# Descent for solid quasi-coherent sheaves on perfectoid spaces 

Johannes Anschütz<br>Lucas Mann

March 5, 2024


#### Abstract

We prove $v$-descent for solid quasi-coherent sheaves on perfectoid spaces as a key technical input for the development of a 6 -functor formalism with values in solid quasi-coherent sheaves on relative Fargues-Fontaine curves.


## Contents

1 Introduction 1
1.1 Acknowledgements . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
1.2 Notations and conventions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
$2 \mathcal{D}_{\hat{\wedge}}(A)$ for adic rings 5
2.1 Adic rings . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
2.2 Definition of $\mathcal{D}_{\hat{\wedge}}(A)$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
2.3 Complete descent . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
2.4 Globalization for stably uniform adic spaces . . . . . . . . . . . . . . . . . . . 19
2.5 Universal descent . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
$3 \mathcal{D}_{\Delta}^{a}\left(\mathcal{O}_{v}^{+}\right)$for perfectoid spaces 23
3.1 Definition of $\mathcal{D}_{\Delta}^{a}\left(A^{+}\right)$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
3.2 Descent on totallv disconnected spaces . . . . . . . . . . . . . . . . . . . . . . 25
3.3 Boundedness conditions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 30
3.4 Descent for relative compactifications of totally disconnected spaces . . . . . 33
3.5 Families of Riemann-Zariski spaces . . . . . . . . . . . . . . . . . . . . . . . . 35

4 Overconvergent and nuclear sheaves 39
$4.1 \quad \omega_{1}$-solid sheaves . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 39
4.2 Overconvergent solid sheaves . . . . . . . . . . . . . . . . . . . . . . . . . . . 43
4.3 Nuclear sheaves . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 48
$4.4 \mathcal{C}$-valued nuclear sheaves . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 55
5 Main theorem: $\mathcal{D}_{\ominus}^{a}\left(\mathcal{O}_{X}^{+}\right)$on +-bounded affinoid perfectoid spaces 57
References 62

## 1 Introduction

Let $p$ be a prime, and let Perfd be the category of perfectoid spaces over $\mathbb{Z}_{p}$. We equip Perfd with the $v$-topology, i.e., the Grothendieck topology generated by surjections of affinoid
perfectoid spaces. The main result of this paper is the following theorem:
Theorem 1.1 (Theorem 3.7, Theorem [5.6). There exists a unique hypercomplete $v$-sheaf of $\infty$-categories

$$
\operatorname{Perfd}^{\mathrm{op}} \rightarrow \mathcal{C}^{2} t_{\infty}, X \mapsto \mathcal{D}_{\hat{\AA}}^{a}\left(\mathcal{O}_{X}^{+}\right),
$$

such that for each + -bounded affinoid perfectoid space $X=\operatorname{Spa}\left(A, A^{+}\right)$we have $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right) \cong$ $\mathcal{D}_{\hat{0}}^{a}\left(A^{+}\right)$compatibly with pullback.

Here, $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$refers to a(n almost version of a) slight modification of the category $\mathcal{D}_{\square}\left(A^{+}\right)$ of solid $A^{+}$-modules introduced in [23] and [1] (the exact definition is Definition [2.1, Definition (3.2). Roughly the potentially non-complete compact generators of $\mathcal{D}_{\square}\left(A^{+}\right)$get replaced in $\mathcal{D}_{\hat{\wedge}}\left(A^{+}\right)$by their $\pi$-adic completions for a pseudo-uniformizer $\pi \in A$. Theorem 1.1 formally implies $v$-hyperdescent of the functor $X \mapsto \mathcal{D}_{\hat{\emptyset}}\left(\mathcal{O}_{X}\right)$ on perfectoid spaces because $\mathcal{D}_{\hat{\ominus}}\left(\mathcal{O}_{X}\right):=\operatorname{Mod}_{\mathcal{O}_{X}}\left(\mathcal{D}_{\hat{\ominus}}^{a}\left(\mathcal{O}_{X}^{+}\right)\right)$.
The notion of + -boundedness (Definition 3.14) is a mild cohomological finiteness condition on perfectoid spaces, which only depends on the tilt to characteristic $p$ and is satisfied for affinoid perfectoid spaces of characteristic $p$ that are quasi-pro-étale or weakly of perfectly finite type over a totally disconnected space (Example (3.16). In particular, this yields the uniqueness of the $v$-sheaf $X \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$in Theorem 1.1.
Theorem 1.1 will be used in [2] as the essential technical ingredient for setting up a 6 functor formalism with values in (modified) solid quasi-coherent sheaves on the FarguesFontaine curve. This 6 -functor formalism implies finiteness and duality results for the proétale cohomology of general pro-étale $\mathbb{Q}_{p}$-local systems on smooth rigid-analytic varieties, and we refer to [2] for more details and motivation for the study of these assertions (and hence of Theorem (1.1). We note that our proof of Theorem [1.1 would not simplify if the pro-étale topology is considered instead of the $v$-topology. Moreover, the identification of $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$ for $X$ a +-bounded affinoid perfectoid space (and not merely in the easier case if $X$ totally disconnected), is critical to establish existence of the 6 -functor formalism, e.g., the projection formula, and some of its properties, e.g., cohomological smoothness for smooth rigid-analytic varieties over $\mathbb{C}_{p}$.
We now give a rough sketch of the proof of Theorem [1.1, which at the same time will summarize the content of the different sections in this paper. First of all, Theorem 1.1 is an $\mathcal{O}^{+}$-analog of a main theorem of [16], namely [16, Theorem 1.8.3], which we recall due to its importance for this paper:

Theorem 1.2 ([16, Theorem 1.8.2., Theorem 1.8.3., Theorem 3.5.21]). There exists a unique hypercomplete $v$-sheaf of $\infty$-categories

$$
\operatorname{Perfd}_{\pi}^{\mathrm{op}} \rightarrow \mathcal{C} a t_{\infty}, \quad X \mapsto \mathcal{D}_{\square}^{a}\left(\mathcal{O}_{X}^{+} / \pi\right),
$$

such that for each +-bounded affinoid perfectoid space $X=\operatorname{Spa}\left(A, A^{+}\right)$with uniformizer $\pi$ we have $\mathcal{D}_{\square}^{a}\left(\mathcal{O}_{X}^{+} / \pi\right) \cong \mathcal{D}_{\square}^{a}\left(A^{+} / \pi\right)$ compatibly with pullback.

Here, $\operatorname{Perfd}_{\pi}$ denotes the category of perfectoid spaces $X$ with a choice of a pseudouniformizer $\pi$ and morphisms respecting $\pi$. We note that here $\mathcal{D}_{\square}^{a}\left(A^{+} / \pi\right)$ is (an almost version of) the solid category for the discrete ring $A^{+} / \pi$ introduced in [23].
Theorem 1.1 and Theorem 1.2 are tightly related. On the one hand, Remark [3.5 implies that

$$
\operatorname{Mod}_{\mathcal{O}_{X}^{+} / \pi}\left(\mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)\right) \cong \mathcal{D}_{\square}^{a}\left(\mathcal{O}_{X}^{+} / \pi\right)
$$

for each perfectoid space $X$ with a pseudo-uniformizer $\pi$, which shows that Theorem 1.1 implies Theorem [1.2. On the other hand our definition of $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$is made to use Theorem 1.2
in our proof of Theorem 1.1. We note that this reduction does not seem possible for the usual category $\mathcal{D}_{\square}^{a}\left(A^{+}\right)$instead of $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$- we heavily use that the compact generators of the latter are $\pi$-complete.

In Section 2.2 we define the modification $\mathcal{D}_{\hat{\mathrm{\rightharpoonup}}}(A)$ of $\mathcal{D}_{\square}(A)$ for any adic analytic ring $A$ (in the general sense of Definition (2.1) by $I$-completing the compact generators of $\mathcal{D}_{\square}(A)$ with respect to some ideal of definition $I \subseteq \pi_{0}(A)$, and study its basic properties. In particular, we prove the following comparison statements.

Theorem 1.3 (Proposition [2.17, Lemma [2.9, Lemma 2.18.(i)]. Let $A$ be an adic analytic ring, and $\alpha^{*}: \mathcal{D}_{\square}(A) \rightarrow \mathcal{D}_{\hat{\ominus}}(A)$ the natural functor.
(i) If $A^{+}$is of finite type over $\mathbb{Z}_{p}$, then $\alpha^{*}$ is an equivalence.
(ii) The functor $\alpha^{*}$ induces an equivalence $\mathcal{D}_{\square}(A)^{\text {nuc }} \cong \mathcal{D}_{\hat{\ominus}}(A)^{\text {nuc }}$ on nuclear objects.
(iii) $\mathcal{D}_{\square}(A)^{\mathrm{nuc}} \subseteq \mathcal{D}_{\square}(A)$ is the full subcategory generated under colimits by I-complete objects $M \in \mathcal{D}_{\square}(A)$, which are discrete mod $I$.

In Theorem 2.30 we prove a descent theorem for $\mathcal{D}_{\hat{\mathrm{O}}}(-)$ as a functor on adic rings, by reducing the descent question modulo all $I^{n}$ :

Theorem 1.4 (Theorem (2.30). Let $A \rightarrow B$ be an adic morphism of adic rings, and $I \subseteq \pi_{0}(A)$ an ideal of definition. Assume that $A / I^{n} \rightarrow B / I^{n}, n \geq 0$, is descendable of index independent of $n$ (in the sense of [16, Definition 2.6.7]). Then $A \rightarrow B$ satisfies descent for $\mathcal{D}_{\hat{\emptyset}}(-)$.

Again, such a statement does not seem possible for $\mathcal{D}_{\square}(-)$. Morever, following [1] we globalize $\mathcal{D}_{\hat{\circ}}$ to stably uniform analytic adic spaces in Section [2.4 with an eye towards our applications in [2].
Theorem 1.5 (Theorem (2.38). The functor $\mathrm{Spa}\left(B, B^{+}\right) \mapsto \operatorname{Mod}_{B}\left(\mathcal{D}_{\hat{\wedge}}\left(B^{\circ}, B^{+}\right)\right)$satisfies analytic (resp. étale) descent on stably uniform (resp. sousperfectoid) affinoid analytic adic spaces.

In Section 3.1 we move to perfectoid spaces, where we define $\mathcal{D}_{\hat{\hat{\circ}}}^{a}\left(A^{+}\right)$for any affinoid perfectoid space $X=\operatorname{Spa}\left(A, A^{+}\right)$(Definition 3.2). Then we apply the complete descent results from Section 2.3 to obtain $v$-descent of $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$on totally disconnected perfectoid spaces (Theorem 3.7). From here, we can define $\mathcal{D}_{\hat{\ominus}}^{a}\left(\mathcal{O}_{X}^{+}\right)$for any perfectoid space $X$ (or even small $v$-stack on Perfd) by $v$-descent. This proves the existence part of Theorem [1.1, with the exception of the identification $\mathcal{D}_{\hat{0}( }^{a}\left(\mathcal{O}_{X}^{+}\right) \cong \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$if $X=\operatorname{Spa}\left(A, A^{+}\right)$is a + -bounded affinoid perfectoid space. This question we adress next.

There exists an evident functor

$$
\widetilde{(-)}: \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right),
$$

and this functor admits a right adjoint, denoted by $\Gamma(X,-)$. Let $\pi \in A$ be a pseudouniformizer. Using Theorem 1.2 (and in particular the assumption that $X$ is + -bounded), it is fairly easy to see that $\overline{(-)}$ is an equivalence if (and only if) the abstractly defined category $\mathcal{D}_{\hat{0}}^{a}\left(\mathcal{O}_{X}^{+}\right)$is generated under colimits by (suitably bounded) $\pi$-complete objects, and $\Gamma(X,-)$ preserves colimits (Lemma 3.22). Both of these properties of $\mathcal{D}_{\hat{\varrho}}^{a}\left(\mathcal{O}_{X}^{+}\right)$are proven by a detour through a category $\mathcal{D}_{\text {nuc }}\left(X, \mathbb{Z}_{p}[[\pi]]\right)$ of (overconvergent) nuclear sheaves of $\mathbb{Z}_{p}[[\pi]]$ modules on $X_{\text {qproet }}$ that we discuss in Section 4 following [15] and (9]. The discussion of $\mathcal{D}_{\text {nuc }}\left(X, \mathbb{Z}_{p}[[\pi]]\right)$ involves a rather long and detailed analysis of solid, $\omega_{1}$-solid and overconvergent objects in $\mathcal{D}\left(X_{\text {qproet }}, \mathbb{Z}_{p}[[\pi]]\right)$ for a general $p$-bounded spatial diamond. The criticial assertion in this section concerns the nuclear objects in the category of $\omega_{1}$-solid objects $\mathcal{D}_{\square}\left(X, \mathbb{Z}_{p}[[\pi]]\right)_{\omega_{1}} \subseteq \mathcal{D}\left(X_{\text {qproet }}, \mathbb{Z}_{p}[[\pi]]\right)$. We stress that nuclearity here refers to the abstract notion from [6, Lecture VIII], and not to the more geometric notion introduced in [15].

Lemma 1.6 (Lemma 4.15, Lemma 4.20). Let $\ell$ be a prime, and let $X$ be an $\ell$-bounded affinoid perfectoid space, and $\Lambda$ an adic profinite $\mathbb{Z}_{\ell}$-algebra, e.g., $\Lambda=\mathbb{Z}_{p}[[\pi]]$ if $\ell=p$. Then the category $\mathcal{D}_{\text {nuc }}(X, \Lambda)$ of nuclear objects in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is generated under colimits by overconvergent, complete, right-bounded objects.

Especially the generation by complete objects is non-obvious. Together with the good cohomological properties of $\omega_{1}$-solid $\mathbb{Z}_{p}[[\pi]]$-sheaves on $X_{\text {qproet }}$ this is the source of the desired properties of $\mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)$for a + -bounded affinoid perfectoid space $X$. The precise implementation of these properties involves Corollary [5.5, which realizes $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$as a category of $\mathcal{O}^{+a}$-modules in the abstract $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear category

$$
\mathcal{D}_{\mathrm{nuc}}\left(X,\left(A^{+}\right)_{\hat{\mathrm{a}}}^{a}\right):=\mathcal{D}_{\mathrm{nuc}}\left(X, \mathbb{Z}_{p}[[\pi]]\right) \otimes_{\mathcal{D}_{\mathrm{nuc}}\left(\mathbb{Z}_{p}[[\pi]]\right)} \mathcal{D}_{\hat{仓}}^{a}\left(A^{+}\right) .
$$

This latter category is well-behaved thanks to the rigidity of $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$ (Remark 4.22) and dualizability of $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$as a $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-module (Lemma 5.2$)$. However, the equivalence

$$
\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right) \cong \operatorname{Mod}_{\mathcal{O}^{+a}}\left(\mathcal{D}_{\mathrm{nuc}}\left(X,\left(A^{+}\right)_{\hat{\circ}}^{a}\right)\right) .
$$

of Corollary 5.5 needs as an input the equivalence $\mathcal{D}_{\hat{ธ}}^{a}\left(\mathcal{O}_{\bar{Z}^{\prime}}^{+}\right) \cong \mathcal{D}_{\hat{0}}^{a}\left(B^{+}, A^{+}\right)$for the relative compactification $\bar{Z}^{/ X}$ of a strictly totally disconnected space $Z=\operatorname{Spa}\left(B, B^{+}\right)$, which is quasi-pro-étale over $X$. As $\bar{Z}^{/ X}$ is not necessarily totally disconnected, this does not follow from the descent of totally disconnected spaces proven in Theorem 3.7, and we supply an argument in Theorem [3.23 using an analysis of families of Riemann-Zariski spaces (Lemma 3.26).

### 1.1 Acknowledgements

We thank Arthur-César Le Bras heartily for all his feedback, and for all discussions in an early stage of this project. For personal, Monadic reasons he unfortunately had to leave this project in a later stage. We moreover thank Gregory Andreychev, Johan de Jong, Juan Esteban Rodriguez Camargo and Peter Scholze for discussions related to this paper.

### 1.2 Notations and conventions

For technical convenience we fix an implicit cut-off cardinal $\kappa$ (in the sense of [22, Section 4]), and assume all our perfectoid spaces, and condensed sets to be $\kappa$-small. In particular, for a Huber pair $\left(A, A^{+}\right)$its associated category $\mathcal{D}_{\square}\left(A, A^{+}\right)$( $\mathbb{1}$, Theorem 3.28]) is generated by a set of compact objects. Passing to the filtered colimit over all $\kappa$ 's implies the descent statement in general.

We work implicitly "condensed" and "animated". More precisely, a ring $R$ will mean a condensed animated ring. It is called static if $R \cong \pi_{0}(R)$ (as a condensed ring), and discrete if $R \cong R(*)_{\text {disc }}$, where $R(*)$ is the "underlying animated ring", or equivalently, $R$ is discrete if the functor $S \mapsto R(S)$ on profinite sets maps cofiltered inverse limits to filtered colimits.

Given a ring $R$ and an ideal $I \subseteq \pi_{0}(R(*))$ generated by elements $f_{1}, \ldots, f_{n} \in I$, we set $M / I:=\mathbb{Z} \otimes_{\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]} M$ for any $R$-module $M$ where the tensor product is implicitly derived. Here, the map $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right] \rightarrow R$ making $M$ into an $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$-module is classified by the elements $f_{1}, \ldots, f_{n}$. In particular, the quotient $M / I$ depends on the choice of elements, and we will carefully make this clear when this may create confusion. Concretely, the quotient $M / I$ is calculated by the tensor product of the complexes $M \xrightarrow{f_{i}} M, i=1, \ldots, n$.

Given a $\mathbb{Z}[x]$-linear category $\mathcal{C}$ with sequential limits we call $X \in \mathcal{C} x$-complete if the inverse limit

$$
\ldots \rightarrow X \xrightarrow{x} X
$$

vanishes.

## $2 \mathcal{D}_{\hat{\ominus}}(A)$ for adic rings

When proving descent for solid modules over $p$-complete rings, one of the main challenges is that in general the compact generators of $\mathcal{D}_{\square}(A)$ are not $p$-complete, even if $A$ itself is so. Indeed, by the very definition the compact generators of $\mathcal{D}_{\square}(A)$ are colimits over the category of maps $A^{\prime} \rightarrow A$ with $A^{\prime}$ a finitely generated $\mathbb{Z}$-algebra ( $\mathbb{1}$, Definition 3.20]). This defect breaks the usual strategy of reducing descent questions to the reductions mod $p^{n}$ and therefore makes it hard to get any good general descent results on $p$-adically complete rings and general solid modules. To overcome this issue, we present a slight modification of $\mathcal{D}_{\square}(A)$, denoted $\mathcal{D}_{\hat{\mathrm{\circ}}}(A)$ in the following (Definition [2.4), where we replace the compact generators of $\mathcal{D}_{\square}(A)$ by their $p$-adic completions. This construction may seem a bit artificial, but the results in this section show that we do not lose much and still have a very tight relation to $\mathcal{D}_{\square}(A)$. With this modified version of solid modules we then explain how descent questions can indeed be reduced modulo $p^{n}$ (Theorem (2.30).
Of course the construction of $\mathcal{D}_{\hat{\imath}}(A)$ works for any adically complete rings (instead of just $p$-complete rings), so we perform it in the natural generality. Hence, we start by discussing adic analytic rings.

### 2.1 Adic rings

We fix the following terminology. Note that we adhere to our conventions from Section 1.2,
We let $\mathbb{Z}_{\square}$ be the solid analytic ring from [23, Definition 5.1] and $\mathbb{Z}[T]_{\square}$ the analytic ring from [23, Theorem 8.2]. If $A$ is any solid ring, and $S \rightarrow \pi_{0}(A)(*)$ a map of sets, then we denote by $(A, S)_{\square}$ the analytic ring over $\mathbb{Z}_{\square}$ such that an object $M \in \operatorname{Mod}_{A}\left(D\left(\mathbb{Z}_{\square}\right)\right)$ is $(A, S)_{\square^{-}}$ complete if and only if $M$ is $\mathbb{Z}[s]_{\square}$-complete for any $s \in S$ with induced map $\mathbb{Z}[s] \rightarrow A$ of rings. For $(A, S)=\left(A, A^{+}\right)$a classical complete Huber pair, this analytic ring structure has been analyzed in [1. We note that by [24, Appendix to lecture XII] and [1, Proposition 3.32] $(A, S)_{\square} \cong(A, \widetilde{S})_{\square}$ if $\widetilde{S} \subseteq \pi_{0}(A)(*)$ denotes the smallest integrally closed subring containing each topologically nilpotent element and the image of $S]^{1}$

Definition 2.1. (a) We say that a ring $A$ is adic if there is some finitely generated ideal $I \subseteq \pi_{0}(A)(*)$ such that $A / I$ is discrete and $A$ is $I$-adically complete. We call any such ideal $I$ an ideal of definition of $A$.
(b) An adic analytic ring is an analytic ring $\mathcal{A}$ of the form $\mathcal{A}=\left(A, A^{+}\right)_{\square}$, where $A$ is an adic ring and $A^{+} \subseteq \pi_{0}(A)(*)$ is a subring. We denote by

$$
\text { AdicRing } \subseteq \text { AnRing }
$$

the full subcategory spanned by the adic analytic rings. Given an adic analytic ring $\mathcal{A}$ we usually write $\underline{\mathcal{A}}=\mathcal{A}[*]$ for its underlying adic ring.
(c) A map $f: A \rightarrow B$ of adic rings is called adic if for some ideal of definition $I$ of $A, f(I)$ generates an ideal of definition of $B$. A map of adic analytic rings is called adic if the map of underlying adic rings is adic.

Given an adic ring $A$ with ideal of definition $I$, we denote by $\mathcal{D}(A):=\operatorname{Mod}_{A}(\mathcal{D}(\operatorname{Cond}(\mathrm{Ab}))$ its (stable $\infty$-)category of modules in condensed abelian groups, and we let $\mathcal{D}(A)_{\hat{I}} \subseteq \mathcal{D}(A)$ be the full subcategory of $I$-adically complete $A$-modules, i.e., those which are $x$-complete for any $x \in I$. This notion does not depend on the choice of $I$ or is generators:

[^0]Lemma 2.2. Let $A$ be an adic ring.
(i) For any two ideals of definition $I$, $I^{\prime}$ of $A$ we have $\mathcal{D}(A)_{\hat{I}}=\mathcal{D}(A)_{\hat{I}^{\prime}}$.
(ii) A map $f: A \rightarrow B$ is adic if and only if for every ideal of definition $I$ of $A, f(I) \subseteq$ $\pi_{0}(B)(*)$ generates an ideal of definition of $B$.

Proof. We first prove (i), so let $I$ and $I^{\prime}$ be given. Without loss of generality we can assume $I \subseteq I^{\prime}$ because the intersection of two ideals of definition is an ideal of definition. Pick any $a \in I^{\prime}$. Then $A / I$ is $a$-adically complete (as a finite limit of copies of $A$ 's), and hence the induced map $\mathbb{Z}[x] \rightarrow A / I$ factors over $\mathbb{Z}[[x]] \rightarrow A / I$. Since $A / I$ is discrete and in particular nuclear as a $\mathbb{Z}_{\square}$-module, we have $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[[x]], A / I)={\underset{\longrightarrow}{\lim }}^{\operatorname{Hom}_{\mathbb{Z}}}\left(\mathbb{Z}[x] / x^{n}, A / I\right)$, which implies that $a$ is nilpotent in $\pi_{0}(A / I)(*)$. But then any $I$-adically complete $A$-module is also $a$-adically complete. Since $a \in I^{\prime}$ was arbitrary, we immediately arrive at the claimed identity.

We now prove (ii), so let $f: A \rightarrow B$ be an adic map of adic rings and $I$ some ideal of definition of $A$. By assumption there is some ideal of definition $I^{\prime}$ of $A$ such that $f\left(I^{\prime}\right)$ generates an ideal of definition of $B$. If $I^{\prime \prime}$ is any ideal of definition of $A$ containing $I^{\prime}$ then by the proof of (i), every element in $I^{\prime \prime}$ has some power which lies in $I^{\prime}$; hence also $f\left(I^{\prime \prime}\right)$ generates an ideal of definition of $B$. We apply this to $I^{\prime \prime}=I^{\prime}+I$. But by reversing this argument we also deduce that $f(I)$ generates an ideal of definition of $B$, as desired. The converse direction is clear.

Definition 2.3. Let $A$ be an adic ring. We say that an $A$-module $M \in \mathcal{D}(A)$ is adically complete if it is $I$-adically complete for some (equivalently every) ideal of definition $I$ of $A$. We denote by $\mathcal{D}_{\mathrm{cpl}}(A) \subseteq \mathcal{D}(A)$ the full subcategory spanned by the adically complete $A$-modules and by

$$
(-)_{\mathrm{cpl}}: \mathcal{D}(A) \rightarrow \mathcal{D}_{\mathrm{cpl}}(A)
$$

the left-adjoint of the inclusion. For an analytic adic $\operatorname{ring} \mathcal{A}$ we similarly denote $\mathcal{D}_{\mathrm{cpl}}(\mathcal{A}):=$ $\mathcal{D}(\mathcal{A}) \cap \mathcal{D}_{\mathrm{cpl}}(\underline{\mathcal{A}})$.

Here, the existence of $(-)_{\mathrm{cpl}}$ is guaranteed by the adjoint functor theorem, which applies here as all categories here are presentable (recall that we have fixed a cut-off cardinal $\kappa$ in Section (1.2).

