Prof. Dr. P. Scholze
Dr. J. Anschütz **WS 2016/17**

Algebraic Geometry I

5. Exercise sheet

Exercise 1 (4 Points):
Let R be a local ring, $S := \mathrm{Spec}(R)$ and $s \in S$ the (unique) closed point. Prove that for every scheme X the map

$$\mathrm{Hom}(S, X) \to \{(x \in X, \varphi \colon \mathcal{O}_{X,x} \to R) \mid \varphi \text{ is a local ring homomorphism}\}$$

sending a morphism $f \colon S \to X$ of schemes to the pair $(f(s), \mathcal{O}_{X,f(s)} \overset{f^\#}{\to} \mathcal{O}_{S,s} \cong R)$ is a bijection.

Exercise 2 (4 Points):
Let $\mathcal{C}$ be a category. For $X \in \mathcal{C}$ let $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be the associated functor. Let $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be an arbitrary functor.
i) Prove the Yoneda lemma, i.e., that the map

$$\mathrm{Hom}(h_X, F) \to F(X),\ \eta \mapsto \eta_X(\mathrm{Id}_X)$$

is a bijection, natural in X and F.
ii) Let S be a scheme and let $X \to S, Y \to S$ be two schemes over S. Let $\mathcal{C} = (\mathrm{Sch}/S)$ be the category of schemes over S and let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory consisting of objects $Z \to S \in \mathcal{C}$ with Z affine. Let $\mathrm{Hom}_S(X, Y)$ be the set of morphisms $f \colon X \to Y$ of schemes over S. Show that there are bijections

$$\mathrm{Hom}_S(X, Y) \cong \mathrm{Hom}(h_X, h_Y) \cong \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}})$$

where $F_{|\mathcal{D}}$ denotes the restriction of a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ to $\mathcal{D}^{\mathrm{op}}$.

Exercise 3 (4 Points):
For a scheme X we denote its underlying topological space by $|X|$. Let X, Y, S be three schemes and assume that there are morphisms $f \colon X \to S$, $g \colon Y \to S$. Prove that the canonical morphism

$$\pi \colon |X \times_S Y| \to |X| \times_{|S|} |Y|$$

is surjective and determine its fibres. Show, by giving examples, that a fiber of π can be infinite or disconnected.

Exercise 4 (4 Points):
Let k be a field. Describe the fibers in all points of the following morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponding in each case to the canonical morphism $A \to B$.
i) $\mathrm{Spec}(k[T, U]/(TU - 1)) \to \mathrm{Spec}(k[T])$
ii) $\mathrm{Spec}(k[T, U]/(T^2 - U^2)) \to \mathrm{Spec}(k[T])$
iii) $\mathrm{Spec}(k[T, U]/(T^2 + U^2)) \to \mathrm{Spec}(k[T])$
iv) $\mathrm{Spec}(k[T, U]/(TU)) \to \mathrm{Spec}(k[T])$

To be handed in on: Tuesday, 22. November 2016.