Prof. Dr. P. Scholze
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Algebraic Geometry II

3. Exercise sheet

Exercise 1 (4 points):
Let A be a ring, B a faithfully flat A-algebra and M an A-module. Recall the exact sequence

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B$$

from the lecture.
i) Use this exact sequence to reprove that the assignment

$$D(f) \mapsto M \otimes_A A[1/f]$$

defines a sheaf on $\mathrm{Spec}(A)$.
ii) Let $f_1, \ldots, f_n \in A$ be elements generating the unit ideal of A and set $B := \prod_{i=1}^{n} A[1/f_i]$. Prove that B is faithfully flat over A and identify the descent data for B/A with glueing data as defined in Algebraic Geometry I, cf. Lemma 10.10 in the notes.

Exercise 2 (4 points):
Let K be a field and let L/K be a finite Galois extension with group G.
i) Prove that the morphism

$$\begin{array}{ccc} \mathrm{Spec}(L) \times G \cong \coprod_{g \in G} \mathrm{Spec}(L) & \to & \mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \\ (x, g) & \mapsto & (x, xg) \end{array}$$

is an isomorphism.
ii) Prove Hilbert's theorem 90: Sending a K-vector space V to $L \otimes_K V$ defines an equivalence

$$\{K\text{-vector spaces }\} \to \{L\text{-vector spaces together with a semilinear } G-\text{action}\}$$

(A G-action on an L-vector space W is called semilinear if $g(lw) = g(l)g(w)$ for all $g \in G$, $l \in L$ and $w \in W$.)
Hint: Identify L-vector spaces with a semilinar G-action with descent data for $\mathrm{Spec}(L)/\mathrm{Spec}(K)$.

Exercise 3 (4 points):
Let A be a ring and let $A \to B$ be a faithfully flat morphism.
i) Prove descent for affine schemes, i.e., prove that the category of A-algebras is equivalent to the category of B-algebras C equipped with an isomorphism

$$C \otimes_{B,\iota_1} (B \otimes_A B) \cong C \otimes_{B,\iota_2} (B \otimes_A B)$$

satisfying the cocycle condition. Here $\iota_1(b) = b \otimes 1$ and $\iota_2(b) = 1 \otimes b$ for $b \in B$.
ii) For a scheme S let $C(S)$ denote the set of closed subschemes of S. Prove the exactness of the natural sequence

$$C(\mathrm{Spec}(A)) \to C(\mathrm{Spec}(B)) \rightrightarrows C(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)).$$

Exercise 4 (4 points):
Let A be a ring, let $I \subseteq A$ be an ideal and let M be an A-module. We set

$$\mathrm{gr}_I(M) := \bigoplus_{i=0}^{\infty} I^n M / I^{n+1} M.$$

i) Assume that M is flat over A. Prove that the canonical map

$$\eta_{M,I} \colon \mathrm{gr}_I(A) \otimes_{A/I} M/IM \to \mathrm{gr}_I(M),\ \overline{f} \otimes \overline{m} \mapsto \overline{fm}$$

is bijective.
Hint: Use the exact sequences

$$0 \to I^n/I^{n+1} \to A/I^{n+1} \to A/I^n \to 0.$$

ii) Let $A = k[x, y]$, $B := k[x', y']$ and define $f \colon A \to B$ by sending $x \mapsto x'y'$, $y \mapsto y'$. Compute the kernel of the map $\eta_{B,(x,y)}$, where B is considered as an A-module via f (and $I = (x, y)$).
iii) Let $A := k[x]$, $B := k[x, y]/(xy)$ an define $f \colon A \to B$ by sending $x \mapsto x$. Compute the kernel of $\eta_{B,(x)}$, where B is considered as an A-module via f (and $I = (x)$).

To be handed in on: Monday, 8. May 2017 (in the lecture).