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WS 2016/17

Algebraic Geometry I

3. Exercise sheet

Exercise 1 (4 Points):

Let k be an algebraically closed field, $I \subseteq k[X_1, \ldots, X_n]$ an ideal and set $A := k[X_1, \ldots, X_n]/I$, $X := \mathrm{Spec}(A)$. Let $Y \subseteq \mathbb{A}^n(k) = k^n$ be the affine algebraic set defined by the ideal I and let $X_0 \subseteq X$ be the set of closed points in X, equipped with the subspace topology.

i) Prove that Y and X_0 are homeomorphic.
ii) Prove that the map $\mathrm{Ouv}(X) \to \mathrm{Ouv}(X_0)$, $U \mapsto U \cap X_0$ is a bijection. Conclude that $\mathrm{Sh}(X) \cong \mathrm{Sh}(X_0)$.
iii) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Assume that the stalk $\mathcal{F}_x$ is zero for every $x \in X_0$. Show that $\mathcal{F} = 0$.

Exercise 2 (4 Points):

Let X be a topological space and let $\mathcal{B}$ be a basis of the topology of X, stable under intersections. Let $\mathrm{Sh}_{\mathcal{B}}(X)$ be the category of sheaves on the basis $\mathcal{B}$ as defined in the lecture. Prove that the functors

$$\mathrm{Sh}(X) \to \mathrm{Sh}_{\mathcal{B}}(X),\ (\mathcal{F}\colon \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Sets}) \mapsto (\mathcal{F}_{|\mathcal{B}^{\mathrm{op}}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Sets})$$

and

$$\mathrm{Sh}_{\mathcal{B}}(X) \to \mathrm{Sh}(X),\ \mathcal{F} \mapsto (U \mapsto \varprojlim_{V \subseteq U, V \in \mathcal{B}} \mathcal{F}(V))$$

are inverse equivalences of categories.

Exercise 3 (4 Points):

Let A be a commutative ring, $X := \mathrm{Spec}(A)$ and let M be an A-module. Show that the assignment

$$D(f) \mapsto M[f^{-1}]$$

defines a sheaf on the basis $\mathcal{B} := \{D(f) \subseteq X \mid f \in A\}$ of X.
Hint: Follow the proof for $M = A$ which was presented in the lecture.

Exercise 4 (4 Points):

Let $X \subseteq \mathbb{C}$ be an open subset.
i) Show that sending $U \subseteq X$ open to $\mathcal{O}_X(U) := \{f\colon U \to \mathbb{C} \text{ holomorphic}\}$ defines a sheaf on X.
ii) For a holomorphic function $f\colon U \to \mathbb{C}$ let f' be its derivative. Show that $f \mapsto f'$ defines a surjective morphism $D\colon \mathcal{O}_X \to \mathcal{O}_X$ of sheaves. Give an example of an open set $X \subseteq \mathbb{C}$ such that $D\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ is not surjective.

To be handed in on: Tuesday, 8. November 2016.