### 2.2 Definition of $\mathcal{D}_{\hat{\wedge}}(A)$

With a good notion of adic rings and adically complete modules at hand, we can now construct the promised modification of $\mathcal{D}(\mathcal{A})$. The definition is very simple, albeit somewhat artificial:

Definition 2.4. Given $\mathcal{A}$ an adic analytic ring, let $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$ be the full subcategory generated under finite (co)limits and retracts from the essential image of the full subcategory $\mathcal{D}(\mathcal{A})^{\omega}$ of compact $\mathcal{A}$-modules under $(-)_{\mathrm{cpl}}$. Then we define

$$
\mathcal{D}(\hat{\mathcal{A}}):=\operatorname{Ind}\left(\mathcal{C}_{\mathcal{A}}\right)
$$

Writing $\mathcal{A}=\left(A, A^{+}\right)_{\square}$, we also use the notation $\mathcal{D}_{\hat{\emptyset}}\left(A, A^{+}\right):=\mathcal{D}(\hat{\mathcal{A}})$.
The following results show that $\mathcal{D}(\hat{\mathcal{A}})$ is quite close to $\mathcal{D}(\mathcal{A})$ and in particular inherits all of its nice properties:

Lemma 2.5. Let $\mathcal{A}$ be an adic analytic ring.
(i) $\mathcal{D}(\hat{\mathcal{A}})$ is a stable $\infty$-category that can naturally be equipped with a closed symmetric monoidal structure and a t-structure.
(ii) There is a natural colimit-preserving, symmetric monoidal and right t-exact functor $\alpha^{*}: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})$.
(iii) The functor $\alpha^{*}$ has a right adjoint $\alpha_{*}: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\mathcal{A})$ which is $t$-exact, conservative, preserves all small limits and colimits and commutes with truncations.
(iv) For every compact $\mathcal{A}$-module $P \in \mathcal{D}(\mathcal{A})$ we have $\alpha_{*} \alpha^{*} P=P_{\mathrm{cpl}}$.

Proof. Let $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$ be as in Definition 2.4. Recall that there is a symmetric monoidal structure $\hat{\otimes}_{\mathcal{A}}$ on $\mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$ given by the completed tensor product $2^{2}$ Since compact objects in $\mathcal{D}(\mathcal{A})$ are stable under $\otimes_{\mathcal{A}}$ (here we use that $\mathcal{A}$ is an analytic ring over $\mathbb{Z}_{\square}$ ) it follows that $\mathcal{C}_{\mathcal{A}}$ is stable under $\hat{\otimes}_{\mathcal{A}}$, so that we naturally get a symmetric monoidal structure on $\mathcal{C}_{\mathcal{A}}$. By [12, Corollary 4.8.1.14] we can uniquely extend this symmetric monoidal structure to $\mathcal{D}(\hat{\mathcal{A}})=\operatorname{Ind}\left(\mathcal{C}_{\mathcal{A}}\right)$ so that it preserves colimits in each argument.

The functor $\alpha^{*}$ is the natural functor colimit $\mathcal{D}(\mathcal{A})=\operatorname{Ind}\left(\mathcal{D}(\mathcal{A})^{\omega}\right) \rightarrow \operatorname{Ind}\left(\mathcal{C}_{\mathcal{A}}\right)=\mathcal{D}(\hat{\mathcal{A}})$ induced by the completion functor $\mathcal{D}(\mathcal{A})^{\omega} \rightarrow \mathcal{C}_{\mathcal{A}}$. By [12, Corollary 4.8.1.14] $\alpha^{*}$ can be upgraded uniquely to a symmetric monoidal functor. The right adjoint $\alpha_{*}$ exists by the adjoint functor theorem (using that we have fixed a cut-off cardinal). Since $\alpha^{*}$ preserves compact objects (by definition) it follows that $\alpha_{*}$ commutes with filtered colimits and hence with all colimits. It is also easy to see that $\alpha_{*}$ is conservative: Given any $M \in \mathcal{D}(\hat{\mathcal{A}})$ with $\alpha_{*} M=0$, pick any compact $P \in \mathcal{D}(\mathcal{A})$; then $0=\operatorname{Hom}\left(P, \alpha_{*} M\right)=\operatorname{Hom}\left(\alpha^{*} P, M\right)$, so it is enough to show that the family of functors $\operatorname{Hom}\left(\alpha^{*} P,-\right)$ is conservative (with $P$ ranging over compact $\mathcal{A}$-modules). But this follows immediately from the fact that the $\alpha^{*} P$ form compact generators of $\mathcal{D}(\hat{\mathcal{A}})$ by construction.

We now prove (iv), so fix any compact $\mathcal{A}$-module $P$. Note that $\alpha^{*} P$ is adically complete: For any ideal of definition $I$ for $\underline{\mathcal{A}}$ and any $x \in I$ we need to check that $\lim _{ڭ_{x}} \alpha^{*} P=0$ with $\lim _{\leftrightarrows}$ taken in $\mathcal{D}(\hat{\mathcal{A}})$, and for this it is enough to check that $\lim _{x} \operatorname{Hom}\left(Q, \alpha^{*} P\right)=0$ for every $\overleftarrow{Q} \in \mathcal{C}_{\mathcal{A}}$; but this statement depends only on $\mathcal{C}_{\mathcal{A}}$, which is a full subcategory of $\mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$. Thus, we have checked that $\alpha^{*} P$ is adically complete. Since $\alpha_{*}$ preserves all small limits, we deduce that also $\alpha_{*} \alpha^{*} P$ is adically complete, i.e. lies in $\mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$. Hence the unit $P \rightarrow \alpha_{*} \alpha^{*} P$ induces a map $P_{\mathrm{cpl}} \rightarrow \alpha_{*} \alpha^{*} P$. To show that this map is an isomorphism, it is enough to do so after applying $\operatorname{Hom}(Q,-)$ for any compact $Q \in \mathcal{D}(\mathcal{A})$; we have:

$$
\operatorname{Hom}\left(Q, P_{\mathrm{cpl}}\right)=\operatorname{Hom}\left(Q_{\mathrm{cpl}}, P_{\mathrm{cpl}}\right)=\operatorname{Hom}\left(\alpha^{*} Q, \alpha^{*} P\right)=\operatorname{Hom}\left(Q, \alpha_{*} \alpha^{*} P\right)
$$

where in the second step we used that $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{D}(\mathcal{A})$ is a full subcategory.
We have now proved everything apart from the claims about the $t$-structure. We define $\mathcal{D}_{\geq 0}(\hat{\mathcal{A}}) \subseteq \mathcal{D}(\hat{\mathcal{A}})$ to be the full subcategory spanned by those objects $M \in \mathcal{D}(\hat{\mathcal{A}})$ such that $\alpha_{*} M \in \mathcal{D}_{\geq 0}(\mathcal{A})$. This subcategory is clearly stable under colimits and extensions and by (iv) it contains $\alpha^{*} \mathcal{A}[S]$ for all profinite sets $S$. It follows that the $\alpha^{*} \mathcal{A}[S]$ are compact generators of $\mathcal{D}_{\geq 0}(\hat{\mathcal{A}})$, so by [12, Proposition 1.4.4.11.(i)] this subcategory defines a $t$-structure on $\mathcal{D}(\hat{\mathcal{A}})$. Since every $M \in \mathcal{D}_{\geq 0}(\mathcal{A})$ is a (sifted) colimit of the free generators $\mathcal{A}[S]$ it follows easily from (iv) that $\alpha^{*}$ is right $t$-exact (because $(-)_{\text {cpl }}$ is so). Moreover, $\alpha_{*}$ is right $t$-exact by construction and it is left $t$-exact because $\alpha^{*}$ is right $t$-exact. Finally, one checks easily that $\alpha_{*}$ commutes with $\tau_{\geq 0}$ by passing to left adjoints; from the usual truncation triangle it then follows that $\alpha_{*}$ also commutes with $\tau_{\leq 0}$.

[^1]Remark 2.6. By Lemma 2.5 the adjunction of $\alpha^{*}$ and $\alpha_{*}$ is monadic and hence identifies $\mathcal{D}(\hat{\mathcal{A}})$ with a category of modules over some monad on $\mathcal{D}(\mathcal{A})$. Thus one may think of $\hat{\mathcal{A}}$ as some form of generalized analytic ring and $\mathcal{D}(\hat{\mathcal{A}})$ as its category of modules. One may attempt to formalize this idea in order to get a general theory of generalized analytic rings, but we will not pursue this further. We note that $\alpha_{*}: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\mathcal{A})$ is in general not $\mathcal{D}(\mathcal{A})$ linear, or equivalently $\alpha_{*}$ does not satisfy the projection formula (otherwise for each compact object $P \in \mathcal{A}$ one has $P \otimes \alpha_{*}\left(\alpha^{*}(\underline{\mathcal{A}}) \cong \alpha_{*}\left(\alpha^{*} P\right) \cong P_{\text {cpl }}\right)$. Thus, modules over the monad $\alpha_{*}\left(\alpha^{*}(-)\right)$ do in general not identify with modules under some ring object in $\mathcal{D}(\mathcal{A})$.

Before we continue, let us make everything more functorial. We have the following straightforward result:

Lemma 2.7. The construction $\mathcal{A} \mapsto \mathcal{D}(\hat{\mathcal{A}})$ defines a functor $\operatorname{AdicRing}{ }^{\mathrm{op}} \rightarrow \mathrm{CAlg}\left(\mathcal{C}\right.$ at $\left.{ }_{\infty}\right)$ and $\alpha^{*}: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})$ defines a natural transformation of such functors.

Proof. The assignment $\mathcal{A} \mapsto \mathcal{D}(\mathcal{A})$ defines a functor AdicRing ${ }^{\text {op }} \rightarrow \operatorname{CAlg}\left(\mathcal{C}\right.$ at $\left.{ }_{\infty}\right)$ (see e.g. [16, Proposition 2.3.26]). By looking at the associated cocartesian family of $\infty$-operads and restricting to a full subcategory of that, we can construct the functor $\mathcal{A} \mapsto \mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$ (see [16, Lemma 2.2.22] for the general strategy of this argument). By restricting to even small full subcategories we obtain the functor which associates to each $\mathcal{A}$ the full subcategory $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$ from Definition 2.4, By the remark right before [12, Proposition 4.8.1.10] the assignment $\mathcal{C} \mapsto \operatorname{Ind}(\mathcal{C})$ defines an endofunctor of symmetric monoidal $\infty$-categories. This finishes the construction of the functor $\mathcal{A} \mapsto \mathcal{D}(\hat{\mathcal{A}})$.

The construction of the natural transformation $\alpha^{*}$ works in a very similar fashion, by making everything relative over $\Delta^{1}$.

Definition 2.8. (a) For every $\mathcal{A} \in$ AdicRing we denote by $-\otimes_{\mathcal{A}} \hat{\mathcal{A}}: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})$ the functor $\alpha^{*}$ from Lemma 2.5.(ii).
(b) For every map $\mathcal{A} \rightarrow \mathcal{B}$ in AdicRing we denote by $-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{B}})$ the induced functor from Lemma 2.7.

We will write $\alpha_{\mathcal{A}}^{*}, \alpha_{\mathcal{A}, *}$ in case we need to clarify the dependence of $\alpha^{*}, \alpha_{*}$ on $\mathcal{A}$. If $\mathcal{A} \rightarrow \mathcal{B}$ is a morphism of adic analytic rings, then $\alpha_{\mathcal{B}}^{*}\left((-) \otimes_{\mathcal{A}} \mathcal{B}\right) \cong \alpha_{\mathcal{A}}^{*}(-) \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ by Lemma 2.7.

We remark that $-\otimes_{\mathcal{A}} \hat{\mathcal{A}}$ and $-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ are symmetric monodial. While for general adic analytic rings $\mathcal{A}, \mathcal{D}(\hat{\mathcal{A}})$ differs from $\mathcal{D}(\mathcal{A})$, in praxis these two categories are often the same:

Lemma 2.9. Let $\left(A, A^{+}\right)_{\square}$ be an adic analytic ring. If either $A$ is discrete or $A^{+}$is finitely generated over $\mathbb{Z}$, then $\mathcal{D}(\hat{\mathcal{A}})=\mathcal{D}(\mathcal{A})$.

Proof. We use the notation from Lemma 2.5. Since $\alpha_{*}$ is conservative, in order to prove the desired equivalence of categories it is enough to show that $\alpha^{*}$ is fully faithful, i.e. that for all $M \in \mathcal{D}(\mathcal{A})$ the unit of the adjunction $M \xrightarrow{\sim} \alpha_{*} \alpha^{*} M$ is an isomorphism. Since $\alpha_{*}$ and $\alpha^{*}$ preserve colimits, this reduces to the case that $M=P$ is compact, in which case by Lemma 2.5.(iv) the claim reduces to showing that $P$ is adically complete. This is evident if $A$ is discrete, so it remains to treat the case where $A^{+}$is finitely generated.

We can assume that $A^{+}$is a finitely generated polynomial algebra over $\mathbb{Z}$ and that there is a map $A^{+} \rightarrow A$. By choosing generators of an ideal of definition of $A$ we construct a map $A^{+}\left[x_{\bullet}\right]:=A^{+}\left[x_{1}, \ldots, x_{n}\right] \rightarrow A$, which then automatically factors as $A^{+}\left[\left[x_{\bullet}\right]\right] \rightarrow A$. Note that for every profinite set $S$ we have

$$
\left(A, A^{+}\right)_{\square}[S]=A \otimes_{A_{\square}^{+}} A_{\square}^{+}[S]=A \otimes_{\left(A^{+}\left[\left[x_{\bullet}\right]\right], A^{+}\right)_{\square}}\left(A^{+}\left[\left[x_{\bullet}\right]\right], A^{+}\right)_{\square}[S]
$$

On the other hand, $\left(A^{+}\left[\left[x_{\bullet}\right]\right], A^{+}\right)_{\square}[S] \cong \prod_{I} A^{+}\left[\left[x_{\bullet}\right]\right]$ for some set $I$ (as $A^{+}$is finitely generated over $\mathbb{Z}$ ), so in particular it is $\left(x_{1}, \ldots, x_{n}\right)$-adically complete. It follows from [16, Proposition 2.12.10] that $\left(A, A^{+}\right)_{\square}[S]$ is $\left(x_{1}, \ldots, x_{n}\right)$-adically complete, i.e. adically complete as an $A$ module. This proves the claim.

Lemma 2.10. Let $\mathcal{A}$ be an adic analytic ring with ideal of definition $I$. Then $-\otimes_{\mathcal{A}} \hat{\mathcal{A}}$ induces an equivalence of categories

$$
\mathcal{D}(\mathcal{A} / I)=\operatorname{Mod}_{\underline{\mathcal{A}} / I}(\mathcal{D}(\hat{\mathcal{A}}))
$$

Here, the $\mathcal{D}(\mathcal{A} / I)$ denotes analytic ring structure on $\underline{\mathcal{A}} / I$ with its induced analytic ring structure along $\underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}} / I$.

Proof. Let us write $\alpha^{*}:=-\otimes_{\mathcal{A}} \hat{\mathcal{A}}$. We have $\mathcal{D}(\mathcal{A} / I)=\operatorname{Mod}_{\underline{\mathcal{A}} / I}(\mathcal{D}(\mathcal{A}))$, so since $\alpha^{*}$ is symmetric monoidal, it induces a functor $\mathcal{D}(\mathcal{A} / I) \rightarrow \operatorname{Mod}_{\underline{\mathcal{A}} / I}(\underline{\mathcal{D}}(\hat{\mathcal{A}}))$. Its right adjoint is given by $\alpha_{*}$, which is conservative, so the desired equivalence reduces to showing that $\alpha^{*}$ is fully faithful on $\underline{\mathcal{A}} / I$-modules, i.e. that id $\xrightarrow{\sim} \alpha_{*} \alpha^{*}$ is an isomorphism. This in turn can be checked on compact generators, i.e. for $P=(\mathcal{A} / I)[S]$. But such a $P$ is also compact as $\mathcal{A}$ module (by our conventions Section 1.2), hence by Lemma 2.5.(iv) we have $\alpha_{*} \alpha^{*} P=P_{\text {cpl }}=P$, as desired.

Next we study adic completeness in $\mathcal{D}(\hat{\mathcal{A}})$. Since this category is $\mathcal{D}(\mathcal{A})$-linear via the symmetric monoidal functor $-\otimes_{\mathcal{A}} \hat{\mathcal{A}}$, there is a good notion of adically complete modules in $\mathcal{D}(\hat{\mathcal{A}})$ :

Definition 2.11. Let $\mathcal{A}$ be an adic analytic ring. An $\hat{\mathcal{A}}$-module $M \in \mathcal{D}(\hat{\mathcal{A}})$ is called adically complete if for some (equivalently every) ideal of definition $I$ of $\underline{\mathcal{A}}$ and every $x \in I$ we have $\lim _{x} M=0$. We denote by $\mathcal{D}_{\mathrm{cpl}}(\hat{\mathcal{A}}) \subseteq \mathcal{D}(\hat{\mathcal{A}})$ the full subcategory spanned by the adically complete modules. Similarly, we let $\mathcal{D}_{\mathrm{cpl}}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ be the full subcategory of adically complete objects in $\mathcal{D}(\mathcal{A}) \cdot 3$

The following result is a first indicator for why we work with $\hat{\mathcal{A}}$ in this paper: The complete objects in $\mathcal{D}(\hat{\mathcal{A}})$ behave much better with respect to pullbacks and tensor products than those in $\mathcal{D}(\mathcal{A})$.

Lemma 2.12. Let $\mathcal{A}$ be an adic analytic ring.
(i) An $\hat{\mathcal{A}}$-module is adically complete if and only if the underlying $\mathcal{A}$-module is adically complete. All the equivalent characterizations from [16, Lemma 2.12.4] apply to adically complete modules in $\mathcal{D}(\hat{\mathcal{A}})$, Most notably, an $\hat{\mathcal{A}}$-module $M$ is adically complete if and only if all its homotopy objects $\pi_{i}(M), i \in \mathbb{Z}$, are.
(ii) The $t$-structure on $\mathcal{D}(\hat{\mathcal{A}})$ restricts to a $t$-structure on $\mathcal{D}_{\mathrm{cpl}}(\hat{\mathcal{A}})$.
(iii) $\mathcal{D}_{\text {cpl }}^{-}(\hat{\mathcal{A}})$ is stable under the symmetric monoidal structure $\otimes_{\hat{\mathcal{A}}}$ on $\mathcal{D}(\hat{\mathcal{A}})$ from Lemma 2.5 .
(iv) Let $\mathcal{A} \rightarrow \mathcal{B}$ be an adic map of adic analytic rings. Then $-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ restricts to a functor $\mathcal{D}_{\text {cpl }}^{-}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}_{\text {cpl }}^{-}(\hat{\mathcal{B}})$.

Proof. We use the notation $\alpha^{*}$ and $\alpha_{*}$ from Lemma 2.5. Then (i) follows easily from Lemma 2.5.(iii) and (ii) is just a reformulation of the fact that $M$ is adically complete if and only if all $\pi_{i} M$ are adically complete (which is part of (i)).

[^2]We now prove (iv), so let $M \in \mathcal{D}_{\text {cpl }}^{-}(\hat{\mathcal{A}})$ be given. We need to show that $M \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ is adically complete. For some $n \geq 0$ we can find an adic map $\mathcal{A}_{0}:=\left(\mathbb{Z}\left[\left[x_{1}, \ldots, x_{n}\right]\right], \mathbb{Z}\right)_{\square} \rightarrow \mathcal{A}$. By using the bar resolution for the monadic adjunction between base-change and forgetful functor along $\mathcal{A}_{0}=\hat{\mathcal{A}}_{0} \rightarrow \hat{\mathcal{A}}$ (using Lemma [2.9) we can write $M$ as a uniformly right-bounded geometric realizations of modules of the form $M_{0} \otimes_{\mathcal{A}_{0}} \hat{\mathcal{A}}$ for some $M_{0} \in \mathcal{D}\left(\mathcal{A}_{0}\right)$. Since adic completion is bounded and hence commutes with uniformly right-bounded geometric realizations, adic completeness is stable under uniformly right-bounded geometric realizations. We can thus reduce the claim to the case $\mathcal{A}=\mathcal{A}_{0}$. We can assume that $M$ is connective and thus write it as a geometric realization of objects of the form $\bigoplus_{i \in I} \mathcal{A}\left[S_{i}\right]$ for some profinite sets $S_{i}$. By using again the fact that adic completeness is stable under uniformly right-bounded geometric realizations, and also under $\omega_{1}$-filtered colimits, we reduce to the case that $M=\widehat{\bigoplus}_{k \in \mathbb{N}} \mathcal{A}\left[S_{k}\right]$ for some profinite sets $S_{k}$. Note that it is enough to show that $-\otimes_{\mathcal{A}} \hat{\mathcal{B}}$ preserves $x$-complete objects for all $x \in I$, so from now on we work only with $x$-completions. Then we can write $M=\lim _{\alpha} \Pi_{k} x^{\alpha_{k}} \mathcal{A}\left[S_{k}\right]$, where the colimit is taken over all monotonous sequences $\alpha: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ converging to $\infty$. We now claim that the natural map

$$
M \otimes_{\mathcal{A}} \hat{\mathcal{B}}=\underset{\alpha}{\lim }\left(\left(\prod_{k} x^{\alpha_{k}} \mathcal{A}\left[S_{k}\right]\right) \otimes_{\mathcal{A}} \hat{\mathcal{B}} \xrightarrow{\sim} \underset{\alpha}{\lim } \prod_{k} x^{\alpha_{k}}\left(\mathcal{A}\left[S_{k}\right] \otimes_{\mathcal{A}} \hat{\mathcal{B}}\right)\right.
$$

is an isomorphism. The argument is similar to the proof of [15, Lemma 3.7.(ii)], where the crucial input is that the constituents $\left(\prod_{k} x^{\alpha_{k}} \mathcal{A}\left[S_{k}\right]\right) \otimes_{\mathcal{A}} \hat{\mathcal{B}}$ are $x$-adically complete - this is true because $\prod_{k} \mathcal{A}\left[S_{k}\right]$ is compact in $\mathcal{D}(\mathcal{A})$ and thus $\left(\prod_{k} x^{\alpha_{k}} \mathcal{A}\left[S_{k}\right]\right) \otimes_{\mathcal{A}} \hat{\mathcal{B}}$ is compact in $\mathcal{D}(\hat{\mathcal{B}})$ (as the right adjoint to $-\otimes_{\mathcal{A}} \hat{\mathcal{B}}$ commutes with colimits). Now compact objects in $\mathcal{D}(\hat{\mathcal{B}})$ are adically complete (by Lemma 2.5.(iv)). To finish the proof of (iv) it remains to see that the natural map

$$
\underset{\alpha}{\lim } \prod_{k} x^{\alpha_{k}}\left(\mathcal{A}\left[S_{k}\right] \otimes_{\mathcal{A}} \hat{\mathcal{B}}\right) \xrightarrow{\sim} \widehat{\bigoplus}_{k}\left(\mathcal{A}\left[S_{k}\right] \otimes_{\mathcal{A}} \hat{\mathcal{B}}\right)
$$

is an isomorphism, or equivalently that the left-hand side is $x$-adically complete. This can be checked after applying the forgetful functor to $\mathcal{D}(\mathcal{A})$ and on homology, where it is a straightforward computation. This finishes the proof of (iv).

It remains to prove (iii), so let $M, N \in \mathcal{D}_{\text {cpl }}^{-}(\hat{\mathcal{A}})$ be given. By the same bar resolution argument as in the proof of (iv) we can assume that $M$ and $N$ come via pullback from adically complete right-bounded $\mathcal{A}_{0}:=\left(\mathbb{Z}\left[\left[x_{1}, \ldots, x_{n}\right]\right], \mathbb{Z}\right)_{\square}$-modules for some $n \geq 0$. But then by (iv) the claim reduces to the case $\mathcal{A}=\mathcal{A}_{0}$, where it follows from [16, Proposition 2.12.10].

Example 2.13. The boundedness assumption in Lemma 2.12.(iv) is necessary: If $M=N=$ $\underset{i \in \mathbb{Z}}{\oplus} \underline{\mathcal{A}}[i]$, then $M \otimes_{\hat{\mathcal{A}}} N$ is not adically complete (if $\underline{\mathcal{A}}$ is not discrete), even though $M, N$ are $i \in \mathbb{Z}$ adically complete. Namely, the tensor product contains the direct summand $\underset{i \in \mathbb{Z}}{ } \mathcal{A}$.

Similarly, the preservation of completeness is wrong for the base change $-\underset{i \in \mathbb{Z}}{-} \otimes_{\mathcal{A}} \mathcal{B}$ of adic analytic rings with the usual definition of $D_{\square}(-)$.

In the presence of finite Tor dimension, the preservation of adic completeness can be strengthened to unbounded complexes:

Corollary 2.14. Let $\mathcal{A} \rightarrow \mathcal{B}$ be an adic map of adic analytic rings such that for some ideal of definition I the map $\mathcal{A} / I \rightarrow \mathcal{B} / I$ has finite Tor dimension. Then $\hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$ has finite Tor dimension and the functor $-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ preserves adic completeness.

Proof. We first prove the claim about Tor dimension, so let $M \in \mathcal{D}^{\ominus}(\hat{\mathcal{A}})$ be given. Then we can write $M$ as a filtered colimit of adically complete modules; by truncating we can assume
that these adically complete modules lie in $\mathcal{D}^{\ominus}$. We can thus assume that $M$ itself is adically complete, hence the same is true for $N:=M \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ by Lemma 2.12.(iv). But by assumption $N / I$ is bounded to the left (independent of $M$ ), hence so is $N$.
The second claim follows from the finite Tor dimension by writing any adically complete module as a filtered colimit over its truncations $\tau_{\geq n}$ for $n \rightarrow-\infty$.

Unsurprisingly, the difference between $\mathcal{A}$ and $\hat{\mathcal{A}}$ vanishes on complete objects; more precisely we have:

Lemma 2.15. Let $\mathcal{A}$ be an adic analytic ring. Then the functor $\alpha_{*}: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\mathcal{A})$ restricts to an equivalence $\mathcal{D}_{\mathrm{cpl}}(\hat{\mathcal{A}}) \cong \mathcal{D}_{\mathrm{cpl}}(\mathcal{A})$.

Proof. By Lemma 2.12 the restriction $\alpha_{*}: \mathcal{D}_{\text {cpl }}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}_{\text {cpl }}(\mathcal{A})$ is well-defined. By general theory of completion, it admits as a left adjoint the functor $\hat{\alpha}^{*}:=(-)_{\mathrm{cpl}} \circ \alpha^{*}$. As both $\alpha_{*}, \hat{\alpha}^{*}$ commute with the reduction $\mathcal{A} \rightarrow \mathcal{A} / I$ for some (finitely generated) ideal of definition $I$, the unit/counit of this adjunction are isomorphisms by Lemma 2.10. This finishes the proof.

