Prof. Dr. P. Scholze
Dr. J. Anschütz **SS 2017**

Algebraic Geometry II

4. Exercise sheet

For a ring A we set $A[\varepsilon] := A[t]/t^2$.

Exercise 1 (4 points):

Let $A \to B \to C$ be morphisms of rings.
i) Show that there exists an exact sequence

$$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

of C-modules.
ii) Assume that $B \to C$ is surjective with kernel $I := \ker(B \to C)$. Prove that there exists an exact sequence (of C-modules)

$$C \otimes_B I \cong I/I^2 \xrightarrow{\alpha} C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to 0$$

where $\alpha(g) := dg$ for $g \in I$.

Exercise 2 (4 points):

Compute $\Omega^1_{B/A}$ for
i) $B = A[X]/(f(X))$ with $f(X) \in A[X]$ a polynomial.
ii) $B = \mathbb{Z}[i]$, $A = \mathbb{Z}$.
iii) $B = k[x, y]/(y^2 - x^3 - x)$ with $A = k$ not of characteristic 2.
iv) $B = k[x, y]/(xy)$, $A = k$.
Hint: Use exercise 2.ii).

Exercise 3 (4 points):

Let A be a perfect $\mathbb{F}_p$-algebra, i.e., the Frobenius $\mathrm{Fr}_A \colon A \to A,\ x \mapsto x^p$ of A is bijective. Prove that $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{F}_p)$ is formally étale.
Hint: If R is a ring of characteristic p, $I \subseteq R$ an ideal with $I^2 = 0$ and $R_0 = R/I$, then for $x \in R$ the element x^p only depends on the image $x_0 \in R_0$ of x.

Exercise 4 (4 points):

Let $X \to S$ be a morphism of schemes. We define a functor, the "tangent bundle" $\mathcal{T}_{X/S}$ of X/S, by sending an affine scheme $\mathrm{Spec}(A)$ over S to the set $X(A[\varepsilon])$. Prove that $\mathcal{T}_{X/S}$ is representable by the relative spectrum $\underline{\mathrm{Spec}}_{\mathcal{O}_X}(\mathrm{Sym}^\bullet(\Omega^1_{X/S}))$.
Hint: Note that there is a natural morphism $X(A[\varepsilon]) \to X(A)$. Use this to reduce to the case that $X = Spec(R)$ and S are affine.

To be handed in on: Monday, 15. May 2017.