Prof. Dr. P. Scholze
Dr. J. Anschütz **WS 2016/17**

Algebraic Geometry I

12. Exercise sheet

Exercise 1 (4 Points):
Let k be a field and let $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n_k(k)$ be a k-rational point. Define the line $\ell := \langle (x_0, \ldots, x_n) \rangle \subseteq k^{n+1}$. Show that there exists a canonical isomorphism

$$T_x \mathbb{P}^n_k \cong \mathrm{Hom}_k(\ell, k^{n+1}/\ell).$$

Hint: Use Exercise 4.i) from Exercise sheet 11.

Exercise 2 (4 Points):
Let $X \subseteq \mathbb{P}^2_{\mathbb{R}}$ be the closed subscheme defined by the equation

$$x^2 + y^2 + z^2 = 0.$$

Show that $X \ncong \mathbb{P}^1_{\mathbb{R}}$, but $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathrm{Spec}(\mathbb{C})}$, where $X_{\mathbb{C}} := X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$.

Exercise 3 (4 Points):
Let R be a Dedekind ring with field of fractions K and set $S := \mathrm{Spec}(R)$. Let $X \to S$ be a proper morphism. Show that the canonical map

$$X(S) \to X(\mathrm{Spec}(K))$$

is bijective.

Exercise 4 (4 Points):
Let k be an algebraically closed field and let $X \to \mathrm{Spec}(k)$ be a proper morphism with X reduced and connected. Let $f \in \Gamma(X, \mathcal{O}_X)$ be a function and denote by $g\colon X \to \mathbb{A}^1_k$ the morphism induced by f.
i) Show that the (set-theoretic) image of g does not contain the generic point of $\mathbb{A}^1_k$.
ii) Deduce that $\Gamma(X, \mathcal{O}_X) \cong k$.
Hint: For i) embed $\mathbb{A}^1_k$ into $\mathbb{P}^1_k$.

To be handed in on: Tuesday, 24. January 2017.