For any adic analytic ring $\mathcal{A}, \mathcal{D}(\hat{\mathcal{A}})$ is a compactly generated closed symmetric monoidal category with compact unit object $\underline{\mathcal{A}}$, hence the definition of nuclear objects from [6, Lecture VIII] applies:

Definition 2.16. We denote by $\mathcal{D}(\hat{\mathcal{A}})^{\text {nuc }} \subseteq \mathcal{D}(\hat{\mathcal{A}})$ the full subcategory spanned by the nuclear objects, as defined in [6, Definition 8.5].

Proposition 2.17. For every adic analytic ring $\mathcal{A}$, the functor $-\otimes_{\mathcal{A}} \hat{\mathcal{A}}$ induces an equivalence of symmetric monoidal categories

$$
\mathcal{D}(\mathcal{A})^{\mathrm{nuc}}=\mathcal{D}(\hat{\mathcal{A}})^{\mathrm{nuc}}
$$

Proof. We use the notation $\alpha^{*}=(-) \otimes_{\mathcal{A}} \hat{\mathcal{A}}$ and $\alpha_{*}$ from Lemma 2.5. We first show that $\alpha_{*}$ preserves nuclear modules (for $\alpha^{*}$ this follows by symmetric monoidality). Since $\alpha_{*}$ preserves colimits, this boils down to showing that for any compact objects $P^{\prime}, Q^{\prime} \in \mathcal{D}(\hat{\mathcal{A}})$ and any trace-class map $f: P^{\prime} \rightarrow Q^{\prime}$ in $\mathcal{D}(\hat{\mathcal{A}})$, the induced map $\alpha_{*} f: \alpha_{*} P^{\prime} \rightarrow \alpha_{*} Q^{\prime}$ is trace-class. This follows if we can show that the natural map

$$
\left(\alpha_{*} P^{\prime}\right)^{\vee} \otimes \alpha_{*} Q^{\prime} \xrightarrow{\sim} \alpha_{*}\left(\left(P^{\prime}\right)^{\vee} \otimes Q^{\prime}\right)
$$

is an isomorphism in $\mathcal{D}(\mathcal{A})$. As this claim is stable under finite colimits and retracts in $P^{\prime}, Q^{\prime}$ we may assume that $P^{\prime}=\alpha^{*} P, Q^{\prime}=\alpha^{*} Q$ for two compact objects $P, Q \in \mathcal{D}(\mathcal{A})$. By Lemma [2.5 $\alpha^{*} \alpha_{*} \alpha^{*} P \cong \alpha^{*} P$ as $P \in \mathcal{D}(\mathcal{A})$ is compact. Using adjunctions and $\alpha_{*} \alpha^{*} \underline{\mathcal{A}} \cong \underline{\mathcal{A}}$ shows $\left(\alpha_{*} \alpha^{*} P\right)^{\vee} \cong P^{\vee}$; in particular this object is right bounded and adically complete $4^{4}$ Similarly, $\left(\alpha^{*} P\right)^{\vee}$ is right-bounded because $\alpha_{*}\left(\left(\alpha^{*} P\right)^{\vee}\right) \cong P^{\vee}$ by adjunctions and symmetric monoidality of $\alpha^{*}$ while $P^{\vee}$ is right-bounded. It follows from Lemma 2.12.(iii) that the right-hand side of the above claimed isomorphism is adically complete, and it follows from [16, Proposition 2.12.10] that the left-hand side is adically complete (here we use that $\left(\alpha_{*} \alpha^{*} P\right)^{\vee}=P^{\vee}$ is discrete modulo any ideal of definition as can be checked directly). Hence the above isomorphism can be checked modulo any ideal of definition of $\underline{\mathcal{A}}$, where it follows from Lemma 2.10.
We have established that $\alpha_{*}$ restricts to a functor $\mathcal{D}(\hat{\mathcal{A}})^{\text {nuc }} \rightarrow \mathcal{D}(\mathcal{A})^{\text {nuc }}$, which is automatically conservative. Thus in order to prove the claimed equivalence of categories it is now enough to show that $\alpha^{*}$ is fully faithful on nuclear modules, i.e. for any nuclear $M$ the natural

[^3]map $M \xrightarrow{\sim} \alpha_{*} \alpha^{*} M$ is an isomorphism. Since $\alpha_{*} \alpha^{*}$ preserves colimits, this reduces to showing the following: Let $\left(P_{n}\right)_{n}$ be a sequence of compact objects in $\mathcal{D}(\mathcal{A})$ with trace-class transition maps; then $\underline{l i m}_{n} P_{n}=\underline{\longrightarrow i m}_{n}\left(P_{n}\right)_{\text {cpl }}$. To prove this, it suffices to check that the map $P_{n} \rightarrow P_{n+1}$ factors as $\overrightarrow{P_{n}} \rightarrow\left(P_{n}\right)_{\text {cpl }} \xrightarrow{\rightarrow} P_{n+1}$. But this follows easily from the definition of trace-class maps by observing that $\left(P_{n}\right)^{\vee}=\left(\left(P_{n}\right)_{\mathrm{cpl}}\right)^{\vee}$ as $\underline{\mathcal{A}}$ is complete.

The following result provides a concrete description of nuclear $\underline{\mathcal{A}}$-modules. Namely, they are exactly the ones that can be written as colimits of "Banach" modules:

Lemma 2.18. Let $\mathcal{A}$ be an adic analytic ring with ideal of definition $I$.
(i) If $M \in \mathcal{D}(\mathcal{A})$ is adically complete and $M / I$ is discrete, then $M$ is nuclear.
(ii) The category $\mathcal{D}^{\text {nuc }}(\hat{\mathcal{A}})=\mathcal{D}^{\text {nuc }}(\mathcal{A})$ is generated under colimits by adically complete objects $M \in \mathcal{D}(\mathcal{A})$ with $M / I$ discrete.

Proof. Let $M$ as in (i). Writing $M$ as a colimit of its truncations $\tau_{\geq n} M, n \rightarrow-\infty$, we can reduce to the case that $M$ is right-bounded. Let $P \in \mathcal{D}(\mathcal{A})$ be compact. We need to see that the map

$$
P^{\vee} \otimes M \rightarrow \underline{\operatorname{Hom}}(P, M)
$$

is an isomorphism. The right hand side is adically complete, because $M$ is, and the left hand side is adically complete by [16, Proposition 2.12 .10 .(ii)] as $P^{\vee}=\left(P_{\text {cpl }}\right)^{\vee}$ and $M$ are adically complete, and $M / I$ is discrete. Hence, it suffices to check the claim modulo $I$, where the claim follows from discreteness of $M / I$ as in [15, Proposition 2.9.7.(i)].

Now we show (ii). Let $M \in \mathcal{D}(\mathcal{A})$ be nuclear. By [6, Theorem 8.6.(2)] we may assume that $M$ is basic nuclear. Then it is sufficient (by the last part of the proof of Proposition 2.17) to show the following: If $f: P_{1} \rightarrow P_{2}$ is any trace class morphism between compact objects in $\mathcal{D}(\mathcal{A})$, then $f_{\mathrm{cpl}}: P_{1, \mathrm{cpl}} \rightarrow P_{2, \mathrm{cpl}}$ factors over the completion $Q$ of $P_{2, \mathrm{cpl}}(*)$. Here, $(-)$ denotes the functor from $\underline{\mathcal{A}}(*)$-modules to $\mathcal{D}(\mathcal{A})$, which is left adjoint to the functor $N \mapsto \overline{N( } *)$. This last claim follows if we can show

$$
\operatorname{Hom}\left(P_{1}, Q\right) \cong\left(P_{1}^{\vee} \otimes Q\right)(*) \cong\left(P_{1}^{\vee} \otimes P_{2, \mathrm{cpl}}\right)(*)
$$

with the first isomorphism implied by nuclearity of $Q$ as shown in (i). As the functor $(-)(*)$ preserves $I$-adic completeness both sides are adically complete as $A(*)$-modules, by arguments as in Proposition 2.17. Hence, it suffices to check the statement modulo $I$. But $P_{1}^{\vee}$ is discrete modulo $I$, and more generally for any discrete $\mathcal{A} / I$-module $D$ the map

$$
(D \otimes Q)(*) \cong\left(D \otimes P_{2, \mathrm{cpl}}\right)(*)
$$

is an isomorphism. Indeed, this statement commutes with colimits in $D$ and is clear in the case $D=\mathcal{A} / I$. This finishes the proof.

As the full subcategory $\mathcal{D}(\hat{\mathcal{A}})^{\text {nuc }} \subseteq \mathcal{D}(\hat{\mathcal{A}})$ is stable under all colimits, it admits a right adjoint $(-)_{\text {nuc }}: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})^{\text {nuc }}$. The next result provides an explicit description of $(-)_{\text {nuc }}$, similar to [15, Proposition 3.12]. Note that it is not clear to us how to explicitly compute the analogous nuclearization functor for $\mathcal{A}$ in place of $\hat{\mathcal{A}}$.

Proposition 2.19. Let $\mathcal{A}$ be an adic analytic ring.
(i) $(-)_{\mathrm{nuc}}: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})^{\mathrm{nuc}}$ preserves all colimits.
(ii) If $P \in \mathcal{D}(\hat{\mathcal{A}})$ is compact, then $P_{\mathrm{nuc}}$ is naturally isomorphic to the completion of $\alpha^{*}\left(\underline{\alpha_{*} P(*)}\right)$.

Proof. Sending $P \in \mathcal{D}(\hat{\mathcal{A}})$ compact to the completion $\alpha^{*}\left(\underline{\alpha}_{*} P(*)\right)^{\wedge}$ extends uniquely to a colimit preserving functor

$$
F: \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})^{\mathrm{nuc}}
$$

(using Lemma 2.18.(i) to see that value is indeed nuclear). Let $\iota: \mathcal{D}(\mathcal{A})^{\text {nuc }} \rightarrow \mathcal{D}(\mathcal{A})$ be the fully faithful inclusion. As the compact generators of $\mathcal{D}(\hat{\mathcal{A}})$ are complete (by Lemma 2.12) there exists a natural transformation, $\iota \circ F \rightarrow$ Id. Indeed, it suffices to construct it on compact objects, and there it suffices to construct a natural map $\alpha^{*}\left(\underline{\alpha_{*} P(*)}\right) \rightarrow P$, which exists as $\alpha^{*} \circ(-)$ is left adjoint to $\left(\alpha_{*}(-)\right)(*)$. Applying $(-)_{\text {nuc }}$ to this transformation yields a natural transformation $F \rightarrow(-)_{\text {nuc }}$. Let $\mathcal{L}=\underset{i \in J}{\lim } \mathcal{L}_{i}$ in $\mathcal{D}(\hat{\mathcal{A}})$ with $\mathcal{L}_{i}$ compact, and let $\mathcal{M} \in \mathcal{D}(\hat{\mathcal{A}})$. Then

$$
\operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}(\mathcal{L}, \iota \circ F(\mathcal{M}))=\lim _{\grave{i \in J}} \operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}\left(\mathcal{L}_{i}, \iota \circ F(\mathcal{M})\right)=\lim _{\overleftarrow{i \in J}}\left(\mathcal{L}_{i}^{\vee} \otimes \iota \circ F(\mathcal{M})\right)(*)
$$

using nuclearity of $\iota \circ F(M)$ (and the notations $(-)(*)=\operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}(\underline{\mathcal{A}},-),(-)^{\vee}=\underline{\operatorname{Hom}}_{\mathcal{D}(\hat{\mathcal{A}})}(-, \underline{\mathcal{A}})$ ). On the other hand,

$$
\operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}\left(\mathcal{L}, \iota\left(\mathcal{M}_{\mathrm{nuc}}\right)\right)=\lim _{\overleftarrow{i \in J}} \operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}\left(\mathcal{L}_{i}, \mathcal{M}\right)
$$

We have to see that both inverse limits agree if $\mathcal{L}$ is nuclear, or even basic nuclear, i.e., we can assume that $\mathcal{L}=\underset{n \in \mathbb{N}}{\lim } \mathcal{L}_{n}$ with $\mathcal{L}_{n}$ compact and the transition maps trace-class. This assumption implies that each map

$$
\operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}\left(\mathcal{L}_{n+1}, \mathcal{M}\right) \rightarrow \operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}})}\left(\mathcal{L}_{n}, \mathcal{M}\right)
$$

factors over the space of trace class maps $\left(\mathcal{L}_{n}^{\vee} \otimes \mathcal{M}\right)(*)$ from $\mathcal{L}_{n}$ to $\mathcal{M}([6$, Lemma 8.2.(3)]). Hence, it suffices to show that, similarly to the last assertion in the proof of Lemma 2.18.(i), the map

$$
\left(P^{\vee} \otimes \iota \circ F(\mathcal{M})\right)(*) \rightarrow\left(P^{\vee} \otimes M\right)(*)
$$

is an isomorphism for any $P, M \in \mathcal{D}(\hat{\mathcal{A}})$ with $P$ compact. As both sides commute with colimits in $M$ we may assume that $M$ is compact. Then both sides are $I$-adically complete over $A(*)$ for some ideal of definition $I \subseteq A(*)$, which reduces us to the case that $I=\{0\}$ by passing to the reduction $\bmod I$. In this case $\alpha^{*}, \alpha_{*}$ are inverse equivalences (by Lemma 2.10), and the assertion follows from the last assertion in the proof of Lemma 2.18.(i).

From Proposition 2.17 and Lemma 2.18 we can produce a weaker form of a projection formula for $\alpha_{*}$.

Corollary 2.20. Let $\mathcal{A}$ be an adic analytic ring. Let $M \in \mathcal{D}(\mathcal{A})^{\text {nuc }}$. Then the natural map

$$
M \otimes \alpha_{*}(N) \rightarrow \alpha_{*}\left(\alpha^{*} M \otimes N\right)
$$

is an isomorphism for any $N \in \mathcal{D}(\hat{\mathcal{A}})$.
Proof. Both sides commute with colimits in $N$. Hence, we may assume that $N \in \mathcal{D}(\hat{\mathcal{A}})$ is compact (and hence complete by Lemma (2.5). As $M \cong M_{\text {nuc }}$ and the nuclearization commutes with colimits, and sends compacts of bounded complete objects, which are discrete modulo some ideal of definition $I$, we may assume that $M$ is complete and discrete modulo $I$. Then the left hand side is complete by [16, Proposition 2.12.10.(ii)], and it is by reduction to the discrete case enough to show that the right hand side is complete, too. By Lemma 2.12 it is sufficient to show that $\alpha^{*} M$ is complete (as $N$ is complete and right bounded). Now, $\alpha^{*} M \in \mathcal{D}(\hat{\mathcal{A}})$ is nuclear, which implies $\alpha_{*}\left(\alpha^{*} M\right) \cong M$ by Proposition 2.17. In particular, $\alpha^{*} M$ is complete by Lemma 2.12.

We now prove a relative variant of Lemma [2.9. We use the terminology that a morphism $\mathcal{A}=\left(A, A^{+}\right)_{\square} \rightarrow \mathcal{B}=\left(B, B^{+}\right)_{\square}$ of adic analytic rings is of +-finite type, if $\left(B, B^{+}\right)_{\square} \cong$ $(B, A \cup\{S\})_{\square}$ for some finite set $S$.

Proposition 2.21. Let $f: \mathcal{A}=\left(A, A^{+}\right)_{\square} \rightarrow \mathcal{B}=\left(B, B^{+}\right)_{\square}$ be an adic morphism of adic analytic rings of +-finite type.
(i) The diagram

$$
\begin{array}{cc}
\mathcal{D}(\hat{\mathcal{A}}) \xrightarrow{-\otimes_{\mathcal{A}} \hat{\mathcal{B}}} \mathcal{D}(\hat{\mathcal{B}}) \\
\alpha_{\mathcal{A}, *} \downarrow & { }^{\alpha_{\mathcal{B}, *}} \\
\mathcal{D}(\mathcal{A}) \xrightarrow{-\otimes_{\mathcal{A}} \mathcal{B}} \mathcal{D}(\mathcal{B})
\end{array}
$$

naturally commutes, i.e., base change holds for $\alpha$.
(ii) The natural functor

$$
\mathcal{D}(\hat{\mathcal{A}}) \otimes_{\mathcal{D}(\mathcal{A})} \mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\hat{\mathcal{B}})
$$

is an equivalence.
Proof. For (i) we have to see that the natural morphism

$$
\mathcal{B} \otimes_{\mathcal{A}} \alpha_{\mathcal{A}, *}(M) \rightarrow \alpha_{\mathcal{B}, *}\left(\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} M\right)
$$

is an isomorphism for any $M \in \mathcal{D}(\hat{\mathcal{A}})$. Both sides commute with colimits in $M$, which reduces to the case $M=\alpha_{\mathcal{A}}^{*} P$ for some compact object $P \in \mathcal{D}(\mathcal{A})$. Then the left hand side equates to $\mathcal{B} \otimes_{\mathcal{A}} P_{\text {cpl }}$ by Lemma 2.5 while the right hand side is

$$
\alpha_{\mathcal{B}, *}\left(\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} \alpha_{A}^{*}(P)\right) \cong \alpha_{\mathcal{B}, *}\left(\alpha_{\mathcal{B}}^{*}\left(\mathcal{B} \otimes_{\mathcal{A}} P\right) \cong\left(\mathcal{B} \otimes_{\mathcal{A}} P\right)_{\mathrm{cpl}}\right.
$$

by Lemma 2.7 and Lemma [2.5, and this object is complete. Hence, it is sufficient to show that $\mathcal{B} \otimes_{\mathcal{A}} P_{\text {cpl }}$ is complete because then one can check the statement modulo some ideal of definition, where it is clear. If $S=\varnothing$, then the statement follows from [16, Proposition 2.12.10]. Using induction one reduces to the case that $S=\{s\}$. Then

$$
\mathcal{B} \otimes_{\mathcal{A}} N=\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}\left(\left(T^{-1}\right)\right) / \mathbb{Z}[T][-1], \underline{\mathcal{B}} \otimes_{\mathcal{A}} N\right)
$$

for any $N \in \mathcal{D}(\mathcal{A})$ by the right adjoint assertion to [23, Observation 8.11]. Here, $\mathbb{Z}[T] \rightarrow \underline{\mathcal{B}}$ is the map classified by $s$. Again using [16, Proposition 2.12.10] we can conclude that

$$
\begin{aligned}
\mathcal{B} \otimes_{\mathcal{A}} N_{\mathrm{cpl}} & \cong \underline{\operatorname{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}\left(\left(T^{-1}\right)\right) / \mathbb{Z}[T][-1], \underline{\mathcal{B}} \otimes_{\mathcal{A}} N_{\mathrm{cpl}}\right) \\
& \cong \underline{\operatorname{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}\left(\left(T^{-1}\right)\right) / \mathbb{Z}[T][-1],\left(\underline{\mathcal{B}} \otimes_{\mathcal{A}} N\right)_{\mathrm{cpl}}\right) \\
& \left(\mathcal{B} \otimes_{\mathcal{A}} N\right)_{\mathrm{cpl}}
\end{aligned}
$$

for any $N \in \mathcal{D}(\mathcal{A})$.
Now we prove (ii). The natural functor

$$
\Phi: \mathcal{D}(\hat{\mathcal{A}}) \otimes_{\mathcal{D}(\mathcal{A})} \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\hat{\mathcal{B}})
$$

commutes with colimits (by definition), and hence has a right adjoint $\Psi$. As the image of the functor $\alpha_{\mathcal{B}}^{*}: \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\hat{\mathcal{B}})$ generates the target under colimits, the same is true for $\Phi$. This implies formally that the functor $\Psi$ is conservative. Hence, it is sufficient to show that $\Phi$ is fully faithful. We may reduce to the cases $B^{+}=A^{+}$or $B=A$ as the statement is stable under composition of + -finite morphisms. We first deal with the case $B^{+}=A^{+}$. Then
$\mathcal{D}(\mathcal{B}) \cong \operatorname{Mod}_{B}\left(\mathcal{D}(\mathcal{A})\right.$, and hence $\mathcal{D}(\hat{\mathcal{A}}) \otimes_{\mathcal{D}(\mathcal{A})} \mathcal{D}(\mathcal{B}) \cong \operatorname{Mod}_{\alpha_{A}^{*} B}(\mathcal{D}(\hat{\mathcal{A}})$. To get fully faithfulness of $\Phi$ it suffices to see that

$$
\operatorname{Hom}_{\mathcal{D}(B)}\left(\left(B \otimes_{A} P\right)_{\mathrm{cpl}},\left(B \otimes_{A} Q\right)_{\mathrm{cpl}}\right) \cong \operatorname{Hom}_{\mathcal{D}(\hat{\mathcal{A}}), \alpha_{A}^{*} B}\left(\alpha_{A}^{*} B \otimes \alpha^{*} P, \alpha_{A}^{*} B \otimes \alpha^{*} Q\right)
$$

for any $P, Q \in \mathcal{D}(\mathcal{A})$ compact. The left hand side simplifies to $\operatorname{Hom}_{\mathcal{D}(A)}\left(P,\left(B \otimes_{A} Q\right)_{\mathrm{cpl}}\right)$, while the right hand side is by adjunctions isomorphic to $\operatorname{Hom}_{\mathcal{D}(A)}\left(P, \alpha_{A, *}\left(\alpha_{A}^{*} B \otimes \alpha_{A}^{*}(Q)\right)\right)$. Now, $\left(B \otimes_{A} Q\right)_{\mathrm{cpl}} \cong B \otimes_{A} Q_{\mathrm{cpl}}$ by [16, Proposition 2.12.10.(ii)], which agrees with $\alpha_{A, *}\left(\alpha_{A}^{*} B \otimes\right.$ $\left.\alpha_{A}^{*}(Q)\right) \cong B \otimes \alpha_{A, *} \alpha_{A}^{*}(Q)$ by Corollary 2.20 and Lemma [2.5] This finishes the proof in the case that $B^{+}=A^{+}$. Hence, assume that $B=A$. As $f: \mathcal{A} \rightarrow \mathcal{B}$ is of +-finite type this implies that the functor $f^{*}:=\mathcal{B} \otimes_{\mathcal{A}}(-) \rightarrow \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{B})$ admits a fully faithful left adjoint $f$ ! satisfying the projection formula. Indeed, it is sufficient to treat the case $S=\{s\}$, where it follows by base change from the discussion in [23, Lecture 8]. We first check that $\hat{f}^{*}:=\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}}(-): \mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{B}})$ commutes with products, so that $\hat{f}^{*}$ admits a left adjoint $\hat{f_{!}}$. As $\alpha_{B, *}$ is conservative, it is sufficient to show that $\alpha_{B, *} \hat{f}^{*} \cong f^{*} \alpha_{A, *}$ (by the proven assertion (i)) commutes with products. This is clear as $f^{*}, \alpha_{A, *}$ are right adjoints. Applying (i) again shows that $\hat{f}!\alpha_{A}^{*} \cong \alpha_{B}^{*} f_{!}$. As the essential image of $\alpha_{B}^{*}$ generates $\mathcal{D}(\hat{\mathcal{B}})$ under colimits this implies that the natural transformation

$$
\mathrm{Id} \rightarrow \hat{f}^{*} \hat{f}!
$$

is an isomorphism because $f^{*} f_{!} \cong$ Id. Similarly, one checks that $\hat{f}$ satisfies the projection formula. This implies that $\hat{f}^{*}$ is the open localization associated with the idempotent algebra $\operatorname{cone}\left(\hat{f}_{!}(1) \rightarrow 1\right) \in \mathcal{D}(\hat{\mathcal{A}})(\underline{6}$, Proposition 6.5]). However, this algebra is the pullback of the idempotent algebra cone $\left(f_{!}(1) \rightarrow 1\right)$ because $\alpha_{B}^{*} f_{!} \cong \hat{f}_{!} \alpha_{A}^{*}$. By Lemma 2.35 this implies the claim.

### 2.3 Complete descent

Next we discuss base-change in the setting of adic rings, in particular with our new modified version of modules. As we will show below, base-change holds in great generality as long as the involved maps are adic.

Lemma 2.22. Let $\left(A, A^{+}\right)_{\square}$ be an adic analytic ring.
(i) For every element $a \in \pi_{0} A$ with induced map $\mathbb{Z}[x] \rightarrow A$ we have

$$
\left(A, A^{+}\right)_{\square} \otimes_{(\mathbb{Z}[x], \mathbb{Z})_{\square}} \mathbb{Z}[x]_{\square}=\left(A, A^{+}[a]\right)_{\square} .
$$

(ii) We have

$$
\left(A, A^{+}\right)_{\square} \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[x]_{\square}=\left(A\langle x\rangle, A^{+}[x]\right)_{\square} .
$$

Proof. Part (i) follows easily from the definitions by using that $A$ is solid over any discrete ring (because it is a limit of discrete rings). For (ii) we note that on modules the functor $-\otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[x]_{\square}$ preserves limits (see [16, Lemma 2.9.5]), so in particular it preserves adic completeness. Then the claim reduces to the observations that the right-hand side is an analytic ring and that the statement is true modulo $I$.

For adic analytic rings we have the following characterization of steady maps (see [24, Definition 12.13]) in terms of adicness:

Lemma 2.23. A map of adic analytic rings is steady if and only if it is adic.

