Prof. Dr. P. Scholze
Dr. J. Anschütz

WS 2016/17

Algebraic Geometry I

8. Exercise sheet

Exercise 1 (4 Points):

Let $n \geq d \geq 0$.
i) Prove that the functor sending a scheme X to the isomorphism classes of quotients $\mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ with $\mathcal{E}$ locally free of rank d is representable by a scheme $\mathrm{Grass}_{n,d}$ over $\mathrm{Spec}(\mathbb{Z})$, called the Grassmannian.
ii) Prove that the morphism

$$\mathrm{Grass}_{n,d} \to \mathbb{P}^{\binom{n}{d}-1},\ (\mathcal{O}_X^n \twoheadrightarrow \mathcal{E}) \mapsto (\mathcal{O}_X^{\binom{n}{d}} \cong \Lambda^d \mathcal{O}_X^n \to \Lambda^d \mathcal{E}),$$

where $\Lambda^d \mathcal{F}$ denotes the d-the exterior power of a vector bundle $\mathcal{F}$, is well-defined and that it is a closed embedding if $n = 4, d = 2$.
Remark: The map, called the Plücker embedding, is a closed embedding in general.

Exercise 2 (4 Points):

i) Let $n \geq 0$. Prove that the functor

$$X \mapsto \{a\colon \mathcal{O}_X^n \to \mathcal{O}_X^n \mid a \text{ is an isomorphism}\}$$

is representable by a scheme, called GL_n, over $\mathrm{Spec}(\mathbb{Z})$.
ii) Let S be a scheme and let $\mathcal{E}$ be a vector bundle on S. Prove that the functor

$$(f\colon X \to S) \mapsto \{a\colon f^*\mathcal{E} \to f^*\mathcal{E} \mid a \text{ is an isomorphism}\}$$

is representable by a scheme, called $\mathrm{Aut}(\mathcal{E})$, over S, which is locally on S isomorphic to $\mathrm{GL}_n \times S$.

Exercise 3 (4 Points):

Let X be a scheme.
i) Prove that the category $\mathrm{QCoh}(X)$ of quasi-coherent modules on X is a full abelian subcategory of the category $\mathcal{O}_X$-Mod of $\mathcal{O}_X$-modules that is closed under extensions. Show that the functor $\mathrm{QCoh}(X) \to \mathcal{O}_X$-Mod is exact.
ii) Assume that X is noetherian. Prove the same statements as in i) with the category $\mathrm{QCoh}(X)$ replaced by the category $\mathrm{Coh}(X)$ of coherent $\mathcal{O}_X$-modules.

Exercise 4 (4 Points):

Let k be a field and let $n \geq 0$. Let $U_0 = \operatorname{Spec}(k[T])$, $U_1 = \operatorname{Spec}(k[T^{-1}])$ with intersection $U_0 \cap U_1 = \operatorname{Spec}(k[T^{\pm 1}])$ be the standard covering of $\mathbb{P}^1_k$.

i) Prove that the map

$$\begin{array}{ccc} \{\text{iso. classes. of rank n vector bundles}\} & \to & \mathrm{GL}_n(k[T]) \setminus \mathrm{GL}_n(k[T^{\pm 1}])/\mathrm{GL}_n(k[T^{-1}]), \\ \mathcal{E} & \mapsto & \alpha^{-1}_{|U_0 \cap U_1} \circ \beta_{|U_0 \cap U_1} \end{array}$$

where $\alpha \colon \mathcal{O}^n_{U_0} \to \mathcal{E}_{|U_0}$ and $\beta \colon \mathcal{O}^n_{U_1} \to \mathcal{E}_{|U_1}$ are two isomorphisms, is well-defined and bijective.

ii) Show that the map

$$\begin{array}{ccc} \{(d_1, \ldots, d_n) \in \mathbb{Z} \mid d_1 \geq \ldots \geq d_n\} & \to & \mathrm{GL}_n(k[T]) \setminus \mathrm{GL}_n(k[T^{\pm 1}])/\mathrm{GL}_n(k[T^{-1}]) \\ d = (d_1, \ldots, d_n) & \mapsto & \mathrm{GL}_n(k[T]) T^d \mathrm{GL}_n(k[T^{-1}]), \end{array}$$

where T^d denotes the diagonal matrix with entries $T^{d_1}, \ldots, T^{d_n}$, is a bijection. Write down all vector bundles on $\mathbb{P}^1_k$ (up to isomorphism).

To be handed in on: Tuesday, 13. December 2016.