Prof. Dr. P. Scholze
Dr. J. Anschütz **WS 2016/17**

Algebraic Geometry I

9. Exercise sheet

Exercise 1 (4 Points):

Let A be a noetherian, factorial ring. Prove that $\mathrm{Pic}(A) = 0$.
Hint: Proposition B.60, B.70, B.72 and remark B.75 in the book "Algebraic Geometry I" by Görtz-Wedhorn might be helpful.

Exercise 2 (4 Points):

Let k be a field. Prove that $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_k)$ via $1 \mapsto \mathcal{O}_{\mathbb{P}^n_k}(1)$.
Hint: Use Exercise 1.

Exercise 3 (4 Points):

Let R be a ring and define

$$X := \{((x_0, \ldots, x_n), (y_0 : \ldots : y_n)) \in \mathbb{A}^{n+1}_R \times_{\mathrm{Spec}(R)} \mathbb{P}^n_R \mid x_i y_j = x_j y_i \text{ for all } i, j\}.$$

i) Show that X is a closed subscheme in $\mathbb{A}^{n+1}_R \times_{\mathrm{Spec}(R)} \mathbb{P}^n_R$.
ii) Prove that X is isomorphic (as a $\mathbb{P}^n_R$-scheme) to the total space $\mathbb{V}(\mathcal{O}_{\mathbb{P}^n_R}(-1))$ of the line bundle $\mathcal{O}_{\mathbb{P}^n_R}(-1)$.
iii) Let $\pi \colon X \to \mathbb{A}^{n+1}_R$ be the canonical projection. Show that

$$\pi_{|\pi^{-1}(\mathbb{A}^{n+1}_R \setminus \{0\})} \colon \pi^{-1}(\mathbb{A}^{n+1}_R \setminus \{0\}) \to \mathbb{A}^{n+1}_R \setminus \{0\}$$

is an isomorphism and that $\pi^{-1}(0) \cong \mathbb{P}^n_R$.
iv) Define $\mathcal{I} := (x_0, \ldots, x_n) \subseteq \mathcal{O}_{\mathbb{A}^{n+1}_R}$ and let $f \colon S \to \mathbb{A}^{n+1}_R$ be a morphism such that the ideal sheaf $f^{-1}(\mathcal{I})\mathcal{O}_S \subseteq \mathcal{O}_S$ is invertible. Prove that f factors uniquely through π.
Remark: The scheme X is called the blow-up of $\mathbb{A}^{n+1}_R$ in the closed subscheme $0 \in \mathbb{A}^{n+1}_R$.

Exercise 4 (4 Points):

i) Show that closed immersions are stable under composition and base change. Show that if $f \colon X \to Y$, $g \colon Y \to S$ are morphisms of schemes such that g and $g \circ f$ are closed immersions, then also f is a closed immersion.
ii) Let $f \colon X \to Y$, $g \colon Y \to S$ be two morphisms of schemes. Assume that $g \circ f$ is proper and that g is separated. Prove that f is proper.

To be handed in on: Tuesday, 20. December 2016.