Proof. Let $f:\left(A, A^{+}\right)_{\square} \rightarrow\left(B, B^{+}\right)_{\square}$ be a map of adic analytic rings. First assume that $f$ is adic. In order to show that $f$ is steady, it is enough to show that $\left(A, A^{+}\right)_{\square} \rightarrow\left(B, A^{+}\right)_{\square}$ and $\left(B, A^{+}\right)_{\square} \rightarrow\left(B, B^{+}\right)$are steady. For the second map this follows from Lemma 2.22.(i) and [16, Proposition 2.9.7.(ii)] by stability of steadiness under base change. The first map is an induced analytic ring structure, so the statement reduces to showing that $B$ is nuclear in $\mathcal{D}_{\square}\left(A, A^{+}\right)$(see [16, Corollary 2.3.23]), which follows directly from Lemma 2.18.(i)] by adicness of $f$. We now prove the converse, so assume that $f$ is steady and let $I$ be an ideal of definition for $A$. Consider the map $g:\left(A, A^{+}\right) \rightarrow\left(A\langle x\rangle, A^{+}[x]\right)$. Then by Lemma 2.22.(ii) the basechange of $g$ along $f$ is given by the map $g^{\prime}:\left(B, B^{+}\right) \rightarrow\left(B\langle x\rangle, B^{+}[x]\right)$. Note that " $\langle x\rangle$ " means adically completed polynomials with respect to the adic topology on $A$ respectively $B$. By steadiness of $f$ we have $B\langle x\rangle=A\langle x\rangle \otimes_{\left(A, A^{+}\right)_{\square}}\left(B, B^{+}\right)_{\square}$. Now $B$ is adically complete as an $A$-module. Indeed, to see this we may assume that $B$ is discrete, in which case it is an $A / I$-module for some ideal of definition $I \subseteq \pi_{0} A(*)$ by nuclearity of $B$ as an $A$-module, and we have shown the completeness of $B$ as an $A$-module. Hence by [16, Proposition 2.12.10] $A\langle x\rangle \otimes_{\left(A, A^{+}\right)_{\square}}\left(B, B^{+}\right)_{\square}$ computes the $I$-adic completion of $B[x]$ (note that this tensor product does not depend on $B^{+}$because the $I$-adic completion of $B[x]$ is solid for any choice of $B^{+}$). For this $I$-adic completion to be the same as the $J$-adic completion of $B[x]$ for some ideal of definition $J$ of $B$ we must have that $f$ is adic. Indeed, base change to $\mathcal{A} / I$ reduces to the case $I=\{0\}$, and then $B[x] \cong \bigoplus_{n \geq 0} B \cdot x^{n}$ can be $J$-adically complete if and only if $J$ is nilpotent. 5

Lemma 2.24. Let $\left(B, B^{+}\right)_{\square} \leftarrow\left(A, A^{+}\right)_{\square} \rightarrow\left(C, C^{+}\right)_{\square}$ be a diagram of adic analytic rings and adic maps. Then

$$
\left(B, B^{+}\right)_{\square} \otimes_{\left(A, A^{+}\right)_{\square}}\left(C, C^{+}\right)_{\square}=\left(B \hat{\otimes}_{A} C, B^{+} \otimes_{A^{+}} C^{+}\right)_{\square} .
$$

In particular this tensor product is still adic.
Proof. We can reduce to the cases $A^{+}=B^{+}$and $A=B$. The second case follows easily from Lemma 2.22.(i). The first case follows from [16, Proposition 2.12.10].

Remark 2.25. In general, the single notation $\hat{\mathcal{A}}$ is not well-defined, but should be seen as a convenient replacement for the datum $(\mathcal{A}, \mathcal{D}(\hat{\mathcal{A}}))$. As an example, given a diagram $\mathcal{B} \leftarrow \mathcal{A} \rightarrow \mathcal{C}$ of adic analytic rings and adic maps, we denote $\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{C}}:=\left(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}\right)^{\text {a }}$, i.e., it denotes the $\operatorname{datum}\left(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}, \mathcal{D}\left(\left(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}\right)^{\wedge}\right)\right)$.

Corollary 2.26. Let $\mathcal{B} \leftarrow \mathcal{A} \rightarrow \mathcal{C}$ be a diagram of adic analytic rings and adic maps. Then for every $M \in \mathcal{D}(\hat{\mathcal{B}})$ the natural morphism

$$
M \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{C}} \xrightarrow{\sim} M \otimes_{\hat{\mathcal{B}}}\left(\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{C}}\right)
$$

is an isomorphism, i.e. the following base-change diagram commutes:


[^4]Proof. The existence of the natural morphism is formally implied by adjunctions as

$$
\left(-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}\right) \otimes_{\hat{\mathcal{B}}}\left(\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{C}}\right) \cong\left(-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{C}}\right) \otimes_{\hat{\mathcal{C}}}\left(\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{C}}\right)
$$

by Lemma 2.7. Both sides of the claimed isomorphism commute with all colimits, so we can w.l.o.g. assume that $M$ is compact. In particular $M$ is right-bounded and adically complete, hence by Lemma 2.12.(iv) both sides of the claimed isomorphism are adically complete. But then we can check the claimed isomorphism modulo any ideal of definition, so by Lemma 2.10 we reduce to the well-known claim that maps of discrete Huber pairs are steady (see [16, Proposition 2.9.7.(ii)]).

We are finally in the position to discuss descent in the setting of $\hat{\mathcal{A}}$-modules. As promised we will see that descendability can be checked modulo all powers of an ideal of definition.

Definition 2.27. A map $\mathcal{A} \rightarrow \mathcal{B}$ of adic analytic rings is called adically descendable of index $\leq d$ if it is adic and for some ideal of definition $I$ and all $n \geq 1$ the map $\mathcal{A} / I^{n} \rightarrow \mathcal{B} / I^{n}$ is descendable of index $\leq d$ in the sense of [16, Definition 2.6.7].

Remark 2.28. If $\mathcal{A} \rightarrow \mathcal{B}$ is a descendable morphism of adic analytic rings then it is in particular adically descendable because descendability is stable under the base-change along $\mathcal{A} \rightarrow \mathcal{A} / I^{n}$ (see [16, Lemma 2.6.9]). The converse seems to be wrong unless the compact objects in $\mathcal{D}(\mathcal{A})$ are adically complete. This is fixed by $\mathcal{D}(\hat{\mathcal{A}})$, as the following result shows.

Example 2.29. A typical example for an adically descendable map $\mathcal{A} \rightarrow \mathcal{B}$ of index $\leq 2$ is an adic $I$-completely faithfully flat and $I$-completely finitely presented map, i.e., $I$ is an ideal of definition of $\mathcal{A}$ and $\underline{\mathcal{A}} / I^{n} \rightarrow \underline{\mathcal{B}} / I^{n}$ is faithfully flat and finitely presented (on $\pi_{0}$ ), cf. [16, Lemma 2.10.6] for the static case, and [19, Theorem 4.15] for the general case.

Theorem 2.30. Let $\mathcal{A} \rightarrow \mathcal{B}$ be an adically descendable morphism of adic analytic rings. Then modules descend along $\hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$, i.e. the functor

$$
\mathcal{D}(\hat{\mathcal{A}}) \stackrel{\sim}{\rightarrow}{\underset{n \in \Delta}{\leftrightarrows}}_{\lim _{n}} \mathcal{D}\left(\hat{\mathcal{B}} \otimes_{\hat{\mathcal{A}}} \ldots \otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}\right)
$$

is an equivalence of categories.
Proof. Let $\operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$ denote the category of $\mathcal{D}(\hat{\mathcal{A}})$-enriched colimit-preserving functors $\mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})(c f . \quad 16, \S 2.5, \S A .4])$. It comes equipped with the composition monoidal structure and there is an embedding $\eta: \mathcal{D}(\hat{\mathcal{A}}) \hookrightarrow \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$ via $M \mapsto-\otimes_{\hat{\mathcal{A}}} M$. As in [16, Lemma 2.5.6] the map $f: \mathcal{A} \rightarrow \mathcal{B}$ induces a pair of adjoint functors

$$
f^{\natural}: \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}})) \rightleftarrows \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{B}})): f_{\natural},
$$

where $f_{\natural}$ acts on underlying functors as $F \mapsto f_{*} \circ F \circ f^{*}$; here $f_{*}$ denotes the forgetful functor and $f^{*}:=-\otimes_{\hat{\mathcal{A}}} \hat{\mathcal{B}}$ is the base-change. Note that $f^{\natural}$ exists by the adjoint functor theorem in our case, as we chose a cutoff cardinal $\kappa$ in the beginning so that $\operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$ is presentable. Also, via compatibility with $\eta$ one sees that $f^{\natural}$ id $=\mathrm{id}$.

Note that for every $x \in \pi_{0} A$ and every $F \in \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$, the pointwise multiplication by $x$ induces an endomorphism of $F$; indeed, via the embedding $\eta$ we can define this endomorphism on id and then also on $F=\mathrm{id} \circ F$. It thus makes sense to speak of adically complete endofunctors in $\operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$. We now observe:
(a) An endofunctor $F \in \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$ is adically complete if and only if for every compact $P \in \mathcal{D}(\hat{\mathcal{A}}), F(P) \in \mathcal{D}(\hat{\mathcal{A}})$ is adically complete.

Indeed, adic completeness of enriched endofunctors can be checked on the underlying ordinary functors (see [16, Lemma A.4.5.(ii)]), i.e., in the category $\operatorname{Fun}^{L}(\mathcal{D}(\hat{\mathcal{A}}), \mathcal{D}(\hat{\mathcal{A}}))$ of colimitpreserving functors $\mathcal{D}(\hat{\mathcal{A}}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})$. If $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{D}(\hat{\mathcal{A}})$ denotes the full subcategory of compact objects, then by the universal property of Ind-completions we have

$$
\operatorname{Fun}^{L}(\mathcal{D}(\hat{\mathcal{A}}), \mathcal{D}(\hat{\mathcal{A}}))=\operatorname{Fun}^{\operatorname{ex}}\left(\mathcal{C}_{\mathcal{A}}, \mathcal{D}(\hat{\mathcal{A}})\right),
$$

where the right-hand side denotes exact functors $\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{D}(\hat{\mathcal{A}})$. Since limits of exact functors between stable categories are exact, limits on the right-hand side are computed pointwise. But this immediately implies (a). From (a) we immediately deduce that id $\in \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}}))$ is adically complete; here it is crucial to work with $\mathcal{D}(\hat{\mathcal{A}})$ instead of $\mathcal{D}(\mathcal{A})$ !
Now let

$$
\mathcal{K}:=\operatorname{fib}\left(\mathrm{id} \rightarrow f_{\mathrm{\natural}} \mathrm{id}\right) \in \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}})),
$$

where the map id $\rightarrow f_{\mathrm{G}} \mathrm{id}=f_{\mathrm{G}} \mathrm{f}^{\natural}$ id comes from the unit of the adjunction. We now claim:
(b) For some $d \geq 0$ the natural map $\mathcal{K}^{d} \rightarrow \mathrm{id}$ is zero.

To prove this, let $d^{\prime} \geq 0$ be such that each map $f_{n}: \mathcal{A} / I^{n} \rightarrow \mathcal{B} / I^{n}$ is descendable of index $\leq d^{\prime}$. We claim that $d=2 d^{\prime}$ works. To see this, denote $g_{n}: \mathcal{A} \rightarrow \mathcal{A} / I^{n}$ the projection, so that we get adjoint functors

$$
g_{n}^{\natural}: \operatorname{End}^{L}(\mathcal{D}(\hat{\mathcal{A}})) \rightleftarrows \operatorname{End}^{L}\left(\mathcal{D}\left(\mathcal{A} / I^{n}\right)\right): g_{n \sharp} .
$$

(Here we implicitly use Lemma 2.10). The adic completeness of id implies that the natural map id $\xrightarrow{\sim} \varliminf_{\ddagger}{ }_{n} g_{n \sharp}$ id is an isomorphism. We deduce the following identity of spectra:

$$
\operatorname{Hom}\left(\mathcal{K}^{d^{\prime}}, \mathrm{id}\right)=\underset{n}{\lim _{n}} \operatorname{Hom}\left(\mathcal{K}^{d^{\prime}}, g_{n \sharp} \mathrm{id}\right)={\underset{\check{~}}{n}}^{\lim _{n}} \operatorname{Hom}\left(g_{n}^{\natural} \mathcal{K}^{d^{\prime}}, \mathrm{id}\right)
$$

The induced map $\operatorname{Hom}\left(\mathcal{K}^{d^{\prime}}, \mathrm{id}\right) \rightarrow \operatorname{Hom}\left(g_{n}^{\natural} \mathcal{K}^{d^{\prime}}, g_{n}^{\natural}\right.$ id $)$ is the one coming from the functoriality of $g_{n}^{\natural}$. By [16, Proposition A.4.17] the functor $g_{n}^{\natural}$ is naturally a monoidal functor, so that if we denote $\mathcal{K}_{n}:=\operatorname{fib}\left(\mathrm{id} \rightarrow f_{n \mathrm{q}} \mathrm{id}\right) \in \operatorname{End}^{L}\left(\mathcal{D}\left(\mathcal{A} / I^{n}\right)\right.$ then $g_{n}^{\natural} \mathcal{K}^{d^{\prime}}=\mathcal{K}_{n}^{d^{\prime}}$. We deduce that the natural map $\mathcal{K}^{d^{\prime}} \rightarrow$ id gets sent to the natural map $\mathcal{K}_{n}^{d^{\prime}} \rightarrow$ id under $g_{n}^{\natural}$. The latter map is zero by choice of $d^{\prime}$. Now (b) follows completely analogous to the argument in [16, Proposition 2.7.2].

With (b) at hand, the claimed descent of modules is now formal: By the same procedure as in [16, Proposition 2.6.5] we deduce from (b) that id can be obtained from $f_{\mathrm{\natural}}$ id in a finite number of steps, each of which consists of a composition, a finite limit or a retract. Using Lemma [2.24 and Corollary [2.26, we can thus apply the argument from [16, Proposition 2.6.3] verbatim.

We also need a way to pass to filtered colimits, similar to [16, §2.7]. This is fairly straightforward since adic completions are countable limits and therefore commute with $\omega_{1}$-filtered colimits (in the following result the superscript $\omega_{1}$ refers to $\omega_{1}$-compact objects):

Lemma 2.31. Let $\left(\mathcal{A}_{i}\right)_{j \in J}$ be an $\omega_{1}$-filtered diagram of adic analytic rings and adic transition maps. Then $\mathcal{A}:=\lim _{\longrightarrow} \mathcal{A}_{j}$ is an adic analytic ring and the natural functor

$$
\underset{j}{\lim _{\vec{j}}} \mathcal{D}\left(\hat{\mathcal{A}}_{j}\right)^{\omega_{1}} \xrightarrow{\sim} \mathcal{D}(\hat{\mathcal{A}})^{\omega_{1}}
$$

is an equivalence of categories.

Proof. We can assume that $J$ has some initial element $0 \in J$. Let $I$ be some ideal of definition of $\underline{\mathcal{A}}_{0}$. Then each $\underline{\mathcal{A}}_{j}$ is $I$-adically complete and since countable limits commute with $\omega_{1-}$ filtered colimits in $\mathcal{D}(\operatorname{Cond}(\mathrm{Ab}))$, we deduce that $\underset{\mathcal{A}}{\mathcal{A}} \underset{\overrightarrow{\lim }}{\boldsymbol{\operatorname { A }}} \underline{\mathcal{A}}_{j}$ is $I$-adically complete. Clearly, $\underline{\mathcal{A}}$ is discrete $\bmod I$ as all transition maps are adic. This shows that $\mathcal{A}$ is indeed an adic analytic ring. By a similar argument we see that for any $\omega_{1}$-compact $P \in \mathcal{D}\left(\hat{\mathcal{A}}_{0}\right)^{\omega_{1}}$ the natural map

$$
\underset{j}{\lim }\left(P \otimes_{\hat{\mathcal{A}}_{0}} \hat{\mathcal{A}}_{j}\right) \xrightarrow{\sim} P \otimes_{\hat{\mathcal{A}}_{0}} \hat{\mathcal{A}}
$$

is an isomorphism in $\mathcal{D}\left(\hat{\mathcal{A}}_{0}\right)$. Indeed, this claim is stable under colimits in $P$, so by [16, Lemma A.2.1] we can w.l.o.g. assume that $P=\mathcal{A}_{0}[S]_{\hat{I}}$ for some profinite set $S$. Then the claim boils down to showing that the map $\lim _{\longrightarrow} \mathcal{A}_{j}[S]_{\hat{I}} \sim \mathcal{} \rightarrow \mathcal{A}[S]_{\hat{I}}$ is an isomorphism. This is true without $I$-completions by [16, Proposition 2.3 .15 .(ii)] and then follows for the $I$-completed version because $I$-completions commute with $\omega_{1}$-filtered colimits. With the above isomorphism at hand, the rest of the argument works exactly as in [16, Lemma 2.7.4].

Definition 2.32. A map $f: \mathcal{A} \rightarrow \mathcal{B}$ of adic analytic rings is called weakly adically descendable if it is an iterated $\omega_{1}$-filtered colimit of adically descendable maps of adic analytic rings along adic maps ${ }^{66}$ If all the maps in this $\omega_{1}$-filtered colimit are descendable of index $\leq d$, for some fixed $d \geq 0$, then we say that $f$ has index $\leq d$.

Proposition 2.33. (i) If $\mathcal{A} \rightarrow \mathcal{B}$ is weakly adically descendable then it is adic and modules descend along $\hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$.
(ii) Every adically descendable morphism of adic analytic rings is weakly adically descendable, and weakly adically descendable morphisms are stable under adic base-change and $\omega_{1}$-filtered colimits.
(iii) A completed filtered colimit of adically descendable morphisms of index $\leq d$ along adic maps of adic analytic rings is weakly adically descendable of index $\leq 2 d$.

Proof. Part (i) follows from Theorem 2.30 and Lemma 2.31 by the same argument as in [16, Proposition 2.7.5] (use also Lemma 2.24 in order to reduce to the case that all maps in the $\omega_{1}$-filtered colimit have the same source). In part (ii) the only non-trivial claim is the one about base-change, and this follows from Lemma 2.24 and [16, Lemma 2.6.9]. Part (iii) follows by the same argument as in [16, Theorem 2.7.8.(iii)].

### 2.4 Globalization for stably uniform adic spaces

We finish our general discussion of the modified category $\mathcal{D}_{\hat{\circ}}(\mathcal{A})$ for an adic analytic ring $\mathcal{A}$ with a globalization to (classical) stably uniform analytic adic spaces. Let $\left(A, A^{+}\right)$be a classical complete analytic Huber pair, and $X:=\mathrm{Spa}\left(A, A^{+}\right)$. Recall that $\left(A, A^{+}\right)$is called uniform if $A^{\circ}$ is a ring of definition, and that $\left(A, A^{+}\right)$is called stably uniform if for each rational open $U \subseteq X$ the analytic Huber pair $\left(\mathcal{O}_{X}(U), \mathcal{O}_{X}^{+}(U)\right)$ is uniform, in which case $\left(A, A^{+}\right)$is automatically sheafy ([25, Definition 5.2.4., Theorem 5.2.5.]). In [1, Theorem 4.1] Andreychev has proven that if ( $A, A^{+}$) is sheafy (not necessarily uniform), then the functor $U \mapsto \mathcal{D}_{\square}\left(\mathcal{O}_{X}(U), \mathcal{O}_{X}^{+}(U)\right)$ on rational opens $U \subseteq X$ satisfies descent for the analytic topology on $X$.

[^5]Definition 2.34. (a) For any classical complete analytic uniform Huber pair $\left(B, B^{+}\right)$set

$$
\mathcal{D}_{\hat{\imath}}\left(B, B^{+}\right):=\operatorname{Mod}_{B}\left(\mathcal{D}_{\hat{\imath}}\left(B^{\circ}, B^{+}\right)\right) .
$$

Here, we implicitly view $B, B^{\circ}$ as condensed static rings (although $B^{+}$is only treated as a discrete ring). Note that we need uniformity to achieve that $B^{\circ}$ is an adic ring in the sense of Definition 2.1.
(b) Given a map $f: Y=\operatorname{Spa}\left(B, B^{+}\right) \rightarrow X=\operatorname{Spa}\left(A, A^{+}\right)$of stably uniform analytic adic spaces, we let $f^{*}:=\widehat{\left(B^{\circ}, B^{+}\right)_{\square}} \otimes \overline{\left(B^{\circ}, B^{+}\right)_{\square}}(-): \mathcal{D}_{\hat{\emptyset}}(X):=\mathcal{D}_{\hat{\wedge}}\left(A, A^{+}\right) \rightarrow \mathcal{D}_{\hat{\wedge}}(Y):=$ $\mathcal{D}_{\hat{\emptyset}}\left(B, B^{+}\right), f_{*}: \mathcal{D}_{\hat{\emptyset}}(Y) \rightarrow \mathcal{D}_{\hat{\emptyset}}(X)$ be the induced pair of adjoint functors. If we want to distinguish these functors two functors $f^{*}, f_{*}$ from their counterparts $f^{*}=\left(B, B^{+}\right)_{\square} \otimes_{\left(A, A^{+}\right)}$ $(-): \mathcal{D}_{\square}(X):=\mathcal{D}_{\square}\left(A, A^{+}\right) \rightarrow \mathcal{D}_{\square}(Y):=\mathcal{D}_{\square}\left(B, B^{+}\right)$, we denote them by $\hat{f}^{*}, \hat{f}_{*}$.
(c) We let $\alpha_{X}^{*}: \mathcal{D}_{\square}(X)=\operatorname{Mod}_{A}\left(\mathcal{D}_{\square}\left(A^{\circ}, A^{+}\right)\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}(X)$ be the functor induced on module categories by the symmetric monoidal functor $\widehat{\left(A^{\circ}, A^{+}\right)_{\square}} \otimes_{\left(A^{\circ}, A^{+}\right)_{\square}}(-)$ from Definition 2.4. We let $\alpha_{X, *}$ be the right adjoint of $\alpha_{X}^{*}$.

Let Sym be the $\infty$-category of cocomplete closed symmetric monoidal stable $\infty$-categories with morphisms given by cocontinuous, symmetric monoidal functors, cf. [6, Definition 6.3]. We recall from [6, Proposition 6.5] that a morphism $g^{*}: C \rightarrow D$ in Sym is called an open (resp. closed) immersion if $g^{*}$ has a fully faithful left adjoint $g_{!}$(resp. colimit preserving fully faithful right adjoint $g_{*}$ ), which satisfies the projection formula. If $g^{*}$ is a closed immersion, then $g_{*}\left(1_{D}\right) \in C$ is an idempotent algebra, and $D \cong \operatorname{Mod}_{g_{*}\left(1_{D}\right)}(C)$. If $g^{*}$ is an open immersion, then the cofiber $R:=\left[g_{!}\left(1_{D}\right) \rightarrow 1_{C}\right] \in C$ is an idempotent algebra, and $D$ is isomorphic to the quotient $C / \operatorname{Mod}_{R}(C)\left(\right.$ via $\left.g^{*}\right)$. We record the following stability of open/closed immersions under base change with respect to the Lurie tensor product.

Lemma 2.35. Let $h^{*}: C \rightarrow C^{\prime}, g^{*}: C \rightarrow D$ be morphisms in Sym. If $g^{*}$ is an open (resp. closed) immersion, the same is true for the base change $k^{*}: C^{\prime} \rightarrow D^{\prime}:=C^{\prime} \otimes_{C} D$.

Proof. For closed immersions this is clear as $\operatorname{Mod}_{R}(C) \otimes_{C} C^{\prime} \cong \operatorname{Mod}_{h^{*}(R)}\left(C^{\prime}\right)$ for any monoid object $R \in C$, and $h^{*}$ preserves idempotent algebras. Thus, assume that $g^{*}$ is an open immersion, and let $R:=\left[g!\left(1_{D}\right) \rightarrow 1_{C}\right] \in C$ be the associated idempotent algebra. As the base change $C^{\prime} \otimes_{C}(-)$ preserves adjunctions between colimit preserving, $C$-linear functors, the base change $k_{!}$of $g_{!}$is a fully left adjoint of $k^{*}$. By $C^{\prime}$-linearity one checks that $k_{!}$satisfies the projection formula. In particular, $k^{*}$ is an open immersion. Moreover, we note that as in the proof of [6, Proposition 6.5] the projection formula for $k^{*}$ implies that the kernel of the functor $k^{*}$ is exactly the category of $h^{*}(R)$-modules in $C^{\prime}$ because $h^{*}(R)$ is the cofiber of $k_{!}\left(1_{D^{\prime}}\right) \rightarrow 1_{C^{\prime}}$ by exactness of $h^{*}$.

Remark 2.36. The last part of the proof of Lemma 2.35 shows that base change preserves the "excision sequence" $D \xrightarrow{g!} C \rightarrow \operatorname{Mod}_{R}(C)$ for an open immersion $g^{*}: C \rightarrow D$.

We now show that rational localizations yield open immersion in Sym for the completed solid category:

Lemma 2.37. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be a stably uniform analytic adic space, and $j: U:=$ $\operatorname{Spa}\left(B, B^{+}\right) \rightarrow X$ a rational open. Then $\hat{j}^{*}: \mathcal{D}_{\hat{\ominus}}(X) \rightarrow \mathcal{D}_{\hat{\emptyset}}(U)$ is an open immersion in Sym. In fact, $\hat{j}^{*}$ is the base change of $j^{*}: \mathcal{D}_{\hat{\ominus}}(X) \rightarrow \mathcal{D}_{\hat{\emptyset}}(U)$ along the functor $\alpha_{X}^{*}: \mathcal{D}_{\square}(X) \rightarrow \mathcal{D}_{\hat{\emptyset}}(X)$.

Lemma 2.35 and the proof below shows that $\alpha_{X}^{*} \circ j_{!} \cong \hat{j} \circ \alpha_{U}^{*}$ for the left adjoints $j!, \hat{j}$ ! of $j^{*}, \hat{j}^{*}$.

