Prof. Dr. P. Scholze
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SS 2017

Algebraic Geometry II

2. Exercise sheet

Exercise 1 (4 points):

Let A be a Dedekind domain and set $S := \operatorname{Spec}(A)$. Let X be an integral scheme. Prove that a morphism $f\colon X \to S$ is flat if and only if it maps the generic point of X to the generic point of S.
Hint: Use Exercise 1.iii) from Exercise sheet 1.

Exercise 2 (4 points):

i) Let $A \to B$ be a flat, local morphism of local rings. Prove that the morphism

$$\operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

is surjective.
ii) Let $f\colon X \to S$ be a flat morphism of schemes. Prove that f is universally generalising.
Remark: A morphism $g\colon Y \to Z$ of topological spaces is called generalising if for all $y \in Y, z \in Z$ such that z specializes to $g(y)$, i.e. $g(y) \in \overline{\{z\}}$, there exists $y' \in Y$ specializing to y such that $z = g(y')$.

Exercise 3 (4 points):

Let $f\colon X \to S$ be a closed immersion. Then f is flat and locally of finite presentation if and only if f is an open immersion.
Hint: Use that a flat and finitely presented module over a local ring is free.

Exercise 4 (4 points):

i) Let $f\colon X \to S$ be a morphism, which is locally of finite presentation, flat and finite. Prove that the function

$$s \in S \mapsto \dim_{k(s)} \Gamma(X \times_S \operatorname{Spec}(k(s)), \mathcal{O}_{X \times_S \operatorname{Spec}(k(s))})$$

is locally constant.
Hint: You may use that, in this situation, if $S = \operatorname{Spec}(A)$ is affine, then $X = \operatorname{Spec}(B)$ is affine and B a finitely presented A-module.
ii) Let X be an integral scheme of finite type over a field k. Prove that the normalization $f\colon \tilde{X} \to X$ is flat if and only if it is an isomorphism.
Hint: Use the finiteness of the normalization and part i).

To be handed in on: Tuesday, 2. May 2017 till 12h (in the prepared box in front of room 4.027).