Prof. Dr. P. Scholze **WS 2016/17**
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Algebraic Geometry I

1. Exercise sheet

Let k be an algebraically closed field and denote by $\mathbb{A}^n(k) = k^n$ the affine n-space over k together with its Zariski topology.

Exercise 1 (4 Points):

1) Let $Z \subseteq \mathbb{A}^n(k)$ be a finite set. Prove that Z is Zariski closed in $\mathbb{A}^n(k)$.
2) Show that the Zariski topology on $\mathbb{A}^2(k)$ is not given by the product topology on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$.

Exercise 2 (4 Points):

Prove, without using the Nullstellensatz, that every non-constant polynomial $f \in k[X_1, \ldots, X_n]$ has a zero in $\mathbb{A}^n(k)$.

Exercise 3 (4 Points):

Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic set with

$$I(X) := \{f \in k[X_1, \ldots, X_n] \mid f(x_1, \ldots, x_n) = 0 \text{ for all } x = (x_1, \ldots, x_n) \in X\}.$$

and coordinate ring $A := \mathcal{O}(X) = k[X_1, \ldots, X_n]/I(X)$. Let $f \in A$ be any element. Prove that

$$D(f) := \{x = (x_1, \ldots, x_n) \in X \mid f(x_1, \ldots, x_n) \neq 0\}$$

is open in X. Show that

$$D(f) \subseteq \mathbb{A}^{n+1}(k),\ x \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n)^{-1})$$

is again an affine algebraic set with coordinate ring given by the localisation $A[f^{-1}]$.

Exercise 4 (4 Points):

We identify the space $M_{2,2}(k)$ of 2×2-matrices over k with $\mathbb{A}^4(k)$ (with coordinates a, b, c, d). We define ideals

$$\mathfrak{a} := (a^2 + bc, d^2 + bc, (a + d)b, (a + d)c) \subseteq k[a, b, c, d]$$
$$\mathfrak{b} := (ad - bc, a + d) \subseteq k[a, b, c, d]$$

and denote their vanishing loci by $V(\mathfrak{a})$ resp. $V(\mathfrak{b})$. Prove that

$$X := V(\mathfrak{a}) = V(\mathfrak{b}) = \{A \in M_{2,2}(k) \mid A \text{ is nilpotent}\}$$

i.e., that a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nilpotent if and only if $A^2 = 0$ if and only if the determinant and the trace of A are zero. Moreover, prove that $\operatorname{rad} \mathfrak{a} = \mathfrak{b}$, but $\mathfrak{a} \neq \mathfrak{b}$.
The affine algebraic set X is called the nilpotent cone.

To be handed in: Tuesday, 25. October 2016.