Proof. This follows from Proposition 2.21 and Lemma 2.35 the morphism $\left(A^{\circ}, A^{+}\right)_{\square} \rightarrow$ $\left(B^{\circ}, B^{+}\right)_{\square}$ is of +-finite type, and therefore

$$
\begin{equation*}
\mathcal{D}_{\hat{\imath}}\left(A^{\circ}, A^{+}\right) \otimes_{\mathcal{D}_{\square}\left(A^{\circ}, A^{+}\right)} \mathcal{D}_{\square}\left(\mathcal{O}_{X}(U)^{\circ}, \mathcal{O}_{X}^{+}(U)\right) \xrightarrow{\sim} \mathcal{D}_{\hat{\imath}}\left(\mathcal{O}_{X}(U)^{\circ}, \mathcal{O}_{X}^{+}(U)\right) . \tag{2.37.1}
\end{equation*}
$$

We claim that the natural morphism $A \otimes_{A^{\circ}} B^{\circ} \rightarrow B$ is an isomorphism in $\mathcal{D}_{\square}\left(A^{\circ}, A^{+}\right)$. Namely, by [1, Theorem 1.6] the claim is local on $X=\operatorname{Spa}\left(A, A^{+}\right)$, which reduces to the case that $A$ is Tate, where the claim is clear. Given this claim, we can conclude

$$
\mathcal{D}_{\hat{\wedge}}(X) \otimes_{\mathcal{D}_{\square}(X)} \mathcal{D}_{\square}(U) \xrightarrow{\sim} \mathcal{D}_{\hat{\imath}}(U)
$$

by tensoring (2.37.1) over $\mathcal{D}_{\square}\left(A^{\circ}, A^{+}\right)$with $\operatorname{Mod}_{A}\left(\mathcal{D}_{\square}\left(\left(A^{\circ}, A^{+}\right)\right)\right.$. From here Lemma 2.35 implies the claim.

We now establish the following version of Andreychev's analytic descent theorem for the completed solid category:

Theorem 2.38. Let $\left(A, A^{+}\right)$be a classical stably uniform analytic Huber pair, and $X:=$ $\operatorname{Spa}\left(A, A^{+}\right)$.
(i) The functor $U \mapsto \mathcal{D}_{\hat{\wedge}}\left(\mathcal{O}_{X}(U), \mathcal{O}_{X}^{+}(U)\right)$ on rational open subsets satisfies descent for the analytic topology.
(ii) If for any rational open $U \subseteq X$ each finite, étale $\mathcal{O}_{X}(U)$-algebra is stably uniform (e.g., $A$ is sousperfectoid, [25, Proposition 6.3.3.]), then the functor $V \mapsto \mathcal{D}_{\hat{\wedge}}\left(\mathcal{O}_{V}(V), \mathcal{O}_{V}^{+}(V)\right)$ on the category of étale maps $V \rightarrow X$ with $V$ affinoid, satisfies descent for the étale topology.

Here, a morphism $V \rightarrow X$ is étale if it is a composition of open immersions and finite étale maps (in the sense of [21, Definition 7.1], generalized to our situation), and the étale topology is defined by jointly surjective étale morphisms.

Proof. By definition of the étale topology, it suffices to show (i) and descent along finite étale maps $\left(A, A^{+}\right) \rightarrow\left(B, B^{+}\right)$. But by definition of the latter, $\left(B, B^{+}\right)_{\square}=\left(B, A^{+}\right)_{\square}$, which reduces to descent along the map $A \rightarrow B$ of algebras in $\mathcal{D}_{\hat{\wedge}}\left(A, A^{+}\right)$. Now, $A(*) \rightarrow B(*)$ is descendable as a map of classical rings ([18, Corollary 3.33]). As it is moreover finite étale, this implies $B \cong A \otimes_{\underline{A(*)}} \underline{B(*)}$, and thus $A \rightarrow B$ is descendable in $\mathcal{D}_{\hat{\wedge}}\left(A, A^{+}\right)$because the functor $\mathcal{D}(A(*)) \rightarrow \mathcal{D}_{\hat{\imath}}\left(A, \overline{\left.A^{+}\right)}\right.$is symmetric monoidal. Hence, (ii) is implied by (i). In order to show (i) we first note that if $X=\bigcup_{i=1}^{n} U_{i}$ is a finite open cover with $j_{i}: U_{i} \rightarrow X$ rational open, then by Lemma 2.37 the functors $\hat{j}_{i}^{*}: \mathcal{D}_{\hat{\imath}}(X) \rightarrow \mathcal{D}_{\hat{\wedge}}\left(U_{i}\right)$ are open immersions in Sym. Moreover, the collection $\hat{j}_{i}^{*}, i=1, \ldots, n$, is jointly conservative. Indeed, this statement is equivalent to $\hat{j}_{1,!}\left(1_{U_{1}}\right) \otimes \ldots \otimes \hat{j}_{n,!}\left(1_{U_{n}}\right)=0$ and thus is implied by $j_{1}\left(1_{U_{1}}\right) \otimes \ldots \otimes j_{n}\left(1_{U_{n}}\right)=0$ ([1) Proposition 4.12.(v)]) using that $\alpha^{*}$ commutes with the !-functors (Lemma 2.37). Thus, the assertion follows from the general descent theorem [6, Proposition 5.5] (or [1, Proposition 4.13] resp. [23, Proposition 10.5]).

Theorem 2.38 allows us to give a reasonable definition of completed solid sheaves on every stably uniform analytic adic space:

Definition 2.39. Let $X$ be a (classical) stably uniform analytic adic space. Then we set $\mathcal{D}_{\hat{\mathrm{\rightharpoonup}}}(X)$ as the inverse limit of $\mathcal{D}_{\hat{\wedge}}\left(A, A^{+}\right)$over all affinoid opens $\operatorname{Spa}\left(A, A^{+}\right) \subseteq X$, with $\left(A, A^{+}\right)$ a stably uniform analytic Huber pair.

### 2.5 Universal descent

In this section, we show that descent of $\mathcal{D}(-)$ along a steady morphism $\mathcal{A} \rightarrow \mathcal{B}$ of analytic rings automatically implies descent after any base change. We discuss variants of this as well, e.g., for $\mathcal{D}_{\hat{\wedge}}(-)$.

If $f: \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of analytic rings, we will denote by $f^{*}(-):=\mathcal{B} \otimes_{\mathcal{A}}(-): \mathcal{D}(\mathcal{A}) \rightarrow$ $\mathcal{D}(\mathcal{B})$ the associated base change functor, and by $f_{*}$ its right adjoint. Note that $f_{*}$ is always conservative.
We recall that $f$ is steady if for any morphism $g: \mathcal{A} \rightarrow \mathcal{A}^{\prime}$ of analytic rings with base change $f^{\prime}: \mathcal{A}^{\prime} \rightarrow \mathcal{B}^{\prime}:=\mathcal{A}^{\prime} \otimes_{\mathcal{A}} \mathcal{B}, g^{\prime}: \mathcal{B} \rightarrow \mathcal{B}^{\prime}$, the natural transformation

$$
f^{*} g_{*} \rightarrow g_{*}^{\prime} f^{\prime \prime, *}
$$

of functors $\mathcal{D}\left(\mathcal{A}^{\prime}\right) \rightarrow \mathcal{D}(\mathcal{B})$ is an isomorphism ([24, Proposition 12.14]).
Lemma 2.40. Assume that $f: \mathcal{A} \rightarrow \mathcal{B}$ is a steady morphism of analytic rings and that modules descend along $f$, i.e., the natural functor $\mathcal{D}(A) \rightarrow \lim _{[n] \in \Delta} \mathcal{D}\left(\mathcal{B}^{n / \mathcal{A}}\right)$ is an equivalence, where $\mathcal{B}^{\bullet} / \mathcal{A}$ denotes the Čech nerve for $f$. Then for any morphism $g: \mathcal{A} \rightarrow \mathcal{A}^{\prime}$ of analytic rings modules descend along $f^{\prime}: \mathcal{A}^{\prime} \rightarrow \mathcal{B}^{\prime}:=\mathcal{A}^{\prime} \otimes_{\mathcal{A}} \mathcal{B}$.

Proof. This follows from [12, Corollary 5.2.2.37.]. We supplement the details using the notation of [12, Corollary 5.2.2.37]. Thus, we set $\mathcal{C}=\Delta, \chi: \Delta^{\triangleleft} \rightarrow \mathcal{C} a t_{\infty},[n] \mapsto \mathcal{D}\left(\left(\mathcal{B}^{\prime}\right)^{n / \mathcal{A}^{\prime}}\right)$ with $\left(\mathcal{B}^{\prime}\right)^{-1 / \mathcal{A}^{\prime}}:=\mathcal{A}^{\prime}$, and similarly, $\chi^{\prime}([n]):=\mathcal{D}\left(\mathcal{B}^{n / \mathcal{A}}\right)$. To construct the natural transformation $\rho: \chi \rightarrow \chi^{\prime}$, given componentwise by $g_{n, *}$ for $g_{n}: \mathcal{B}^{n / \mathcal{A}} \rightarrow\left(\mathcal{B}^{\prime}\right)^{n / \mathcal{A}^{\prime}}$ is the natural map, one can argue as follows: clearly, the $g_{n}^{*}$ yield a natural transformation $\sigma: \chi^{\prime} \rightarrow \chi$ (by naturality of $\mathcal{D}(-)$ as a functor on the $\infty$-category of analytic rings). As for each morphism $[n] \rightarrow[m]$ in $\Delta^{\triangleleft}$ the induced morphism $\mathcal{B}^{n / \mathcal{A}} \rightarrow \mathcal{B}^{m / \mathcal{A}}$ is steady (using [24, Proposition 12.15]), we can conclud ${ }^{7}$ that the $g_{n, *}$ assemble to a natural transformation $\rho: \chi \rightarrow \chi^{\prime}$. Now condition 1) in [12, 5.2.2.37] is the assumption that modules descend along $f$. Condition 2) is automatic as each $g_{n, *}$ is conservative. Condition 3) is clear as $\mathcal{D}\left(\mathcal{A}^{\prime}\right)$ has all limits and $g_{*}$ preserves these. Condition 4) is again automatic because for each morphism $\alpha:[n] \rightarrow[m]$ in $\Delta^{\triangleleft}$ the functors $\chi([n]) \rightarrow \chi([m])$ resp. $\chi^{\prime}([n]) \rightarrow \chi^{\prime}([m])$ admit right adjoints given by $*$-pushforward, and clearly these right adjoints commute with $\rho$ (which in component $[n]$ is given by $\left.g_{n, *}\right)$. Thus, [12, 5.2.2.37] is applicable, and yields exactly that modules descend along $f^{\prime}$.

Next assume that $f: \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of adic analytic rings. By Lemma 2.23 we know that $f$ is steady if and only if $f$ is adic. We get the following analog of Lemma 2.40 for $\mathcal{D}(\hat{\mathcal{A}})$.

Lemma 2.41. Let $f: \mathcal{A} \rightarrow \mathcal{B}$ be an adic morphism of adic analytic rings, such that the natural functor $\mathcal{D}(\hat{\mathcal{A}}) \rightarrow \lim _{[n] \in \Delta} \mathcal{D}\left(\mathcal{B}^{\hat{n} / \mathcal{A}}\right)$ is an equivalence. Then the same holds true after any base change $f^{\prime}: \mathcal{A}^{\prime} \rightarrow \mathcal{A}^{\prime} \otimes_{\mathcal{A}} \mathcal{B}$ of $f$ along an adic morphism $\mathcal{A} \rightarrow \mathcal{A}^{\prime}$ of adic analytic rings.

Note that adicness of $\mathcal{A} \rightarrow \mathcal{A}^{\prime}$ implies by Lemma 2.24 that all terms in the Čech nerves are adic analytic rings (and thus the assertion is well-defined).

Proof. Replacing $\mathcal{D}((-))$ by $\mathcal{D}(\hat{-})$ we can follow the steps in Lemma 2.40. Using Corollary 2.26 the existence of $\rho$ follows in the same way. The only statement to check is that

[^6]$\hat{f}_{*}: \mathcal{D}(\hat{\mathcal{B}}) \rightarrow \mathcal{D}(\hat{\mathcal{A}})$ is conservative for any morphism of adic analytic rings $f: \mathcal{A} \rightarrow \mathcal{B}$. As $\alpha_{*} \hat{f}_{*} \cong f_{*} \alpha_{*}$ (by Lemma 2.7 and passage to right adjoints) this follows from conservativity of $f_{*}$ and $\alpha_{*}$ (Lemma 2.5).

Remark 2.42. Let $f: Y \rightarrow X$ be any morphism of stably uniform analytic adic spaces. Assume that all terms $Y^{n / X}$ of the Čech nerve of $f$ are again stably uniform (this is a serious assumption, cf. Example [2.43), and that $\mathcal{D}_{\hat{\Delta}}$ satisfies descent for $f$. Then for any morphism $X^{\prime} \rightarrow X$ of stably uniform analytic adic spaces (such that the terms in the Čech nerve of $f^{\prime}: Y \times_{X} X^{\prime} \rightarrow X^{\prime}$ are again stably uniform) the modified quasi-coherent sheaves $\mathcal{D}_{\hat{0}}$ satisfy descent for $f^{\prime}$ (by reducing to the affinoid case and then using the same argument as in Lemma 2.40, Lemma [2.41). Note that the steadiness assumption is automatically satisfied in this case because $X$ is locally Tate.

Example 2.43. Let $\mathbb{Q}_{p}^{\text {cycl }}$ be the completion of $\mathbb{Q}_{p}\left(\mu_{p^{\infty}}\right)$. We mention here the well-known example that the Huber ring $A:=\mathbb{Q}_{p}^{\text {cycl }} \otimes_{\mathbb{Q}_{p}} \mathbb{Q}_{p}^{\text {cycl }}$ (with $\otimes$ refering to the solid or equivalently the Banach space tensor product) is not uniform even though each of the factors is. In fact, the uniform completion of $A$ is the space $B:=C\left(\Gamma, \mathbb{Q}_{p}^{\text {cycl }}\right)$ of continuous functions on $\Gamma=\mathbb{Z}_{p}^{\times}=\operatorname{Gal}\left(\mathbb{Q}_{p}^{\text {cycl }} / \mathbb{Q}_{p}\right)$. The ring $A_{0}:=\mathbb{Z}_{p}^{\text {cycl }} \otimes_{\mathbb{Z}_{p}} \mathbb{Z}_{p}^{\text {cycl }}$ is a ring of definition of $A$ (by definition of the Banach space tensor product), and the ring $B_{0}:=C\left(\Gamma, \mathbb{Z}_{p}^{\text {cycl }}\right)$ is a ring of definition of $B$. Moreover, $B_{0}=B^{0}$ because $\mathbb{Z}_{p}^{\text {cycl }}$ is the ring of power bounded elements in $\mathbb{Q}_{p}^{\text {cycl }}$. In particular, $B$ is uniform. There exists the natural map

$$
\Phi: A_{0} \rightarrow B_{0}, a \otimes b \mapsto(\gamma \mapsto a \gamma(b)) .
$$

From the relation

$$
\sum_{\zeta \in \mu_{p^{n}}} \zeta^{x}=\left\{\begin{array}{lc}
p^{n}, & x \in p^{n} \mathbb{Z}_{p} \\
0, & \text { otherwise }
\end{array}\right.
$$

for $x \in \mathbb{Z}_{p}$ one can conclude that for $a \in \mathbb{Z}_{p}^{\times}$the continuous function

$$
\mathbb{Z}_{p}^{\times} \rightarrow \mathbb{Z}_{p}^{\text {cycl }}, x \mapsto \frac{1}{p^{h}} \sum_{\zeta \in \mu_{p^{n}}} \zeta^{x-a}
$$

is the characteristic function $\chi_{a, n}$ of the clopen subset $a+p^{n} \mathbb{Z}_{p}$ of $\mathbb{Z}_{p}^{\times}=\Gamma$. We can conclude that $\Phi$ is injective with image exactly the closure of the subspace generated by the $p^{n} \chi_{a, n}$ for $a \in \mathbb{Z}_{p}^{\times}$and $n \geq 0$ (cf. [7, Proposition I.3.4]). In particular, $\Phi$ extends to an injection $A \rightarrow B$ with dense image and this identifies $B$ as the uniform completion of $A$. Concretely, for $a \in \mathbb{Z}_{p}^{\times}$and $n \geq 0$ the elements

$$
\frac{1}{p^{n}} \sum_{\zeta \in \mu_{p^{n}}} \zeta^{-a} \otimes \zeta \in A
$$

generate $A^{0} \subseteq A$ as an $A_{0}$-module.

## $3 \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)$for perfectoid spaces

In this section we will prove the first part of Theorem 1.1, i.e., $v$-descent of almost $\mathcal{O}^{+}$-sheaves on perfectoid spaces for our modified version of + -modules from Definition 2.4.

### 3.1 Definition of $\mathcal{D}_{\hat{\ominus}}^{a}\left(A^{+}\right)$

As in [16, Definition 3.1.1] we denote by Perfd ${ }^{\text {aff }}$ the category of affinoid perfectoid spaces over $\operatorname{Spf} \mathbb{Z}_{p}$. We note that each object $\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}$ is classical and static and hence we will often not distinguish between $\left(A, A^{+}\right)$and the classical Tate-Huber pair $\left(A(*), A^{+}(*)\right)$.

Lemma 3.1. Let $\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}$, and let $A^{\circ \circ} \subseteq A^{+}$be the ideal of topologically nilpotent elements. Then $A^{+} / A^{\circ \circ}$ is an idempotent algebra in the classical derived category $\mathcal{D}\left(A^{+}\right)=$ $\mathcal{D}\left(A^{+}(*)\right)$ of $A^{+}$-modules.

Proof. This follows from the fact that $A^{\circ \circ}$ is a filtered colimit of principal ideals generated by non-zero divisors, and that $A^{\circ \circ} \cdot A^{\circ \circ}=A^{\circ \circ}$.

Definition 3.2. Let $\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}$. Then we define $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right):=\mathcal{D}_{\hat{\emptyset}}\left(\left(A^{+}, A^{+}\right)\right) \otimes_{\mathcal{D}\left(A^{+}\right)}$ $\mathcal{D}^{a}\left(A^{+}\right)$, where $\mathcal{D}\left(A^{+}\right) \rightarrow \mathcal{D}^{a}\left(A^{+}\right)$is the open immersion defined by the idempotent algebra $A^{+} / A^{\circ 0}$ from Lemma 3.1.
Remark 3.3. By Lemma 2.35 we have $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)=\mathcal{D}_{\hat{\ominus}}\left(\left(A^{+}, A^{+}\right)\right) / \operatorname{Mod}_{A^{+} / A^{\circ \circ}}\left(\mathcal{D}_{\hat{\imath}}\left(\left(A^{+}, A^{+}\right)\right)\right)$.
Remark 3.4. We note that $A^{+} \cong A^{\circ}$ in $\mathcal{D}_{\hat{\ominus}}^{a}\left(A^{+}\right)$as $A^{\circ \circ} \subseteq A^{+}$. Thus, $\mathcal{D}_{\hat{\wedge}}\left(\left(A^{\circ}, A^{+}\right)\right)$, $\mathcal{D}_{\hat{\wedge}}\left(\left(A^{+}, A^{+}\right)\right.$ become isomorphic after $(-) \otimes_{\mathcal{D}(A)} \mathcal{D}^{a}(A)$. However, in general $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)$and $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{\circ}\right)$ are different.

Remark 3.5. Assume that $\pi \in A \cap A^{+}$is a pseudo-uniformizer. Then $\operatorname{Mod}_{A^{+} / \pi}\left(\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)\right) \cong$ $\mathcal{D}_{\square}^{a}\left(A^{+} / \pi\right)$ is naturally equivalent to the category defined in [16, Definition 3.1.2] because $\operatorname{Mod}_{A^{+} / \pi}\left(\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)\right) \cong \operatorname{Mod}_{A^{+} / \pi}\left(\mathcal{D}_{\square}\left(A^{+}\right)\right) \otimes_{\mathcal{D}\left(A^{+}\right)} \mathcal{D}^{a}\left(A^{+}\right) \cong \mathcal{D}_{\square}^{a}\left(A^{+} / \pi\right)$ using Lemma 2.10 in the first isomorphism.

The following special property is the basic reason why (almost) $v$-descent results for +modules on perfectoid spaces are possible while they fail drastically for non-perfectoid spaces.

Lemma 3.6. Let

be a cartesian diagram in $\operatorname{Perfd}^{\text {aff }}$. Write $X=\operatorname{Spa}\left(A, A^{+}\right), X^{\prime}=\operatorname{Spa}\left(A^{\prime}, A^{\prime+}\right), Y=$ $\operatorname{Spa}\left(B, B^{+}\right)$and $Y^{\prime}=\operatorname{Spa}\left(B^{\prime}, B^{\prime+}\right)$. Then the natural morphism of analytic rings

$$
\left(A^{\prime+}\right)_{\square}^{a} \otimes_{\left(A^{+}\right)_{\square}^{a}}\left(B^{+}\right)_{\square}^{a} \xrightarrow{\sim}\left(B^{\prime+}\right)_{\square}^{a} .
$$

is an isomorphism.
Here, we view $\left(A^{+}\right)_{\square}^{a}$ etc. as analytic rings over the almost setup $\left(A^{+}, A^{\circ \circ}\right)$ following [16, Section 2.3]. More concretely, the assertion means that the natural functor $\mathcal{D}\left(\left(B^{\prime}\right)_{\square}^{+}\right) \rightarrow$ $\mathcal{D}\left(\left(A^{\prime+}\right)_{\square} \otimes_{A_{\square}^{+}} B_{\square}^{+}\right)$becomes an equivalence after $(-) \otimes_{\mathcal{D}\left(A^{+}\right)} \mathcal{D}^{a}\left(A^{+}\right)$.

Proof. Since $B^{\prime+}$ is the (completed) integral closure of the image of $\pi_{0}\left(A^{\prime+} \otimes_{A^{+}} B^{+}\right)$in $\pi_{0}\left(A^{\prime} \otimes_{A} B\right)$, by Lemma 2.24 it is enough to show that the map

$$
A^{\prime+a} \hat{\otimes}_{A^{+a}} B^{+a} \xrightarrow{\sim} B^{\prime+a}
$$

is an isomorphism of almost rings. But both sides of this claimed isomorphism are $\pi$-adically complete for any pseudouniformizer $\pi$ of $A$, hence the claim can be checked modulo $\pi$, where it follows from [16, Lemma 3.1.6].

As a corollary from the proof we see that the derived (completed) tensor product $A^{\prime+a} \otimes_{A^{+a}}$ $B^{+a}$ is in fact static, i.e., concentrated in degre 0 . This statement also holds before passing to the almost category by reduction to the case of perfect rings.

### 3.2 Descent on totally disconnected spaces

In this section, we prove the first part of Theorem 1.1, i.e., the existence of the $v$-sheaf $X \mapsto \mathcal{D}_{\ominus}^{a}\left(\mathcal{O}_{X}\right)$ on perfectoid spaces. More precisely, in this section we prove the following theorem.

Theorem 3.7. There is a unique hypercomplete $v$-sheaf of categories

$$
(\mathrm{Perfd})^{\mathrm{op}} \rightarrow \mathcal{C} a t, \quad X \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)
$$

such that for every $X=\operatorname{Spa}\left(A, A^{+}\right)$which admits a quasi-pro-étale map to a totally disconnected space we have

$$
\mathcal{D}_{\hat{\imath}}^{a}\left(\mathcal{O}_{X}^{+}\right)=\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right) .
$$

We recall that a qcqs perfectoid space $X$ is totally disconnected if each connected component of $X$ has the form $\operatorname{Spa}\left(K, K^{+}\right)$for some perfectoid field $K$ with open and bounded valuation subring $K^{+} \subseteq K\left(\left[22\right.\right.$, Lemma 7.3]), in which case $X=\operatorname{Spa}\left(R, R^{+}\right)$is necessarily affinoid, and for any pseudo-uniformizer $\pi \in R$ we have $\left(|X|, \mathcal{O}_{X}^{+} / \pi\right) \cong \operatorname{Spec}\left(R^{+} / \pi\right)$ as locally ringed spaces ([16, Lemma 3.6.1]).

Totally disconnected perfectoid spaces enjoy the following surprising flatness property ([16, Lemma 3.1.7]).

Lemma 3.8. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be a totally disconnected perfectoid space, and $\pi \in A$ a uniformizer. Let $A^{\prime}$ be a $\pi$-torsion free $A^{+}$-algebra of finite type. Then for every connected component $x=\operatorname{Spec}\left(\left(A^{+} / \pi\right)_{x}\right)$ of $\operatorname{Spec}\left(A^{+} / \pi\right)$ the map $\left(A^{+} / \pi\right)_{x} \rightarrow A^{\prime} \otimes_{A^{+}}\left(A^{+} / \pi\right)_{x}$ is flat and finitely presented.

Proof. This is [16, Lemma 3.1.7].
This result strengthens [22, Proposition 7.23] and is critical for Theorem [1.2 as it implies descent for $\mathcal{O}^{+} / \pi$-modules for a $v$-cover $f: Y \rightarrow X$ with $X$ totally disconnected (16, Lemma 3.1.8]). The proof of loc. cit. uses critically that the (pre)sheaf $\operatorname{Spa}\left(R, R^{+}\right) \mapsto R^{+} / \pi$ on affinoid perfectoid spaces sends cofiltered inverse limits to filtered colimits, roughly to reduce to a single connected component of $X$ where Lemma 3.8 applies. This argument does not work for the sheaf $\mathcal{O}^{+}$. Instead, we roughly spread out the flatness $\bmod \pi$ on a connected components (provided by Lemma (3.8) to some neighborhood, at least up to replacing $Y$ by some disjoint union of some finite open cover. For the precise assertion below, we denote by $Y^{\mathrm{wl}} \in \operatorname{Perfd}^{\text {aff }}$ the w-localization of any $Y \in \operatorname{Perfd}^{\text {aff }}$, as defined in [22, Proposition 7.12].

Proposition 3.9. Let $Y=\operatorname{Spa}\left(B, B^{+}\right) \rightarrow X=\operatorname{Spa}\left(A, A^{+}\right)$be a surjective map of totally disconnected spaces in Perfd ${ }^{\text {aff }}$. Then the map $Y^{\mathrm{wl}} \rightarrow X$ is +-weakly adically descendable of index $\leq 4$, i.e., if $Y^{\mathrm{wl}}=\operatorname{Spa}\left(B_{\mathrm{wl}}, B_{\mathrm{wl}}^{+}\right)$the map $\left(A^{+}\right)_{\square} \rightarrow\left(B_{\mathrm{wl}}^{+}\right)_{\square}$ is weakly adically descendable of index $\leq 4$ in the sense of Definition 2.32.

