Prof. Dr. P. Scholze
Dr. J. Anschütz **WS 2016/17**

Algebraic Geometry I

13. Exercise sheet

Exercise 1:
Let k be a field and let X be a curve over k (in particular, X is integral). Prove that X is proper over k if and only if its normalization $\tilde{X}$ is proper over k.

Exercise 2:
Let k be a field and let X be a curve over k. Show that X contains a dense open subset U such that U is normal.

Exercise 3:
Let $X \subseteq \mathbb{A}^2_{\mathbb{Z}}$ be defined by the equation $y^2 = f(x)$ where $f(x) = x^3 - 10x^2 - 75x \in \mathbb{Z}[x]$. For every point $y \in \mathrm{Spec}(\mathbb{Z})$ determine the normal points and the singular $k(y)$-rational points of $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k(y))$.
Hint: Factorize f. Moreover, base change to $\overline{k(y)}$.

Exercise 4:
Let k be an algebraically closed field of characteristic $\neq 2$ and let $Y \subseteq \mathbb{A}^2_k$ be defined by the polynomial $f(x,y) := y^2 - x^3 + x$.
i) Show that every k-rational point of Y is smooth. Deduce that $A := k[x,y]/f$ is a normal domain.
ii) Let $R := k[x] \subseteq A$. Show that R is a polynomial ring and that A is integral over R.
iii) Show that there exists an automorphism $\sigma\colon A \to A$ which sends y to $-y$, but leaves x fixed.
iv) For $a \in A$ set $N(a) := a\sigma(a)$. Show $N(a) \in R$ and that N is multiplicative.
v) Show that $A^\times = k^\times$ and that y, x are irreducible elements in A. Deduce that A is not factorial and that Y is not isomorphic to $\mathbb{A}^1_k$.

To be handed in on: Tuesday, 31. January 2017.