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WS 2016/17

Algebraic Geometry I

4. Exercise sheet

Exercise 1 (4 Points):
Let X be a topological space and let $\mathcal{F}$ be the presheaf

$$U \mapsto \{f \colon U \to \mathbb{R} \mid f \text{ continuous and bounded }\}$$

on X. Describe the sheafification $\tilde{\mathcal{F}}$ of $\mathcal{F}$.

Exercise 2 (4 Points):
Let $(X, \mathcal{O}_X)$ be a locally ringed space.
i) Let $U \subseteq X$ be an open and closed subset. Show that there exists a unique section $e_U \in \Gamma(X, \mathcal{O}_X)$ such that $e_{U|V} = 1$ for all open subsets $V \subseteq U$ and $e_{U|V} = 0$ for all open subsets $V \subseteq X \setminus U$. Show that $U \mapsto e_U$ yields a bijection

$$\mathrm{OC}(X) \leftrightarrow \mathrm{Idem}(\Gamma(X, \mathcal{O}_X))$$

from the set of open and closed subsets of X to the set of idempotents in $\Gamma(X, \mathcal{O}_X)$.
ii) Show that $e_U e_{U'} = e_{U \cap U'}$ for $U, U' \in \mathrm{OC}(X)$.
iii) Prove that X is connected if and only if $\Gamma(X, \mathcal{O}_X)$ contains no idempotent $e \neq 0, 1$ if and only if $\Gamma(X, \mathcal{O}_X)$ is not isomorphic to $R_1 \times R_2$ for two non-zero rings R_1, R_2.

Exercise 3 (4 Points):
Let p be a prime, let $\mathbb{F}_p$ be the field with p elements, and let $i_{(p)} \colon \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z})$ be the canonical morphism. We call a ring A *of characteristic* p if $p \cdot 1_A = 0$. Let X be a scheme. Prove that the following conditions are equivalent:
i) For every open subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is of characteristic p.
ii) The ring $\Gamma(X, \mathcal{O}_X)$ has characteristic p.
iii) The canonical scheme morphism $X \to \mathrm{Spec}(\mathbb{Z})$ factors through $i_{(p)}$.
Are these conditions equivalent to iv)?
iv) For all $x \in X$ the residue field $k(x)$ is of characteristic p.

Exercise 4 (4 Points):
Let $U_i, i = 1, 2$, be two schemes. Let $V_i \subseteq U_i$ be open subsets and let $\varphi \colon V_1 \xrightarrow{\sim} V_2$ be an isomorphism.
i) Show that there exists a scheme X with open subsets $W_i \subseteq X$ and isomorphisms $\alpha_i \colon U_i \xrightarrow{\sim} W_i$, $i = 1, 2$, such that $\alpha_i^{-1}(W_1 \cap W_2) = V_i$ and $\varphi = \alpha_2^{-1} \circ \alpha_{1|V_1}$.
Remark: We say that X is obtained from U_1 and U_2 by gluing along φ.
ii) Let A be a ring and let $X_{\pm 1}$ be the scheme obtained by glueing $U_1 = U_2 = \mathrm{Spec}(A[T])$ along the isomorphism $\varphi_{\pm 1} \colon \mathrm{Spec}(A[T, T^{-1}]) \to \mathrm{Spec}(A[T, T^{-1}]), T \mapsto T^{\pm 1}$. Show that $X_{\pm 1}$ is not an affine scheme.
Remark: X_1 is called the affine line over A with doubled origin, X_{-1} is called the projective line over A.

To be handed in on: Tuesday, 15. November 2016.