Proof. We need to show that the map $A^{+} \rightarrow B_{\mathrm{wl}}^{+}$of adic rings is an $\omega_{1}$-filtered colimit of adically descendable maps of index $\leq 4$. Let $I$ be the category of factorizations $Y^{\mathrm{wl}} \rightarrow Y_{i} \rightarrow Y$, where $Y_{i}$ is a finite disjoint union of qcqs open subsets of $Y$. Then by [22, Lemma 7.13] $I$ is cofiltered and $Y^{\mathrm{wl}}=\varliminf_{\mathrm{lim}_{i \in I}} Y_{i}$. We denote $Y_{i}=\operatorname{Spa}\left(B_{i}, B_{i}^{+}\right)$, so that $B_{\mathrm{wl}}^{+}$is the completed filtered colimit of the $B_{i}^{+}$.

In the following discussion, all rings and modules will be discrete and static unless stated otherwise. For some fixed pseudouniformizer $\pi \in A$ we let $J_{\pi}$ be the following category: An object of $J_{\pi}$ is a factorization $A^{+} \rightarrow A_{j} \rightarrow B_{\mathrm{wl}}^{+}$of $A^{+}$-algebras such that the map $A^{+} \rightarrow A_{j}$ is a finitely presented map of classical rings which is flat modulo $\pi$ (here we take the underived
quotient $\left.A_{j} / \pi!\right)$ and such that the map $A_{j} \rightarrow B_{\mathrm{wl}}^{+}$factors over some $B_{i}^{+}$. A morphism in $J_{\pi}$ is a morphism $\alpha: A_{j} \rightarrow A_{j^{\prime}}$ over $A^{+}$such that there is some $i \in I$ with the property that both $A_{j} \rightarrow B_{\mathrm{wl}}^{+}$and $A_{j^{\prime}} \rightarrow B_{\mathrm{wl}}^{+}$factor over $B_{i}^{+}$and these factorizations are compatible with $\alpha$. We now claim that $J_{\pi}$ has the following crucial property:
(a) Let $A_{j} \in J_{\pi}$. Pick some $i$ such that $A_{j} \rightarrow B_{\mathrm{wl}}^{+}$factors over $B_{i}^{+}$and some $a \in A_{j}$ which is sent to 0 via $A_{j} \rightarrow B_{i}^{+}$. Then there is some morphism $\alpha: A_{j} \rightarrow A_{j^{\prime}}$ in $J_{\pi}$ such that $\alpha(a)=0$.

We now prove this claim, so fix $A_{j}, i$ and $a$ as in the claim. Let $A_{j}^{\prime} \subseteq B_{i}^{+}$be the image of $A_{j}$ and fix some connected component $x \in \pi_{0}(X)$. Equivalently $x$ is a connected component of $\operatorname{Spec} A^{+} / \pi$, i.e. of the form $x=\operatorname{Spec}\left(A^{+} / \pi\right)_{x}$, as $X$ is totally disconnected. For any $A^{+}$-algebra $A^{\prime}$ we denote $\left(A^{\prime} / \pi\right)_{x}:=A^{\prime} \otimes_{A^{+}}\left(A^{+} / \pi\right)_{x}$. Then by Lemma 3.8 the $\operatorname{map}\left(A^{+} / \pi\right)_{x} \rightarrow\left(A_{j}^{\prime} / \pi\right)_{x}$ is flat and finitely presented for $A_{j}^{\prime}$ as above and in particular the $\operatorname{map} \beta_{x, \pi}:\left(A_{j} / \pi\right)_{x} \rightarrow\left(A_{j}^{\prime} / \pi\right)_{x}$ is finitely presented ([27, Tag 00F4]). Pick some generators $\bar{a}_{1, x}, \ldots, \bar{a}_{n, x}$ of $\operatorname{ker}\left(\beta_{x, \pi}\right)$. They automatically extend over some (pullback of some) clopen neighbourhood of $x$ in $\operatorname{Spec} A^{+} / \pi$ and hence to all of $A_{j} / \pi$ by extending them by 0 on the (pullback of the) clopen complement of that neighbourhood. Moreover, we can guarantee that these extensions $\bar{a}_{1}, \ldots, \bar{a}_{n} \in A_{j} / \pi$ lie in the kernel of the map $A_{j} / \pi \rightarrow A_{j}^{\prime} / \pi$ by shrinking the clopen neighbourhood of $x$ if necessary. Since the map $\beta: A_{j} \rightarrow A_{j}^{\prime}$ is surjective, we can find lifts $a_{1}, \ldots, a_{n} \in A_{j}$ of $\bar{a}_{1}, \ldots, \bar{a}_{n}$ which lie in the kernel of $\beta$. We can assume that $a$ is one of the $a_{k}$ 's as by assumption $a$ maps to 0 in $A_{j}^{\prime} \subseteq B_{i}^{+}$. Now denote

$$
A_{j, x}^{\prime}:=A_{j} /\left(a_{1}, \ldots, a_{n}\right)
$$

Then $A_{j, x}^{\prime}$ is a finitely presented $A^{+}$-algebra with a map to $A_{j}^{\prime}$ such that $\left(A_{j, x}^{\prime} / \pi\right)_{x}=\left(A_{j}^{\prime} / \pi\right)_{x}$ and such that $a=0$ in $A_{j, x}^{\prime}$. By [27, Tag 00RC] the locus $U \subseteq \operatorname{Spec} A_{j, x}^{\prime} / \pi$ which is flat over $A^{+} / \pi$ is an open subset and by construction we have $\operatorname{Spec}\left(A_{j, x}^{\prime} / \pi\right)_{x} \subseteq U$. We can thus find elements $\bar{f}_{1}, \ldots \bar{f}_{m} \in A_{j, x}^{\prime} / \pi$ such that each localization $\left(A_{j, x}^{\prime} / \pi\right)_{\bar{f}_{k}}$ is flat over $A^{+} / \pi$ and such that the associated schemes cover $\operatorname{Spec}\left(A_{j, x}^{\prime} / \pi\right)_{x}$. By the proof of [16, Lemma 3.6.1] the elements $\bar{f}_{1}, \ldots, \bar{f}_{m}$, more precisely their images in $B_{i}^{+} / \pi$, induce qcqs open, even rational, subsets $V_{1, x}, \ldots V_{m, x} \subseteq Y_{i}$ such that $\bar{f}_{k}$ is a unit in $\mathcal{O}_{Y_{i}}^{+}\left(V_{k, x}\right)$ for $k=1, \ldots, m$ (here we use that $Y$ and hence $Y_{i}$ is totally disconnected in order to apply [16, Lemma 3.6.1]). Pick any lifts $f_{1}, \ldots, f_{m} \in A_{j, x}^{\prime}$ and denote

$$
A_{j, x}:=\prod_{k=1}^{m}\left(A_{j, x}^{\prime}\right)_{f_{k}}, \quad Y_{i, x}:=\bigsqcup_{k=1}^{m} V_{k, x}
$$

Writing $Y_{i, x}=\operatorname{Spa}\left(B_{i, x}, B_{i, x}^{+}\right)$we get a map $A_{j, x} \rightarrow B_{i, x}^{+}$of $A^{+}$-algebras which is compatible with the natural maps $A_{j} \rightarrow A_{j, x}$ and $B_{i}^{+} \rightarrow B_{i, x}^{+}$. Moreover, by construction the subsets $Y_{k, x}$ cover the fiber of $Y_{i}$ over $x$ (because the distinguished opens $D\left(\bar{f}_{1}\right), \ldots, D\left(\bar{f}_{m}\right)$ cover $\left.\operatorname{Spec}\left(A_{j, x}^{\prime} / \pi\right)_{x}\right)$, and hence their pullbacks cover $\left.\operatorname{Spec}\left(B_{i}^{+} / \pi\right)\right)$. Since $Y_{i}$ is qcqs, we can find finitely many connected components $x_{1}, \ldots, x_{\ell}$ of $X$ such that the map $Y_{i^{\prime}}:=Y_{i, x_{1}} \sqcup \cdots \sqcup$ $Y_{i, x_{\ell}} \rightarrow Y_{i}$ is a cover, so that $Y_{i^{\prime}} \in I$. Let

$$
A_{j^{\prime}}:=\prod_{k=1}^{\ell} A_{j, x_{k}}
$$

Then $A_{j^{\prime}} \in J_{\pi}$ and the natural map $\alpha: A_{j} \rightarrow A_{j^{\prime}}$ satisfies the claim (a). This proves (a).
We note that we did not use that $A_{j} / \pi$ is flat over $A^{+} / \pi$ in the proof, only flatness of $A_{j}^{\prime} / \pi$, which was supplied by Lemma 3.8 as the $A_{j}^{\prime}$ was a finitely generated, $\pi$-torsion free
$A^{+}$-algebra. Replacing in the argument $\pi$ by $\pi^{n}, n \geq 1$, we see that we can even guarantee that $A_{j^{\prime}} / \pi^{n}$ is flat over $A^{+} / \pi^{n}$.

From now on, we work with derived condensed rings again. With (a) at hand, we can now prove the following claim:
(b) The category $J_{\pi}$ is filtered and $B_{\mathrm{wl}}^{+}$is the (derived) $\pi$-adically completed colimit of the system $\left(\hat{A}_{j}\right)_{j \in J_{\pi}}$, where $\hat{A}_{j}$ denotes the (derived) $\pi$-adic completion of $A_{j}$.

Let us first show that $J_{\pi}$ is filtered. Clearly, $J_{\pi}$ is non-empty as $A^{+} \in J_{\pi}$. Given $A_{j}, A_{j^{\prime}} \in J_{\pi}$, pick any $i \in I$ such that both $A_{j} \rightarrow B_{\mathrm{wl}}^{+}$and $A_{j^{\prime}} \rightarrow B_{\mathrm{wl}}^{+}$factor over $B_{i}^{+}$. Then the static, non-condensed tensor product $A_{j^{\prime \prime}}:=A_{j} \otimes_{A^{+}} A_{j^{\prime}}$ lies in $J_{\pi}$ and the map $A_{j^{\prime \prime}} \rightarrow B_{\mathrm{wl}}^{+}$factors over $B_{i}^{+}$. In particular the natural maps $A_{j} \rightarrow A_{j^{\prime \prime}}$ and $A_{j^{\prime}} \rightarrow A_{j^{\prime \prime}}$ lie in $J_{\pi}$. Now suppose we have two maps $\alpha_{1}, \alpha_{2}: A_{j} \rightarrow A_{j^{\prime}}$ in $J_{\pi}$. By repeatedly applying (a) to the images of polynomial generators of $A_{j}$ under $\alpha_{1}-\alpha_{2}$ we can construct some $A_{j^{\prime \prime}} \in J_{\pi}$ with a map $A_{j^{\prime}} \rightarrow A_{j^{\prime \prime}}$ such that $\alpha_{1}$ and $\alpha_{2}$ become equal on $A_{j^{\prime \prime}}$. This proves that $J_{\pi}$ is indeed filtered.

Now let $B^{\prime}$ be the (derived) $\pi$-adic completion of $\lim _{j \in J_{\pi}} \hat{A}_{j}$. There is a natural map $B^{\prime} \rightarrow B_{\mathrm{wl}}^{+}$of $\pi$-adic rings which we need to show to be an isomorphism. This can be checked modulo $\pi$, so we need so show that the natural map

$$
\underset{j}{\lim } \hat{A}_{j} / \pi=\underset{j}{\lim } A_{j} / \pi \rightarrow B_{\mathrm{wl}}^{+} / \pi=\underset{i}{\lim } B_{i}^{+} / \pi
$$

is an isomorphism of discrete rings (here the tensor product $A_{j} / \pi$ is a priori derived!). For this it is enough to show the stronger claim that the natural map $\underset{\longrightarrow}{\lim _{j}} A_{j} \rightarrow \underset{\longrightarrow}{\lim } B_{i}^{+}$of classical, non-condensed rings is an isomorphism of static discrete rings. This map is clearly surjective, because every $b \in B_{i}^{+}$is hit by some $A_{j} \rightarrow B_{i}^{+}$, e.g., for $A_{j}=A^{+}[x]$ and $x \mapsto b$. The map is injective by claim (a). This proves the desired isomorphism and thus also claim (b).

We have now shown that the map $A^{+} \rightarrow B_{\mathrm{wl}}^{+}$is the $\pi$-completed filtered colimit of $\pi$ complete finitely presented $A^{+}$-algebras whose underived reduction modulo $\pi$ (i.e. $\pi_{0}\left(A_{j} / \pi\right)$ ) is flat over $A^{+} / \pi$. Since we only talk about underived reductions modulo $\pi$, we cannot immediately conclude the desired descendability statement: Since we have no control over the $\pi$-torsion of $A_{j} \in J_{\pi}$, we cannot deduce from the flatness of $\pi_{0}\left(A_{j} / \pi\right)$ that also $\pi_{0}\left(A_{j} / \pi^{n}\right)$ is flat. We therefore need the following additional argument.

From now on we denote $J_{n}:=J_{\pi^{n}}$ for some fixed pseudouniformizer $\pi \in A$. Then $J_{n+1} \subseteq J_{n}$ is a full subcategory for every $n \geq 1$ and as we noted after the proof of (a) this subcategory is even cofinal. Now let $J$ be the following category: The objects of $J$ are the functors $j: \mathbb{N} \rightarrow J_{1}$ such that $j(n) \in J_{n}$ for all $n \geq 1$. For $j, j^{\prime} \in J$ we define

$$
\operatorname{Hom}_{J}\left(j, j^{\prime}\right)=\underset{n \in \mathbb{N}}{\lim } \operatorname{Hom}\left(\left.j\right|_{\mathbb{N}_{\geq n}},\left.j^{\prime}\right|_{\mathbb{N} \geq n}\right)
$$

To every $j \in J$ we associate the $A^{+}$-algebra $A_{j}:=\lim _{\longrightarrow} A_{j(n)}$. It follows from (a) that $J$ is $\omega_{1}$-filtered 8 and it follows as in (b) that $B_{\mathrm{wl}}^{+}$is the $\pi$-completed colimit of $\left(\hat{A}_{j}\right)_{j \in J}$. Thus, to finish the proof it is now enough to show the following claim:
(c) For every $j \in J$ the map $\left(A^{+}\right)_{\square} \rightarrow\left(\hat{A}_{j}\right)_{\square}$ is adically descendable of index $\leq 4$.

[^7]To prove (c), fix $j \in J$ and $n \geq 1$. We need to show that the map $\left(A^{+} / \pi^{n}\right)_{\square} \rightarrow\left(A_{j} / \pi^{n}\right)_{\square}$ is descendable of index $\leq 4$. This map is the filtered colimit of the maps $\left(A^{+} / \pi^{n}\right)_{\square} \rightarrow$ $\left(A_{j(k)} / \pi^{n}\right)_{\square}$, so by [16, Proposition 2.7.2] it is enough to show that for $k \gg 0$ the map $\left(A^{+} / \pi^{n}\right)_{\square} \rightarrow\left(A_{j(k)} / \pi^{n}\right)_{\square}$ is descendable of index $\leq 2$. We claim that this works for $k \geq n$. Indeed, we have the factorization

$$
\alpha:\left(A^{+} / \pi^{n}\right)_{\square} \rightarrow\left(A_{j(k)} / \pi^{n}\right)_{\square} \rightarrow \pi_{0}\left(A_{j(k)} / \pi^{n}\right)_{\square},
$$

and by definition of $J_{n}$ the composed map $\alpha$ is flat and finitely presented. The map $\alpha$ is even surjective on spectra, and thus an fppf cover, because its composition with $\pi_{0}\left(A_{j(k)} / \pi^{n}\right) \rightarrow$ $B_{\mathrm{wl}}^{+} / \pi^{n}$ is $A^{+} / \pi^{n} \rightarrow B_{\mathrm{wl}}^{+} / \pi^{n}$, which is surjective on spectra (as $Y^{\mathrm{wl}} \rightarrow X$ is a $v$-cover of totally disconnected spaces). Thus by [16, Lemma 2.10.6] the map $\alpha$ is descendable of index $\leq 2$, so by [16, Lemma 2.6.10.(ii)] the same is true for the first map $\left(A^{+} / \pi^{n}\right)_{\square} \rightarrow\left(A_{j(k)} / \pi^{n}\right)_{\square}$, as desired.

Proof of Theorem 3.7. It is sufficient to construct $X \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$for $X \in \operatorname{Perfa}^{\text {aff }}$ as it will formally extend via analytic descent. We define the presheaf $X=\operatorname{Spa}\left(A, A^{+}\right) \mapsto F(X):=$ $\mathcal{D}_{\hat{0}}^{a}\left(A^{+}\right)$on Perfd ${ }^{\text {aff }}$. We first observe that for every v-cover $f: Y=\operatorname{Spa}\left(B, B^{+}\right) \rightarrow X=$ $\operatorname{Spa}\left(A, A^{+}\right)$of totally disconnected spaces, the presheaf $F$ satisfies descent along $f$. Indeed, by Proposition 3.9 and Lemma 3.6 the presheaf $F$ satisfies descent along the map $Y^{\mathrm{wl}} \rightarrow X$, even after any base change $X^{\prime} \rightarrow X$ with $X^{\prime}$ any affinoid, not necessarily totally disconnected, perfectoid space 9 Hence, by [11, Lemma 3.1.2.(3)] the same is true for $f: Y \rightarrow X$. Using Lemma [3.6 we can argue as in [16, Proposition 3.1.9] to show that there is a (necessarily unique) quasi-pro-étale sheaf $X \mapsto \mathcal{D}_{\grave{\wedge}}^{a}\left(\mathcal{O}_{X}^{+}\right)$on $\operatorname{Perfd}^{\text {aff }}$ which has the desired form on those spaces which are quasi-pro-étale over some totally disconnected space. We recall the argument for the convenience of the reader: first, the presheaf $F$ defines a quasi-pro-étale sheaf on the pro-étale sit ${ }^{10}$ of strictly totally disconnected perfectoid spaces, i.e., those totally disconnected spaces whose connected components are of the form $\mathrm{Spa}\left(C, C^{+}\right)$with $C$ algebraically closed and $C^{+} \subseteq C$ an open and bounded valuation subring. Formally, this quasi-pro-étale sheaf extends to the quasi-pro-étale sheaf $X \mapsto \mathcal{D}_{\grave{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$on Perfd ${ }^{\text {aff }}$. Assume that $X=\operatorname{Spa}\left(A, A^{+}\right)$is affinoid perfectoid with a quasi-pro-étale map $X \rightarrow Z$ to a totally disconnected space $Z$. If $Y \rightarrow Z$ is a quasi-pro-étale cover with $Y$ strictly totally disconnected, then $Y \rightarrow Z$ satisfies universal descent for $F$ as we have seen above. In particular, $F$ satisfies descent for the cover $Y^{\prime}:=Y \times_{Z} X \rightarrow X$. As the terms of the Čech nerve for $Y^{\prime} \rightarrow X$ are strictly totally disconnected, we can therefore conclude that $F(X)=\mathcal{D}_{0}^{a}\left(\mathcal{O}_{X}^{+}\right)$as claimed.

It remains to show that the sheaf $X \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$satisfies v-hyperdescent. This can be done in the same way as in the proof of [16, Theorem 3.1.27]: One first shows the analog of [16, Lemma 3.1.23] for adic coefficients, where the two crucial inputs are [16, Lemma 3.1.22] (which can be replaced by Lemma 3.10) and the finite Tor dimension (which can be lifted to the adic level by Corollary [2.14). Then the adic version of [16, Corollary 3.1.24] is a formal corollary and consequently we deduce the adic version of [16, Proposition 3.1.25]; here the only non-formal input is the fact that +-modules, i.e., $\mathcal{D}_{\hat{\ominus}}^{+}\left(A^{+}\right)$for $X=\operatorname{Spa}\left(A, A^{+}\right)$, descend along v-covers of totally disconnected spaces, as shown above. In other words, we have now shown that $X \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$is a v-sheaf. To get v-hyperdescent it remains to prove the adic version of [16, Proposition 3.1.26]. But here the only non-formal input is the finite Tor dimension, which can again be deduced from the mod $-\pi$ version via Corollary 2.14.

[^8]The following result was used in the proof of Theorem 3.7 above. It is an interesting statement on its own, so we extract it here:

Lemma 3.10. Let $f: Y \rightarrow X$ be an étale cover in $\operatorname{Perfd}{ }^{\text {aff }}$. Then + -modules descend along $j$, i.e., the functor $X=\operatorname{Spa}\left(A, A^{+}\right) \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$is an étale sheaf on $\operatorname{Perfd}^{\text {aff }}$.
Proof. After inverting a pseudo-uniformizer $\pi \in A$, i.e., for the functor $X=\operatorname{Spa}\left(A, A^{+}\right) \mapsto$ $\operatorname{Mod}_{A}\left(\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)\right)=\operatorname{Mod}_{A}\left(\mathcal{D}_{\hat{\mathrm{\imath}}}\left(A^{+}\right)\right)$this follows from Theorem [2.38. Using the fiber sequence $A^{+} \rightarrow A \rightarrow A / A^{+}=\underset{\rightarrow}{\lim } A^{+} / \pi^{n}$ reduces therefore to the descent for the functor $X=$ $\operatorname{Spa}\left(A, A^{+}\right) \mapsto \operatorname{Mod}_{A^{+} / \pi^{n}}\left(\mathcal{D}_{\hat{1}}^{a}\left(A^{+}\right)\right)=\operatorname{Mod}_{A^{+} / \pi^{n}}\left(\mathcal{D}_{\square}^{a}\left(A^{+}\right)\right)$. Moreover, using induction and the fiber sequence $A^{+} / \pi^{n-1} \rightarrow A^{+} / \pi^{n} \rightarrow A^{+} / \pi$ reduces to the case $n=1$. As $\pi$ was arbitrary, we may even assume that $p \mid \pi$, and then (by tilting) that $X=\operatorname{Spa}\left(A, A^{+}\right), Y=$ $\mathrm{Spa}\left(B, B^{+}\right)$are perfectoid spaces of characteristic $p$. As $f: Y \rightarrow X$ is étale, there exists by [22, Proposition 6.4.(iv)] some étale morphism $f_{0}: Y_{0}=\operatorname{Spa}\left(B_{0}, B_{0}^{+} \rightarrow X_{0}=\operatorname{Spa}\left(A_{0}, A_{0}^{+}\right)\right.$ of affinoid perfectoid spaces, which are weakly of perfectly finite type over $\operatorname{Spa}\left(\mathbb{F}_{p}\left(\left(\pi^{1 / p^{\infty}}\right)\right)\right)$ (in the sense of [16, Definition 3.1.13.]), and morphisms $g: X \rightarrow X_{0}, Y \rightarrow Y_{0}$ such that $Y \cong Y_{0} \times{ }_{X_{0}} X$. By [16, Theorem 3.1.17] the natural functor

$$
\mathcal{D}_{\square}^{a}\left(A_{0}^{+} / \pi\right) \rightarrow \lim _{[n] \in \Delta} \mathcal{D}_{\square}^{a}\left(\left(B_{0}^{n / A_{0}}\right)^{+} / \pi\right)
$$

is an equivalence (here $\left(B_{0}^{n / A_{0}}\right)^{+}$denotes the + -ring for the $n$-th stage of the Čech nerve of $Y_{0} \rightarrow X_{0}$ ). Using the arguments from Lemma 2.40, we see that this implies that the functor

$$
\mathcal{D}_{\square}^{a}\left(A^{+} / \pi\right) \rightarrow \lim _{[n] \in \Delta} \mathcal{D}_{\square}^{a}\left(\left(B^{n / A}\right)^{+} / \pi\right)
$$

is an equivalence as well. More precisely, the critical statement to check is that for any morphism $g: X=\operatorname{Spa}\left(A, A^{+}\right) \rightarrow X_{0}=\operatorname{Spa}\left(A_{0}, A_{0}^{+}\right)$of affinoid perfectoid spaces over $\mathrm{Spa}\left(\mathbb{F}_{p}\left(\left(\pi^{1 / p^{\infty}}\right)\right)\right)$ the pushforward $g_{*}^{a}: \mathcal{D}_{\square}^{a}\left(A^{+} / \pi\right) \rightarrow \mathcal{D}_{\square}^{a}\left(A_{0}^{+} / \pi\right)$ on almost categories is conservative. This however follows formally from conservativity of $g_{*}: \mathcal{D}_{\square}\left(A^{+} / \pi\right) \rightarrow \mathcal{D}_{\square}\left(A_{0}^{+} / \pi\right)$ by embedding the almost categories via $*$-pushforward, i.e., the functors of almost elements over $A^{+} / \pi$ resp. $A_{0}^{+} / \pi$, into $\mathcal{D}_{\square}\left(A^{+} / \pi\right)$ resp. $\mathcal{D}_{\square}\left(A_{0}^{+} / \pi\right)$. This finishes the proof.

Remark 3.11. The same argument as in Theorem 3.7 can be used to show the following: Let $X \rightarrow S$ be a map in Perfd ${ }^{\text {aff }}$. Then there is a unique hypercomplete v -sheaf

$$
\left(\operatorname{Perfd}_{/ S}\right)^{\mathrm{op}} \rightarrow \mathcal{C}^{2} t_{\infty}, \quad T \mapsto \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X_{T}}^{+}\right)
$$

such that if $T$ admits a quasi-pro-étale map to some totally disconnected space then $\mathcal{D}_{乞}^{a}\left(\mathcal{O}_{X_{T}}^{+}\right)=$ $\mathcal{D}_{\hat{\circ}}^{a}\left(B^{+}\right)$, where $X \times_{S} T=\operatorname{Spa}\left(B, B^{+}\right)$. Indeed, Proposition 3.9 implies descent after base change.

We note that logically we did not use Theorem 1.2 in the proof of Theorem 3.7, only in its disguise through the key ingredient Lemma 3.8 and the étale descent from [16, Lemma 3.1.22].

