Prof. Dr. P. Scholze **SS 2017**
Dr. J. Anschütz

Algebraic Geometry II

6. Exercise sheet

Exercise 1 (4 points):

Let X be scheme. For a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ define $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ to be the sheaf of sets

$$V \subseteq X \mapsto \mathrm{Isom}(\mathcal{O}_V, \mathcal{L}_{|V}).$$

i) Prove that $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ is a $\mathcal{O}_X^\times$-torsor, where $\mathcal{O}_X^\times \subseteq \mathcal{O}_X$ denotes the sheaf of units.
ii) Prove that sending $\mathcal{L}$ to $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ defines a bijection $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$.

Exercise 2 (4 points):

Let X be a topological space and let $1 \to N \to G \to Q \to 1$ be a short exact sequence of sheaves of groups on X, i.e., for every $x \in X$ the morphism $G_x \to Q_x$ is surjective with kernel N_x. Prove that there is a natural short exact sequence (of pointed sets)

$$1 \to N(X) \to G(X) \to Q(X) \to H^1(X, N) \to H^1(X, G) \to H^1(X, Q).$$

Exercise 3 (4 points):

Let $f\colon Z \to S$, $g\colon X \to S$ be smooth morphisms of schemes and let $i\colon Z \to X$ be a closed immersion over S. Prove that for all $z \in Z$ there exists a neighborhood $U \subseteq X$ of $i(z)$ and sections $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ such that $Z \cap U = V(f_1, \ldots, f_d)$ for some $d \leq n$ and the induced diagram

$$\begin{tikzcd} Z \cap U \arrow[r] \arrow[d, "(\bar{f}_{d+1}, \ldots, \bar{f}_n)"'] & U \arrow[d, "(f_1, \ldots, f_n)"] \\ \mathbb{A}_S^{n-d} \arrow[r, "\alpha"] & \mathbb{A}_S^n \end{tikzcd}$$

with $\alpha((x_{d+1}, \ldots, x_n)) = (0, \ldots, 0, x_{d+1}, \ldots, x_n)$ is cartesian with both vertical arrows étale.
Hint: Let $\mathcal{I}$ be the ideal sheaf of Z in X. Prove that the sequence $0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$ is an exact sequence of vector bundles and choose, locally around z, sections $f_1, \ldots, f_n \in \mathcal{O}_X$ such that the differentials $df_1, \ldots, df_n$ form an adabted basis of $i^*\Omega^1_{X/S}$.

Exercise 4 (4 points):

Let k be an algebraically closed field. Prove that every finite étale morphism $f\colon X \to Y := \mathbb{P}^1_k$ is trivial, i.e., X is isomorphic to a disjoint union of copies of Y.
Hint: Consider the quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A} := f_*(\mathcal{O}_X)$ with its multiplication $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \to \mathcal{A}$. Use exercise 4 from Algebraic geometry I, exercise sheet 8 to write $\mathcal{A} = \bigoplus \mathcal{A}_i$ with $\mathcal{A}_i \cong \mathcal{O}_Y(i)^{\oplus k_i}, k_i \geq 0$. Use your knowledge about the global sections $\mathcal{O}_Y(i)(Y)$ to prove that each element in $\mathcal{A}_i$ is nilpotent for $i > 0$. Use exercise 4 from exercise sheet 5 to construct a canonical isomorphism $\mathcal{A}_{-i} \cong \mathcal{A}_i, i \in \mathbb{Z}$, and deduce $\mathcal{A} = \mathcal{A}_0$, i.e., $\mathcal{A} \cong \mathcal{O}_Y \otimes_k H^0(X, \mathcal{O}_X)$ with $H^0(X, \mathcal{O}_X)$ a finite dimensional étale k-algebra.

To be handed in on: Monday, 29. May 2017.
Die Fachschaft Mathematik feiert am 1.6. ihre Matheparty in der N8schicht. Der VVK findet am Mo. 29.05., Di. 30.05. und Mi 31.05. in der Mensa Poppelsdorf statt. Alle weitere Infos auch auf fsmath.uni-bonn.de.