Prof. Dr. P. Scholze **SS 2017**
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Algebraic Geometry II

7. Exercise sheet

Exercise 1 (4 points):
Let S be a scheme. Prove that there exists an exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_S/S} \to \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_S/S}(-1) \xrightarrow{(x_0,\ldots,x_n)} \mathcal{O}_{\mathbb{P}^n_S/S} \to 0$$

where $x_0, \ldots, x_n \in H^0(X, \mathcal{O}_{\mathbb{P}^n_S/S}(1)) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n_S/S}(-1), \mathcal{O}_{\mathbb{P}^n_S/S})$ are the canonical sections.
Hint: Either try to construct the sequence by glueing or use Exercise 4 from Exercise sheet 4 (perhaps remembering Exercise 1 from Algebraic Geometry I, Exercise sheet 12). In any case one can reduce to $S = \mathrm{Spec}(\mathbb{Z})$.

Exercise 2 (4 points):
i) Let A be a ring. Prove that the category of A-modules has enough injectives.
ii) Let $(X, \mathcal{O}_X)$ be a ringed space. Prove that the category of $\mathcal{O}_X$-modules has enough injectives.

Exercise 3 (4 points):
Let $(X, \mathcal{O}_X)$ be a ringed space, let $V \subseteq X$ be an open subset and set $\mathcal{O}_V := \mathcal{O}_{X|V}$. Let $j\colon (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ be the canonical morphism of ringed spaces.
i) Let $\mathcal{F}$ be an $\mathcal{O}_V$-module and set $j_!$ as the sheafification of the presheaf

$$U \subseteq X \mapsto \begin{cases} \mathcal{F}(U) & \text{if } U \subseteq V \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the functor $j_!$ from $\mathcal{O}_V$-modules to $\mathcal{O}_X$-modules is left adjoint to the restriction functor j^*.
ii) Let $\mathcal{F}$ be an injective $\mathcal{O}_X$-module. Prove that the restriction $\mathcal{F}_{|V}$ is an injective $\mathcal{O}_V$-module.

Exercise 4 (4 points):
Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{T}$ be an $\mathcal{F}$-torsor with corresponding extension

$$0 \to \mathcal{F} \to \widetilde{\mathcal{T}} \to \mathbb{Z} \to 0$$

constructed similarly to the lecture (for $\mathcal{O}_X$-modules on a scheme). We set $\Phi(\mathcal{T}) := \delta_{\widetilde{\mathcal{T}}}(1)$ where $\delta_{\widetilde{\mathcal{T}}}\colon \mathbb{Z}(X) \to H^1(X, \mathcal{F})$ is the connecting morphism (for the δ-functor $H^i(X, -)$ constructed in the lecture using injective resolutions).
i) Prove that sending $\mathcal{T}$ to $\Phi(\mathcal{T})$ defines a bijection

$$\{\mathcal{F} - \text{torsors}\}/\text{isom.} \cong H^1(X, \mathcal{F}).$$

ii) For two $\mathcal{F}$-torsors $\mathcal{T}, \mathcal{T}'$ construct naturally an $\mathcal{F}$-torsor $\mathcal{T}''$ such that $\Phi(\mathcal{T}'') = \Phi(\mathcal{T}) + \Phi(\mathcal{T}')$.
Hint: For surjectivity in i) take a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ *with* $\mathcal{I}$ *injective and use it to construct, for a given* $a \in H^1(X, \mathcal{F})$*, a suitable extension* $0 \to \mathcal{F} \to \widetilde{\mathcal{T}} \to \mathbb{Z} \to 0$.

To be handed in on: Monday, 12. June 2017.
Die Fachschaft Mathematik feiert am 1.6. ihre Matheparty in der N8schicht. Der VVK findet am Mo. 29.05., Di. 30.05. und Mi 31.05. in der Mensa Poppelsdorf statt. Alle weitere Infos auch auf fsmath.uni-bonn.de.