Remark 3.12. Let $X=\operatorname{Spa}\left(A, A^{+}\right) \in \operatorname{Perfd}{ }^{\text {aff }}$ with pseudo-uniformizer $\pi \in A$. It follows from Theorem 3.7 and Lemma 2.10 that

$$
\operatorname{Mod}_{A^{+} / \pi}\left(\mathcal{D}_{\hat{○}}^{a}\left(\mathcal{O}^{+}\right)\right) \cong \mathcal{D}_{\square}^{a}\left(\mathcal{O}_{X}^{+} / \pi\right),
$$

with the right hand side defined in [16, Definition 3.1.3]. Indeed, both sides satisfy $v$-descent and if $X$ is totally disconnected we can apply Lemma 2.10

Definition 3.13. Let $f: Y \rightarrow X$ be a map of perfectoid spaces.
(a) We let $f^{*}: \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{Y}^{+}\right)$be the restriction map of the sheaf $Z \mapsto \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{Z}^{+}\right)$from Theorem 3.7.
(b) We let $f_{*}$ be the right adjoint to $f^{*} 11$
(c) If $X=\operatorname{Spa}\left(A, A^{+}\right)$is affinoid, we denote by

$$
\widetilde{(-)}: \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)
$$

the natural functor, and by $\Gamma(X,-)$ its right adjoint.

### 3.3 Boundedness conditions

Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be an affinoid perfectoid space. In the rest of this paper, we will be concerned with the question whether the functor

$$
\widetilde{(-)}: \mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)
$$

from Definition 3.13 is an equivalence. This will turn out to be true under certain finiteness assumptions. The critical definition of + -boundedness is the following.

Definition 3.14. (a) A map $f: Y^{\prime} \rightarrow Y$ of affinoid perfectoid spaces in characteristic $p$ is called + -bounded if it is $p$-bounded in the sense of [16, Definition 3.5.5.(b)].
(b) $Y \in \operatorname{Perfd}_{\mathbb{F}_{p}}^{\mathrm{aff}}$ is called + -bounded if there exists a + -bounded map $Y \rightarrow Z$ with $Z$ totally disconnected.
(c) Let $\ell$ be a prime, e.g., $\ell=p$. A spatial diamond $Y$, e.g., an affinoid perfectoid space in characteristic $p$, is called $\ell$-bounded if there exists some integer $d \geq 0$, such that for all (static) $\mathbb{F}_{\ell}$-sheaves $\mathcal{M}$ on $Y_{\text {et }}$ we have $H^{i}\left(Y_{\text {et }}, \mathcal{M}\right)=0$ for $k>d$. Here, $Y_{\text {et }}$ is the étale site of $Y$ as defined in [22, Definition 14.1.(i)].
(d) An affinoid perfectoid space $X \in P$ Perfd is + -bounded (resp. $\ell$-bounded) if its tilt $X^{b} \in \operatorname{Perfd}_{\mathbb{F}_{p}}$ is + -bounded (resp. $\ell$-bounded).

Most notably, $X=\operatorname{Spa}\left(A, A^{+}\right)+$-bounded implies by Theorem 1.2 (and Remark 3.12) that for any pseudo-uniformizer $\pi \in A$

$$
\operatorname{Mod}_{A^{+} / \pi}\left(\mathcal{D}_{\hat{\imath}}^{a}\left(\mathcal{O}_{X}^{+}\right)\right) \cong \mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+} / \pi\right) \cong \operatorname{Mod}_{A^{+} / \pi}\left(\mathcal{D}_{\hat{\ominus}}^{a}\left(A^{+}\right)\right)
$$

and this will be a critical ingredient in our proof of Theorem 1.1. We note that the property of being +-bounded (resp. $\ell$-bounded) depends only on the tilt. This is crucial for our major application.

Remark 3.15. Roughly, a morphism $f: Y^{\prime} \rightarrow Y$ is +-bounded if $f_{*}$ has finite "cohomological dimension on $\mathcal{O}^{+} / \pi$-modules" and $f$ has $\operatorname{dim} . \operatorname{trg}(f)<\infty$. In [16, Definition 3.5.5] the definition of + -bounded from Definition 3.14 is phrased for all (morphisms of) small $v$-stacks. This extra generality is not necessary for this paper. We note that [16, Definition 3.5.5.] uses the terminology " $p$-bounded" for what we call "+-bounded" here, while our use of " $p$-bounded" refers to the different (but related) notion considered in [15, Definition 2.1].

[^9]Example 3.16. Let $f: Y^{\prime} \rightarrow Y$ be a morphism of affinoid perfectoid spaces in characteristic $p$. If $f$ is quasi-pro-étale or weakly of perfectly finite type, then $f$ is + -bounded ([16, Lemma 3.5.10.(v)], [16, Lemma 3.5.13]).

Lemma 3.17. Let $X$ be an $\ell$-bounded spatial diamond, and $f: X^{\prime} \rightarrow X$ a qcqs quasi-pro-étale map. Then $X^{\prime}$ is $\ell$-bounded.

Proof. We note that $f_{*}: \mathcal{D}^{+}\left(X_{\text {et }}^{\prime}, \mathbb{F}_{\ell}\right) \rightarrow \mathcal{D}^{+}\left(X_{\text {et }}, \mathbb{F}_{\ell}\right)$ is $t$-exact by [22, Remark 21.14] if $f$ is separated. This implies that the assertion is local on $X^{\prime}$, and thus we may furthermore assume that $f$ is separated. Then again [22, Remark 21.14] reduces to the $\ell$-boundedness of $X$.

The notions of +-boundedness and $p$-boundedness are closely related. The next result shows that the former implies the latter. We have not investigated to what extend the other implication is valid.

Theorem 3.18. Let $X$ be a +-bounded affinoid perfectoid space. Then $X$ is $p$-bounded. In fact, if $f: X \rightarrow Z$ is a + -bounded map to a totally disconnected space $Z$, then $H^{i}\left(X_{\mathrm{et}}, \mathcal{F}\right)=0$ for $i>3+\operatorname{dim} \cdot \operatorname{trg}(f)$ and any (static) $\mathbb{F}_{p}$-sheaf $\mathcal{F}$ on $X_{\mathrm{et}} 12$
Proof. We may replace $X$ by $X^{b}$ and hence assume that $X$ is of characteristic $p$. We first deal with the case that $X=Z$ is totally disconnected. Let $\pi: X_{\mathrm{et}} \rightarrow \pi_{0}(X)_{\text {et }}$ be the natural morphism of sites. As $\pi_{0}(X)_{\text {et }}$ has cohomological dimension 0 ([27, Lemma 0A3F]), it suffices to show that $R^{i} \pi_{*}(\mathcal{F})=0$ for $i>2$ and any static $\mathbb{F}_{p}$-sheaf on $X_{\text {et }}$. This can be checked on stalks, and hence we may assume that $X=\operatorname{Spa}\left(K, K^{+}\right)$is a perfectoid affinoid field in characteristic $p$, i.e., $K$ is a perfectoid field of characteristic $p$ and $K^{+} \subseteq K$ an open and bounded valuation subring. We may write $\left(K, K^{+}\right)$as a (completed) filtered colimit of perfectoid affinoid fields $\left(L, L^{+}\right)$and therefore reduce (via [22, Proposition 14.9]) to the case that $K^{+}$has finite Krull dimension. Let $j: U \rightarrow X$ be the complement of the closed point. Then $U=\operatorname{Spa}\left(K, \widetilde{K}^{+}\right)$for some valuation ring $K^{+} \subseteq \widetilde{K}^{+}$. By [22, Lemma 21.13] the fiber $\mathcal{G}:=\left[\mathcal{F} \rightarrow R j_{*} j^{*} \mathcal{F}\right]$ lies in $\mathcal{D}^{[0,1]}\left(X_{\mathrm{et}}, \mathbb{F}_{p}\right)$. By [22, Proposition 21.15] we can conclude that $H^{i}\left(X_{\mathrm{et}}, \mathcal{G}\right)=0$ for $i \geq 3$ because $\operatorname{Spa}\left(K, \mathcal{O}_{K}\right)$ has $p$-cohomological dimension $\leq 1$ (using $\operatorname{Spa}\left(K, \mathcal{O}_{K}\right)_{\mathrm{et}} \cong \operatorname{Spec}(K)_{\text {et }}$ and [20, Proposition 6.5.10]). This implies that $H^{i}\left(X_{\mathrm{et}}, \mathcal{F}\right) \cong$ $H^{i}\left(U_{\text {et }}, \mathcal{F}_{\mid U}\right)$ for $i \geq 3$. By induction this reduces to the case $X=\operatorname{Spa}\left(K, \mathcal{O}_{K}\right)$, in which case $H^{i}\left(X_{\text {et }}, \mathcal{F}\right)=0$ for $i \geq 2$.
Now assume that $X=\operatorname{Spa}\left(A, A^{+}\right)$is a general affinoid perfectoid space with a + -bounded morphism $f: X \rightarrow Z$ to a totally disconnected space $Z=\operatorname{Spa}\left(R, R^{+}\right)$(it is actually sufficient to assume that $\operatorname{dim} \cdot \operatorname{trg}(f)<\infty)$. We claim that $R^{i} f_{*}(\mathcal{F})=0$ for any $i>\operatorname{dim} . \operatorname{trg}(f)+1$ and any static $\mathbb{F}_{p}$-sheaf $\mathcal{F}$ on $X_{\text {et }}$. We may write $f$ as a cofiltered inverse limits of morphisms $f_{j}: X_{j}=\operatorname{Spa}\left(A, A_{j}^{+}\right) \rightarrow Z$ and $A_{j}^{+}$finitely generated over $B^{+}$. By [16, Lemma 3.5.10.(iv)] the morphisms $f_{j}$ are +-bounded again as the relative compactifications $\bar{X}^{/ Z}={\overline{X_{j}}}^{\prime Z}$ agree. By definition $\operatorname{dim} \cdot \operatorname{trg}(f)=\operatorname{dim} \cdot \operatorname{trg}\left(f_{j}\right)$ for any $j$. In particular, we may replace $f$ by $f_{j}$ and assume that $f: X \rightarrow Z$ is compactifiable, i.e., $X \rightarrow \bar{X}^{/ Z}$ is a (necessarily quasi-compact) open immersion. Note that $\bar{X}^{/ Z}$ is an affinoid perfectoid space, and thus in particular a spatial diamond. Thus, the claim follows from Theorem 3.19,

The next result Theorem[3.19] is a $\bmod p$ analog of [22, Theorem 22.5]. In [22, Theorem 22.5] $f$ is assumed to be a compactifiable morphism of general small $v$-stacks, which is representable in spatial diamonds. We have to add the stronger assumption that $Y^{\prime}, Y$ are diamonds as the base change results for mod $p$-coefficients are weaker than in the prime-to- $p$-case. However, we follow the strategy of proof of [22, Theorem 22.5] quite closely.

[^10]Theorem 3.19. Let $f: Y^{\prime} \rightarrow Y$ be a compactifiable morphism of spatial diamonds, with canonical compactification $Y^{\prime} \xrightarrow{j} Y^{\prime \prime}:=\overline{Y^{\prime}} / Y \xrightarrow{\bar{f}} Y$. Then $R^{i} f_{*}, R^{i} \bar{f}_{*}$ vanish on static étale $\mathbb{F}_{p}$-sheaves for $i>2 \operatorname{dim} \cdot \operatorname{trg}(f)+1$. If $\bar{f}$ is representable in spatial diamonds, they vanish for $i>\operatorname{dim} . \operatorname{trg}(f)+1$.

Proof. We note that $j_{*}$ is exact by [22, Lemma 21.13]. Hence, it suffices to prove the assertion about $R^{i} \bar{f}_{*}$. Let $X \rightarrow Y$ be a quasi-pro-étale cover with $X$ strictly totally disconnected. Then by [22, Corollary 16.10.(ii)] we may replace $f$ by its base change $Y^{\prime} \times_{Y} X \rightarrow X$, and thus reduce to the case that $Y$ is strictly totally disconnected. As the statement can be checked on stalks we may even assume that $Y=\operatorname{Spa}\left(C, C^{+}\right)$is an affinoid field. Let $s \in Y$ be the closed point, and $U:=Y \backslash\{s\}$ with pullback $V:=\bar{f}^{-1}(U) \xrightarrow{j} Y^{\prime \prime}$. Using Theorem 3.20 we see that for every static $\mathbb{F}_{p}$-sheaf $\mathcal{F}$ on $Y^{\prime \prime}$ we have

$$
R \Gamma\left(Y^{\prime \prime}, j!j^{*} \mathcal{F}\right)=0
$$

which allows us to reduce to the case that $\mathcal{F}_{\mid U}=0$. If $Y^{\prime \prime}$ is a spatial diamond, then [22, Proposition 21.11] applies and yields the cohomological bound dim. $\operatorname{trg}(f)+1\left(\operatorname{as~}_{\operatorname{cd}}^{p}(y) \leq 1\right.$ for all maximal points $y$ of $Y^{\prime \prime}$ and $\left.\operatorname{dim} \cdot \operatorname{trg}(f)=\operatorname{dim} \cdot \operatorname{trg}(\bar{f})\right)$. It $Y^{\prime \prime}$ is not a spatial diamond, then the arguments of [22, Theorem 22.5] go through without change and yield the cohomological bound $2 \operatorname{dim} . \operatorname{trg}(f)+1$.

The following is a mod $p$-analog of [22, Theorem 19.2]. Again the stronger assumption that $Y^{\prime}, Y$ are spatial diamonds avoids the failure of strong base change results mod $p$.

Theorem 3.20. Let $f: Y^{\prime} \rightarrow Y$ be a proper morphism of spatial diamonds, and let $j: U \rightarrow Y$ be an open immersion with pullback $g: U^{\prime}:=Y^{\prime} \times_{Y} U \rightarrow U, j^{\prime}: U^{\prime} \rightarrow Y^{\prime}$. Then for any $A \in \mathcal{D}^{+}\left(U^{\prime}, \mathbb{F}_{p}\right)$ the natural morphism $j!R g_{*}(A) \rightarrow R f_{*}\left(j_{!}^{\prime} A\right)$ is an isomorphism.

Proof. The proof of [22, Theorem 19.2] goes through with a minor change: To avoid the use of [22, Proposition 17.6] take a quasi-pro-étale cover $X \rightarrow Y$ with $X$ strictly totally disconnected (this is possible as $Y$ is a diamond) and apply instead [22, Corollary 16.10.(ii)] to reduce to the case $Y$ strictly totally disconnected. We note that the reduction in [22, Theorem 19.2] to the case that $Y^{\prime}={\overline{X^{\prime}}}^{/ Y}$ for some strictly totally disconnected space $X^{\prime}$ over $Y=X$ does not need [22, Corollary 16.8.(ii)] (or something similar): the $v$-site of $Y^{\prime}$ is replete, and $R \Gamma\left(Y_{\mathrm{et}}^{\prime}, B\right) \cong R \Gamma\left(Y_{v}^{\prime}, \varepsilon^{*} B\right)$ by [22, Proposition 14.10 ] for $B \in \mathcal{D}^{+}\left(Y_{\mathrm{et}}, \mathbb{F}_{p}\right)$, where $\varepsilon: Y_{v}^{\prime} \rightarrow Y_{\text {et }}$ is the natural morphism of sites. This implies the desired cohomological descent. The remaining steps in the proof, e.g., [22, Lemma 19.4], go through without change.

The major goal of this paper is to find a large class of affinoid perfectoid spaces $X=$ $\operatorname{Spa}\left(A, A^{+}\right)$for which we have $\mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)=\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)$. The next two results provide useful abstract criteria for when this identification is valid.

Lemma 3.21. Let $X=\operatorname{Spa}\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}$. Assume that there exists a quasi-pro-étale cover $Y \rightarrow X$ with $Y=\operatorname{Spa}\left(B, B^{+}\right)$totally disconnected almost + -modules (in the sense of Definition (3.2) descent along $\left(A^{+}\right)_{\square} \rightarrow\left(B^{+}\right)_{\square}$. Then the natural functor

$$
\widetilde{(-)}: \mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)
$$

is an equivalence.
Proof. Let $Y_{\bullet} \rightarrow X$ be the Čech nerve of $Y \rightarrow X$ and write $Y_{n}=\operatorname{Spa}\left(B_{n}, B_{n}^{+}\right)$. Note that each $Y_{n}, n \geq 0$, is quasi-pro-étale over the totally disconnected space $Y=Y_{0}$. Then

$$
\mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{Y}^{+}\right) \cong \lim _{[n] \in \Delta} \mathcal{D}_{\hat{\llcorner }}^{a}\left(\mathcal{O}_{Y_{n}}^{+}\right) \cong \lim _{[n] \in \Delta} \mathcal{D}_{\hat{\emptyset}}^{a}\left(B_{n}^{+}\right) \cong \mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right),
$$

using Theorem [3.7 in the second isomorphism and that almost +-modules descent along $\left(A^{+}\right)_{\square} \rightarrow\left(B^{+}\right)_{\square}$ in the third.

Lemma 3.22. Let $X=\operatorname{Spa}\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}$ be + -bounded, and let $\pi \in A$ be a pseudouniformizer. Then the functor $\overline{(-)}: \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$is an equivalence if and only if $\Gamma(X,-): \mathcal{D}_{\hat{\wedge}}^{a}\left(\mathcal{O}_{X}^{+}\right) \rightarrow \mathcal{D}_{\hat{\hat{}}}^{a}\left(A^{+}\right)$preserves colimits and $\mathcal{D}_{\hat{\wedge}}^{a}\left(\mathcal{O}_{X}^{+}\right)$is generated under colimits by $\pi$-complete objects $\mathcal{N}$ with $\Gamma(X, \mathcal{N})$ bounded to the right.
Proof. The necessity is clear (as $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$is generated by $\pi$-complete objects), so let us check the converse. There is the natural adjunction

$$
\widetilde{(-)}: \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right) \rightleftarrows \mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right): \Gamma(X,-)
$$

and we need to show that these functors are inverse to each other. We first show that $\widetilde{(-)}$ is fully faithful, i.e. that for every $M \in \mathcal{D}_{\hat{0}}^{a}\left(A^{+}\right)$we have $\Gamma(X, \widetilde{M})=M$. Since $\Gamma(X,-)$ preserves colimits, we can assume that $M=P^{a}$ for a compact generator $P \in \mathcal{D}_{\hat{\wedge}}\left(A^{+}\right)$, in particular $M$ is $\pi$-complete and bounded to the right. We claim that $\widetilde{M} \in \mathcal{D}_{\ominus}^{a}\left(\mathcal{O}_{X}^{+}\right)$is $\pi$-complete. This can be checked on a $v$-cover $Y \rightarrow X$ with $Y$ totally disconnected. By naturality of $\widetilde{(-)}$ and Lemma [2.12, this reduces to the case that $Y=X$, where $\widetilde{(-)}$ is an equivalence. Thus $\Gamma(X, \widetilde{M})$ and $M$ are $\pi$-complete and hence the identity $\Gamma(X, \widetilde{M})=M$ can be checked modulo $\pi$. But then it follows from [16, Theorem 3.5.21] (note that it is formal that $\widetilde{(-)}, \Gamma(X,-)$ are compatible with passage to $A^{+} / \pi$-modules).

To finish the proof, it is now enough to show that $\widetilde{(-)}$ is essentially surjective. Since by assumption $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$is generated under colimits by $\pi$-complete objects, it is enough to show that every $\pi$-complete object lies in the essential image of $\widetilde{(-)}$. Given a $\pi$-complete $\mathcal{N} \in \mathcal{D}_{\hat{0}}^{a}\left(\mathcal{O}_{X}^{+}\right)$we need to see that the counit $\widehat{\Gamma(X, \mathcal{N})} \xrightarrow{\sim} \mathcal{N}$ is an isomorphism. By assumption we may even assume that $\Gamma(X, \mathcal{N})$ is bounded to the right. Then by the same argument as above both sides of the claimed isomorphism are $\pi$-complete, so we can check the claimed isomorphism modulo $\pi$. Then it reduces to again [16, Theorem 3.5.21].

### 3.4 Descent for relative compactifications of totally disconnected spaces

In this section we want to identify $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$for certain relative compactifications of totally disconnected spaces. The precise statement is the following.
Theorem 3.23. Let $X \in \operatorname{Perfd}^{\text {aff }}$ whose tilt admits a map $f: X^{b} \rightarrow Z$ to some totally disconnected space $Z$ with $\operatorname{dim} \cdot \operatorname{trg} f<\infty$. Let $X^{\prime} \rightarrow X$ be a quasi-pro-étale map, where $X^{\prime}$ is a totally disconnected space. Then for $Y:=\bar{X}^{\prime / X}=\operatorname{Spa}\left(B, B^{+}\right)$the natural functor

$$
\mathcal{D}_{\hat{\emptyset}}^{a}\left(B^{+}\right) \rightarrow \mathcal{D}_{\hat{\imath}}^{a}\left(\mathcal{O}_{Y}^{+}\right)
$$

is an equivalence.
Proof. Set $Z^{\prime}:=\left(X^{\prime}\right)^{b}$. We let $X^{\prime} \rightarrow Y^{\prime}$ be the unique map of perfectoid spaces whose tilt is the map $Z^{\prime} \rightarrow \overline{Z^{\prime}}{ }^{\prime Z}$ (note that $\mathcal{O}\left(X^{\prime}\right)=\mathcal{O}\left(Y^{\prime}\right)$ as this holds for the tilt). We note that there exists a natural inclusion $Y \rightarrow Y^{\prime}$ whose tilt is the natural map $\overline{Z^{\prime}} / X \rightarrow{\overline{Z^{\prime}}}^{/ Z}$. In fact, $Y$ is an intersection of rational open subsets in $Y^{\prime}$ as this holds on the tilt. By Proposition 3.24 there is a pro-étale cover $W^{\prime} \rightarrow Y^{\prime}$ which is of universal almost +-descent, where $W^{\prime}$ is a totally disconnected space, i.e., almost +-modules (in the sense of Definition (3.2) descend along $W^{\prime} \rightarrow Y^{\prime}$ and all its base changes to affinoid perfectoid spaces. Then $W:=Y \times_{Y^{\prime}} W \rightarrow Y$ is a pro-étale cover with +-descent for $Y$, and $W$ is still totally disconnected (as it is an intersection of rational open subsets in the totally disconnected space $W^{\prime}$ ). By Lemma 3.21 this finishes the proof.

In the next proposition, an untilt of a perfectoid space $Z$ of characteristic $p$ is a perfectoid space $X$ together with an identification $X^{b} \cong Z$. Proposition 3.24]is similar to [16, Proposition 3.1.11]. Again the essential point is to show that open covers of relative compactifications are descendable with uniformly bounded index. This will as in [16, Proposition 3.1.11] be reduced to open covers of (families of) Riemann-Zariski spaces. The necessary results on these kind of adic spaces will be discussed in Section 3.5.

Proposition 3.24. Let $f: Z^{\prime} \rightarrow Z$ be a map of totally disconnected spaces in $\operatorname{Perfd}_{\mathbb{F}_{p}}^{\text {aff }}$ with $\operatorname{dim} . \operatorname{trg} f<\infty$. Let $X$ be an untilt of $\bar{Z}^{\prime / Z}$. Then there is a pro-étale cover $Y \rightarrow X$ by a totally disconnected space $Y$ such that almost +-modules descend along $Y \times_{X} X^{\prime} \rightarrow X^{\prime}$ for any morphism $X^{\prime} \rightarrow X$ of affinoid perfectoid spaces.

Proof. Let $Y:=X^{\mathrm{wl}}$ be the w-localization of $X$, which by [22, Lemma 7.13] is a totally disconnected space of the form $Y=\lim _{\leftrightarrows} Y_{i} \rightarrow X$ such that each $Y_{i}$ is a disjoint union of rational open subsets of $X$. We will show that the map $Y \rightarrow X$ has the claimed descent properties and similarly any base change. Write $X=\operatorname{Spa}\left(A, A^{+}\right), Y=\operatorname{Spa}\left(C, C^{+}\right)$. It suffices to check that the map $\left(A^{\circ}, A^{+}\right)_{\square} \rightarrow\left(C^{\circ}, C^{+}\right)_{\square}$ is weakly adically descendable of an index $\leq c(d)$, where $c(d)$ is a constant only depending on $d:=\operatorname{dim} \cdot \operatorname{trg} f$. Indeed, then almost +-modules will descent universally along $Y \rightarrow X$ because $\mathcal{D}_{\hat{\imath}}\left(A^{\circ}, A^{+}\right)$and $\mathcal{D}_{\hat{\mathrm{\theta}}}\left(A^{+}\right)$define the same almost category using $\left(A^{\circ}\right)^{a}=\left(A^{+}\right)^{a}$ (and the similar statement for $\mathcal{D}_{\hat{\mathrm{n}}}\left(C^{\circ}, C^{+}\right)$and $\mathcal{D}_{\hat{\wedge}}\left(C^{+}\right)$). By Proposition 2.33.(iii) it is enough to show that for each map $Y_{i}=\operatorname{Spa}\left(C_{i}, C_{i}^{+}\right) \rightarrow$ $X$ the map $\left(A^{\circ}, A^{+}\right)_{\square} \rightarrow\left(C_{i}^{\circ}, C_{i}^{+}\right)_{\square}$ of adic analytic rings is adically descendable with an index $\leq c(d)$. By definition of adic descendability it is enough to prove this statement modulo $\pi$ for every pseudouniformizer $\pi$ on $X$. Now we can argue as in [16, Proposition 3.1.11], using Lemma 3.26 in place of [16, Proposition 2.10.10] in order to get descendability (and not just fs-descendability) of the covering. In the following we provide the details.

