}_p)$. For infinitely many m , the completion $C_{x,m}^{\text{cl}}$ of $C_{x,m}$ is a projective curve with positive genus.

Let $U \subset \mathcal{M}_{H,\infty}^\circ$ be an affinoid neighborhood of x . Then there exists $N \geq 0$ such that $Z_{x,m} \subset U$ for all $m \geq N$. Let $K \subset \text{SL}_2(\mathbf{Q}_p)$ be a compact open subgroup which stabilizes U , so that U/K is an affinoid subset of the rigid-analytic curve $\mathcal{M}_{H,\infty}^\circ/K$. For each $m \geq N$, let $K_m \subset K$ be the stabilizer of $Z_{x,m}$, so that K_m acts on $C_{x,m}$.

There exists an integral model of U/K whose special fiber contains as a component the completion of each $\bar{Z}_{x,m}/K_m$ which has positive genus. Since there is a nonconstant morphism $\bar{Z}_{x,m}/K_m \rightarrow C_{x,m}/K_m$, we must have

$$\dim_{\mathbb{F}_\ell} H^1(U/K, \mathbb{F}_\ell) \geq \sum_{m \geq N} \dim_{\mathbb{F}_\ell} H^1(C_{x,m}^{\text{cl}}/K_m, \mathbb{F}_\ell).$$

Now we take a limit as K shrinks. Since $U \sim \varprojlim U/K$, we have $H^1(U, \mathbb{F}_\ell) \cong \varinjlim H^1(U/K, \mathbb{F}_\ell)$. Also, for each m , the action of K_m on $C_{x,m}$ is trivial for all sufficiently small K . Therefore

$$\dim_{\mathbb{F}_\ell} H^1(U, \mathbb{F}_\ell) \geq \sum_{m \geq N} \dim_{\mathbb{F}_\ell} H^1(C_{x,m}^{\text{cl}}, \mathbb{F}_\ell) = \infty.$$

This shows that U is not cohomologically smooth.

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