Prof. Dr. P. Scholze **WS 2016/17**
Dr. J. Anschütz

Algebraic Geometry I

2. Exercise sheet

A (non-empty) partially ordered set $(I, \leq)$ is called filtered (resp. cofiltered) if for any two elements $i, j \in I$ there exists an element $k \in I$ such that $i \leq k$ and $j \leq k$ (resp. $k \leq i$ and $k \leq j$).

Exercise 1 (4 Points):

Let $\varphi\colon A \to B$ be a ring homomorphism. Assume that every element $b \in B$ can be written as $b = \varphi(a)u$ for some $a \in A$ and $u \in B^\times$. Show that the induced morphism

$$^a\varphi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is a topological immersion, i.e., that $\mathrm{Spec}(B)$ is homeomorphic to $^a\varphi(\mathrm{Spec}(B))$ with its subspace topology.

Exercise 2 (4 Points):

i) Let $f\colon X \to Y$ be a morphism of topological spaces. Assume that X, Y are irreducible admitting generic points $\xi \in X$ resp. $\eta \in Y$ and that Y is T_0. Show that $f(X)$ is dense in Y if and only if $f(\xi) = \eta$.
ii) Let $X_i, i \in I$, be an inverse system of spectral spaces such that I is cofiltered and for every $i \leq j$ in I the given morphism $f_{i,j}\colon X_i \to X_j$ is quasi-compact, i.e., for every quasi-compact open $U \subseteq X_j$ its preimage $f_{i,j}^{-1}(U)$ is again quasi-compact. Show that the inverse limit

$$X := \varprojlim_I X_i = \{(x_i)_{i\in I} \in \prod X_i \mid f_{i,j}(x_i) = x_j \text{ for all } i, j \in I, i \leq j\}$$

with its inverse limit topology is again a spectral space and that each projection $X \to X_i$ is quasi-compact.

Exercise 3 (4 Points):

i) Show that every finite irreducible topological space admits a generic point. Deduce that finite T_0-spaces are spectral.
ii) Express $\mathrm{Spec}(\mathbb{Z})$ as an inverse limit of finite T_0-spaces.

Exercise 4 (4 Points):

Let I be a filtered partially ordered set. Show that for each inductive system

$$0 \longrightarrow A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \longrightarrow 0$$

of short exact sequences of abelian groups the sequence

$$0 \longrightarrow \varinjlim_I A_i \longrightarrow \varinjlim_I B_i \longrightarrow \varinjlim_I C_i \longrightarrow 0$$

of colimits is again exact.

To be handed in on: Thursday, 3. November 2016.