We are given a map $U:=Y_{i}=\bigsqcup_{j=1}^{n} U_{j} \rightarrow X$ for a cover of $X$ by qcqs open subsets $U_{j} \subseteq X$. Write $X=\operatorname{Spa}\left(A, A^{+}\right), Z=\operatorname{Spa}\left(B, B^{+}\right)$and $Z^{\prime}=\operatorname{Spa}\left(B^{\prime}, B^{\prime+}\right)$, where we note that $B^{\prime}=A^{b}$ and that $A^{b+}$ is the completed integral closure of $B^{+}+B^{100}$ in $B^{\prime+}$. Let $\bar{B}^{\prime}:=B^{10} / B^{100}$, $\bar{B}^{+}:=B^{+} / B^{\circ \circ}$ and $\bar{X}:=\operatorname{Spa}\left(\bar{B}^{\prime}, \bar{B}^{+}\right)$. Note that $\bar{B}^{\prime}=A^{b \circ} / A^{b \circ \circ}$, that $A^{b+} / A^{b \circ \circ}$ is the integral closure of the image of $\bar{B}^{+}$in $\bar{B}^{\prime}$ and that $X^{b}$ satisfies the hypothesis of Lemma 3.27, Hence by Lemma 3.27we obtain a canonical homeomorphism $|X|=\left|X^{b}\right| \cong|\bar{X}|$. In particular, the open cover $X=\bigcup_{j} U_{j}$ corresponds to an open cover $\bar{X}=\bigcup_{j} \bar{U}_{j}$ and we define $\bar{U}:=\bigsqcup_{j} \bar{U}_{j}$.

We now claim that $\bar{X}$ satisfies the condition of Lemma 3.26, Indeed, the connected components of $Z^{\prime}$ are of the form $\operatorname{Spa}\left(K^{\prime}, K^{\prime+}\right)$ for a perfectoid field $K^{\prime}$, hence the connected components of $\operatorname{Spec} \bar{B}^{\prime}$ are of the form $\operatorname{Spec}\left(\mathcal{O}_{K^{\prime}} / \mathfrak{m}_{K^{\prime}}\right)$ (we note for later that this implies that $\left|\operatorname{Spec}\left(\bar{B}^{\prime}\right)\right|$ is a profinite set). Similarly, if a connected component of $Z$ is of the form $\mathrm{Spa}\left(K, K^{+}\right)$then the corresponding connected component of $\operatorname{Spec} \bar{B}^{+}$is of the form $K^{+} / \mathfrak{m}_{K}$, which is a valuation ring. This implies that the condition (a) of Lemma 3.26 is satisfied. Similarly, condition (b) is satisfied using the same $d$ ([4, VI.10.3.Corollaire 1]). Altogether we deduce that the map $\bar{U} \rightarrow \bar{X}$ is descendable of index bounded by a constant only depending on $d$. More precisely, by the proof of Lemma 3.26 there is a reduced projective $\bar{B}^{+}$-scheme $\bar{S}$ together with a dominant map Spec $\bar{B}^{\prime} \rightarrow \bar{S}$ such that the map $\bar{U} \rightarrow \bar{X}$ comes via base-change from an open covering of $\bar{S}$ along the map $\bar{X} \rightarrow \bar{S}$. Moreover, $\bar{S}$ can be covered by $d+1$ open affine subschemes $\bar{W}_{i}=\operatorname{Spec} \bar{R}_{i} \subseteq \bar{S}$. As $\bar{S}$ is separated, the morphism $\operatorname{Spec} \bar{B}^{\prime} \rightarrow \bar{S}$ is affine. Note that we have a surjective map $A^{\circ} / \pi \rightarrow A^{\circ} / A^{\circ \circ}=A^{b \circ} / A^{b \circ 0}=\bar{B}^{\prime}$ with locally nilpotent kernel. By [27, Lemma 07 RT ] we can therefore form the pushout $S:=\operatorname{Spec}\left(A^{\circ} / \pi\right) \cup_{\operatorname{Spec}\left(\bar{B}^{\prime}\right)} \bar{S}$ in the category of schemes. By the construction in [27, Lemma 07RT], the scheme $S$ is covered by the $d+1$ open affine subschemes $\operatorname{Spec}\left(A_{i}^{\prime}\right) \cup_{\operatorname{Spec}\left(\bar{B}_{i}^{\prime}\right)} \bar{W}_{i}$, where $\bar{Z}_{i}=\operatorname{Spec}\left(\bar{B}_{i}^{\prime}\right) \subseteq \operatorname{Spec}\left(\bar{B}^{\prime}\right)$ is the affine open preimage of $W_{i}$ along $\operatorname{Spec}\left(\bar{B}^{\prime}\right) \rightarrow \bar{S}$, and $\operatorname{Spec}\left(A_{i}^{\prime}\right) \subseteq \operatorname{Spec}\left(A^{\circ} / \pi\right)$ is the
unique affine open subscheme with underlying topological space $\bar{Z}_{i}$.
The morphism $\bar{X} \rightarrow \bar{S}$ lifts to a morphism $X_{\pi}:=\operatorname{Spa}\left(A^{\circ} / \pi, A^{+} / \pi\right) \rightarrow S$ which is the same map on underlying topological spaces. Now the given open covering $U=\operatorname{Spa}\left(C_{i}, C_{i}^{+}\right) \rightarrow X$ induces an open covering $U_{\pi}:=\mathrm{Spa}\left(C_{i}^{\circ} / \pi, C_{i}^{+} / \pi\right) \rightarrow X_{\pi}$ (using that Lemma 3.27 applies to $U$ as well), and $U_{\pi} \rightarrow X_{\pi}$ reduces to the covering $\bar{U} \rightarrow \bar{X}$. Thus the covering $U_{\pi} \rightarrow X_{\pi}$ is an open covering of discrete adic spaces which comes via base-change from an open covering of $S$ (as can be checked after base change to $\bar{X}$ ). Since $S$ is covered by $d+1$ open affine subsets, the map $U_{\pi} \rightarrow X_{\pi}$ is therefore descendable of index bounded by a constant only depending on $d$ ([16, Corollary 2.10.7]). This finishes the proof.

### 3.5 Families of Riemann-Zariski spaces

In this section we provide the necessary results on families of Riemann-Zariski spaces, which entered in the proof of Proposition 3.24. Unless stated otherwise, each Huber pair is classical in this section. A related discussion can be found in [26], but we need to ensure that the relative Riemann-Zariski space is a cofiltered inverse limits of projective schemes (and not merely of proper schemes) in order to ensure the uniform bound on descendability in Lemma 3.26.

Lemma 3.25. Let $A^{+} \rightarrow A$ be a map of classical, discrete rings such that each connected component of $X:=\operatorname{Spa}\left(A, A^{+}\right)$is of the form $\operatorname{Spa}\left(K, A_{x}^{+}\right)$for some field $K$ and some subring $A_{x}^{+} \subseteq K$. Let $I$ be the category of factorizations $\operatorname{Spec} A \rightarrow Y_{i} \rightarrow \operatorname{Spec} A^{+}$such that $Y_{i}$ is a reduced projective $A^{+}$-scheme and the map $\operatorname{Spec} A \rightarrow Y_{i}$ is dominant. Then:
(i) $I$ is cofiltered and for every $i \in I$ the map $\operatorname{Spec} A \rightarrow Y_{i}$ factors uniquely over a map $X \rightarrow Y_{i}$ over $\operatorname{Spec}\left(A^{+}\right)$. The induced map $|X| \rightarrow\left|Y_{i}\right|$ is closed.
(ii) The map $|X| \xrightarrow{\sim} \varliminf_{\mathrm{m}_{i}}\left|Y_{i}\right|$ is a homeomorphism of topological spaces.

We note that under the assumptions of this lemma the natural map $\operatorname{Spa}(A, A) \rightarrow \operatorname{Spec}(A)$ is an isomorphism of locally ringed spaces, which makes condition (i) well-defined.

Proof. Let us first show that $I$ is cofiltered. Given any $i, i^{\prime} \in I$ there can be at most one map $Y_{i} \rightarrow Y_{i^{\prime}}$ in $I$. Indeed, this follows immediately from the fact that the map $\operatorname{Spec} A \rightarrow Y_{i}$ is an epimorphism (in general dominant morphisms are epimorphisms in the category of reduced separated schemes, as one easily checks by a standard diagonal argument). Thus to prove that $I$ is cofiltered we only need to check that there is some $i^{\prime \prime} \in I$ such that there are maps $Y_{i^{\prime \prime}} \rightarrow Y_{i}$ and $Y_{i^{\prime \prime}} \rightarrow Y_{i^{\prime}}$. But note that we can take $Y_{i^{\prime \prime}}$ to be the scheme-theoretic image of the natural map $\operatorname{Spec} A \rightarrow Y_{i} \times_{\text {Spec } A^{+}} Y_{i^{\prime}}$.

Now fix some $i \in I$. Note that $X$ is the relative compactification of the map $\operatorname{Spec} A \rightarrow$ Spec $A^{+}$(see [16, Example 2.9.27]), so since $Y_{i} \rightarrow$ Spec $A^{+}$is proper (and thus its own relative compactification) we deduce that the map Spec $A \rightarrow Y_{i}$ extends canonically to a map $X \rightarrow Y_{i}$ over $\operatorname{Spec}\left(A^{+}\right)$. The uniqueness of the extension can be checked on connected components, where it follows from the valuative criterion for properness. We now check that the morphism $X \rightarrow Y_{i}$ is closed. As $A$ is discrete, the morphism $X \rightarrow Y_{i}$ is a quasi-compact morphism of spectral spaces. Because it is as well specializing (by the valuative criterion for properness) this implies that the morphism $X \rightarrow Y_{i}$ is closed. Indeed, by quasi-compactness of $X \rightarrow Y_{i}$ the image of a closed subsets is pro-constructible and stable under specialization, hence closed ([27, Lemma 0903]). This proves (i).

It remains to prove (ii). By (i) we know that the map $|X| \rightarrow \varliminf_{i}\left|Y_{i}\right|$ is closed and continuous, so it is enough to show that this map is bijective. Let us first check that it is injective, so let $y, y^{\prime} \in|X|$ be two given points which get mapped to the same point in each $\left|Y_{i}\right|$. Let $\widetilde{A^{+}}$be the integral closure of the image of $A^{+}$in $A$. Then $\pi_{0}\left(\operatorname{Spec}\left(\widetilde{A^{+}}\right)\right)=\pi_{0}(\operatorname{Spec}(A))$ and
writing $\widetilde{A^{+}}$as a filtered colimit of finite, reduced $A^{+}$-algebras, we see that $y$ and $y^{\prime}$ lie in the same connected component $x=\operatorname{Spa}\left(K, A_{x}^{+}\right)$of $X$ and hence correspond to valuation rings $V_{y}, V_{y^{\prime}} \subseteq K$ containing $A_{x}^{+}$. Assuming $y \neq y^{\prime}$ we can w.l.o.g. assume that there is some $\bar{a} \in V_{y} \backslash V_{y^{\prime}}$. Let $a \in A$ be a lift of $\bar{a}$, which we can assume to be a unit (note that $K$ is the filtered colimit of the ring of functions of clopen neighbourhoods of $\operatorname{Spec} K \subseteq \operatorname{Spec} A$ ). Then the pair $\{1, a\}$ determines a map $f: \operatorname{Spec} A \rightarrow \mathbb{P}_{A^{+}}^{1}$ and we let $Y_{a}$ be the scheme theoretic image of $f$. Then $Y_{a} \in I$ and we claim that the induced map $X \rightarrow Y_{a}$ separates $y$ and $y^{\prime}$. To see this, we can assume that the map $A^{+} \rightarrow A$ is injective, because otherwise the surjection from $A^{+}$to its image in $A$ induces a closed immersion on schemes and in particular an injective map. Now $Y_{a}$ can be described explicitly: It is obtained by gluing $\operatorname{Spec} A^{+}[a]$ and Spec $A^{+}[1 / a]$ along $\operatorname{Spec} A^{+}[a, 1 / a]$. Moreover, the map $X \rightarrow Y_{a}$ sends a valuation ring $V \subseteq K$ to the unique point on $Y_{a}$ such that $V$ dominates the associated local ring. Thus $V_{y^{\prime}}$ is sent to a point in $\operatorname{Spec} A^{+}[1 / a]$ which does not lie in $\operatorname{Spec} A^{+}[a, 1 / a]$, whereas $V_{y}$ is sent to a point in $\operatorname{Spec} A^{+}[a]$.

We now prove surjectivity of the map $|X| \rightarrow \lim _{i}\left|Y_{i}\right|$, so let any $\left(y_{i}\right)_{i} \in \lim _{i}\left|Y_{i}\right|$ be given. Replacing $A^{+}$by $\widetilde{A^{+}}$we may assume that $\pi_{0}(\operatorname{Spec}(A))=\pi_{0}\left(\operatorname{Spec}\left(A^{+}\right)\right)$. Then all $y_{i}$ live over the same connected component of $X$, so by base-changing to this connected component we can assume that $X$ is connected, i.e. of the form $X=\operatorname{Spa}\left(K, A^{+}\right)$for a field $K$. For each $i$ the dominant map $\operatorname{Spec} K \rightarrow Y_{i}$ induces an injection $\mathcal{O}_{Y_{i}, y_{i}} \subseteq K$. Let $R \subseteq K$ be the (filtered) union of all $\mathcal{O}_{Y_{i}, y_{i}}$. Then $R$ is a local ring and hence dominated by some valuation ring $V \subseteq K$. This valuation ring corresponds to a point $y \in|X|$ which maps to $\left(y_{i}\right)_{i}$ (by uniqueness), as desired.

The next lemma yields the crucial uniform descendability bound for Proposition 3.24, It is the analog of [16, Proposition 2.10.10] for families of Riemann-Zariski spaces.

Lemma 3.26. Let $A^{+} \rightarrow A$ be a map of classical, discrete rings and $X=\operatorname{Spa}\left(A, A^{+}\right)$the associated discrete adic space. Assume that:
(a) Every connected component of $\operatorname{Spec} A$ is of the form $\operatorname{Spec} K^{\prime}$ for some field $K^{\prime}$ and every connected component of $\operatorname{Spec} A^{+}$is of the form $\operatorname{Spec} V$ for some valuation ring $V$.
(b) There is an integer $d \geq 0$ such that for every map of connected components $\operatorname{Spec} K^{\prime} \rightarrow$ Spec $V$ induced by $A^{+} \rightarrow A$, the associated map $V \hookrightarrow K^{\prime}$ is injective and the transcendence degree of $K^{\prime}$ over the fraction field of $V$ is $\leq d$.

Let $U=\bigsqcup_{j=1}^{n} U_{j} \rightarrow X$ be an open covering by quasicompact open subsets $U_{j} \subseteq X$. Then $U \rightarrow X$ is descendable of index bounded by a constant only depending on d.

Proof. Let $I$ be as in Lemma 3.25. Then by Lemma 3.25 we have $|X|=\lim _{i}\left|Y_{i}\right|$ and all the maps $|X| \rightarrow\left|Y_{i}\right|$ are closed, which together implies that there is some $i \in I$ and an open covering $Y_{i}=\bigcup_{j=1}^{m} W_{i j}$ by qcqs open subsets $W_{i j} \subseteq Y_{i}$ such that $U \rightarrow X$ is the base-change of the map $W_{i}:=\bigsqcup_{j=1}^{m} W_{i j} \rightarrow Y_{i}$ along the map $X \rightarrow Y_{i}$. By [16, Lemma 2.6.9] it is thus enough to show that the map $W_{i} \rightarrow Y_{i}$ is descendable of bounded index.
To simplify notation let us write $Y:=Y_{i}$ and $W:=W_{i}$. By [16, Corollary 2.10.7] it is enough to show that $Y$ can be covered by $d+1$ affine open subschemes. The map $\pi_{0}(\operatorname{Spec} A) \rightarrow$ $\pi_{0}\left(\operatorname{Spec} A^{+}\right)$is a map of compact Hausdorff spaces and in particular has closed image. This image thus corresponds to an affine subscheme $\operatorname{Spec} A^{\prime+} \subseteq \operatorname{Spec} A^{+}$(the limit over all clopen neighbourhoods of the set-theoretic image inside Spec $A^{+}$) by definition of $I$ we know that $Y$ is supported over Spec $A^{\prime+}$. We can thus replace $A^{+}$by $A^{\prime+}$ in order to assume that the map $\pi_{0}(\operatorname{Spec} A) \rightarrow \pi_{0}\left(\operatorname{Spec} A^{+}\right)$is surjective. Now pick a connected component $x=\operatorname{Spec} V \in$ $\pi_{0}\left(\operatorname{Spec} A^{+}\right)$and let $Y_{x}$ be the fiber of $Y$ over $x$. Denote by $\eta$ and $s$ the open and closed point of $V$ and let $Y_{x, \eta}$ and $Y_{x, s}$ be the the generic and special fiber of $Y_{x}$, respectively. We claim:
(*) $Y_{x, s}$ has dimension $\leq d$.
To see this, we first claim that there are finitely many connected components $y_{1}=\operatorname{Spec} K_{1}^{\prime}$, $\ldots, y_{n}=$ Spec $K_{n}^{\prime}$ over $x$ such that the map $y_{1} \sqcup \cdots \sqcup y_{n} \rightarrow Y_{x}$ is dominant. To see this, fix any irreducible component $Z \subseteq Y_{x, \eta}$. For every (qcqs) open subset $W \subseteq Z$, let $S_{W} \subseteq \operatorname{Spec} A$ be the preimage of $W$ under the map $\operatorname{Spec} A \rightarrow Y$. Then $S_{W}$ is a quasicompact open subset and since the topological space of Spec $A$ is profinite it follows that $S_{W}$ is closed. We claim that the intersection of all $S_{W}$, for $W$ ranging through non-empty subsets of $Z$, is non-empty. Otherwise, by compactness there would be some $W_{1}, \ldots, W_{m} \subseteq Z$ such that $S_{W_{1}} \cap \cdots \cap S_{W_{m}}=\varnothing$. Since $Z$ is irreducible, $W:=W_{1} \cap \cdots \cap W_{m}$ is still a non-empty open subset of $Z$ and we deduce $S_{W}=\varnothing$; but this contradicts the fact that Spec $A \rightarrow Y$ is dominant. Pick any $y$ in the intersection of all $S_{W}$; then the map $y \rightarrow Z$ is dominant. Repeating this procedure for all irreducible components of $Y_{x, \eta}$, we find $y_{1}, \ldots, y_{n} \subseteq \operatorname{Spec} A$ such that the map $y_{*}:=y_{1} \sqcup \cdots \sqcup y_{n} \rightarrow Y_{x, \eta}$ is dominant. By the assumption that the induced maps $V \hookrightarrow K^{\prime}$ are injective (see (b)) we see that every point of $Y_{x}$ that is hit by some $y \in \operatorname{Spec} A$ necessarily lies in $Y_{x, \eta}$. Consequently, since the map $\operatorname{Spec} A \rightarrow Y$ is dominant, we deduce that $y_{*} \rightarrow Y_{x}$ must be dominant.

With $y_{1}, \ldots, y_{n}$ as in the previous paragraph, let $Z_{1}, \ldots, Z_{n} \subseteq Y_{x}$ be the scheme-theoretic images of these points under the map $\operatorname{Spec} A \rightarrow Y$. Then $Y_{x}=Z_{1} \cup \cdots \cup Z_{n}$, so to prove claim (*) it is enough to show that each $Z_{i, s}$ has dimension $\leq d$. Since $Z_{i}$ is integral, by [27, Lemma 0B2J] it is enough to show that the generic fiber $Z_{i, \eta}$ has dimension $\leq d$. But this follows immediately from the fact that $Z_{i}$ is a projective $K$-variety and there is a dominant map Spec $K_{i}^{\prime} \rightarrow Z_{i, \eta}$ such that $K_{i}^{\prime}$ has transcendence degree $\leq d$ over $K$. This finishes the proof of claim (*).

With claim (*) proved, we now proceed with the proof that $Y$ is covered by $d+1$ affine open subschemes. Let $\kappa=\kappa(s)$ be the residue field of $s$. By the proof of [16, Lemma 2.10.9] we can find some $n \geq 0$ such that $Y \subseteq \mathbb{P}_{A^{+}}^{n}$ and homogeneous polynomials $\bar{f}_{0}, \ldots, \bar{f}_{d} \in \kappa\left[t_{1}, \ldots, t_{n}\right]$ such that the intersection of the vanishing loci of the $\overline{f_{j}}$ 's in $\mathbb{P}_{\kappa}^{n}$ is disjoint from $Y_{x, s}$. We can assume that each $\bar{f}_{j}$ has a coefficient 1 and we can then lift each $\bar{f}_{j}$ to a homogeneous polynomial $f_{j} \in A^{+}\left[t_{1}, \ldots, t_{n}\right]$, each of which has one coefficient 1 (first lift the coefficients to $V$ and then use that $V$ is the colimit of the functions on clopen neighbourhoods of $x \subseteq \operatorname{Spec} A^{+}$). Let $H_{j} \subseteq \mathbb{P}_{A^{+}}^{n}$ be the zero locus of $f_{j}$. As in the proof of [16, Lemma 2.10.9] each $\mathbb{P}_{A^{+}}^{n} \backslash H_{j}$ is affine. Now let $Y_{j}:=Y \cap\left(\mathbb{P}_{A^{+}}^{n} \backslash H_{j}\right)$. This is a closed subscheme of the affine scheme $\mathbb{P}_{A^{+}}^{n} \backslash H_{j}$ and hence affine. Let $Y^{\prime}:=\bigcup_{j=0}^{d} Y_{j}$. This is an open subscheme of $Y$ and by the argument in [16, Lemma 2.10.9] it contains $Y_{x}$.

Since the map $Y \rightarrow \operatorname{Spec} A^{+}$is closed, the image $Z^{\prime} \subseteq \operatorname{Spec} A^{+}$of $Y \backslash Y^{\prime}$ under this map is closed. By construction we have $x \notin Z^{\prime}$. By the argument in [22, Lemma 7.5] there exists a clopen neighborhood $U \subseteq \operatorname{Spec} A^{+}$of $x$ contained in $\operatorname{Spec}\left(A^{+} \backslash Z^{\prime}\right.$. Hence $Y_{U}=Y \times_{\text {Spec } A^{+}} U$ is covered by the affine opens $\left(Y_{0}\right)_{U}, \ldots,\left(Y_{d}\right)_{U}$.

The above proof shows that for every $x \in \pi_{0}(X)$ there is a clopen neighbourhood $U_{x} \subseteq$ Spec $A^{+}$of $x$ such that $Y_{U}$ can be covered by $d+1$ affine open subsets. By passing to a finite disjoint cover of Spec $A^{+}$by $U_{x}$ 's we can deduce that $Y$ can be covered be $d+1$ affine open subsets, as desired.

In Proposition 3.24 we used as well the following isomorphism of topological spaces (similar to [16, Lemma 3.6.1]). Note that for any Tate-Huber pair $\left(A, A^{+}\right)$there exists a natural specialization map

$$
\mathrm{sp}=\operatorname{sp}_{\left(A, A^{+}\right)}:\left|\operatorname{Spa}\left(A, A^{+}\right)\right| \rightarrow\left|\operatorname{Spa}\left(A^{\circ} / A^{\circ \circ}, A^{+} / A^{\circ \circ}\right)\right|,
$$

which is uniquely determined by naturality and the requirement that if $\left(A, A^{+}\right)=\left(K, K^{+}\right)$ for a non-archimedean valued field $K$ and an open and bounded valuation subring $K^{+} \subseteq K$,
then $\operatorname{sp}_{\left(K, K^{+}\right)}$sends a point $x \in \operatorname{Spa}\left(K, K^{+}\right)$corresponding to a prime ideal $\mathfrak{p}_{x}$ containing $K^{\circ \circ}$ to the valuation on $\mathcal{O}_{K} / K^{\circ \circ}$ corresponding to the prime ideal $\mathfrak{p}_{x} / K^{\circ \circ}$.

Lemma 3.27. Let $\operatorname{Spa}\left(A, A^{+}\right)$be an affinoid Tate adic space such that every connected component is of the form $\mathrm{Spa}\left(K, A_{K}^{+}\right)$, where $K$ is a non-archimedean field and $A_{K}^{+} \subseteq K$ is a subring. Then the specialiation map

$$
\mathrm{sp}:\left|\operatorname{Spa}\left(A, A^{+}\right)\right| \rightarrow\left|\operatorname{Spa}\left(A^{\circ} / A^{\circ \circ}, A^{+} / A^{\circ \circ}\right)\right|
$$

is a homeomorphism.
Proof. Without loss of generality $A^{+} \subseteq A$ is an open and integrally closed subring in $A^{\circ}$. We denote $X=\operatorname{Spa}\left(A, A^{+}\right), \bar{A}=A^{\circ} / A^{\circ \circ}, \bar{A}^{+}=A^{+} / A^{\circ \circ}$ and $\bar{X}=\operatorname{Spa}\left(\bar{A}, \bar{A}^{+}\right)$. Then the claim is that the specialization map induces a homeomorphism $|X| \cong|\bar{X}|$.

First note that there is a canonical homeomorphism $\pi_{0}(X)=\pi_{0}(\bar{X})$, as both can be derived from the idempotent elements in $A^{+}$. Fix a connected component $c=\operatorname{Spa}\left(K, A_{K}^{+}\right)$of $X$ and let $\bar{c}$ be the corresponding connected component of $\bar{X}$. Let us write $\bar{c}=\operatorname{Spa}\left(\bar{K}, \bar{A}_{K}^{+}\right)$. Then $K^{\circ}$ is the completed filtered colimit of $\mathcal{O}_{X}^{\circ}(W)$ for clopen neighbourhoods $W \subseteq X$ of $c$ as the clopen neighborhoods of $c$ are cofinal among all neighborhoods of $c$ (by the argument in [22, Lemma 7.5]). Similarly, $\bar{K}$ is the filtered colimit of $\mathcal{O}_{\bar{X}}(\bar{W})$ for clopen neighbourhoods $\bar{W} \subseteq \bar{X}$ of $\bar{c}$. By the homeomorphism of spaces of connected components, the $W$ correspond to $\bar{W}$ and under this correspondence we have $\mathcal{O}_{\bar{X}}(\bar{W})=\mathcal{O}_{X}^{\circ}(W) / \mathcal{O}_{X}^{\circ \circ}(W)$. It follows that $\bar{K}=K^{\circ} / K^{\circ \circ}$, which is a discrete field.

With $c$ and $\bar{c}$ as above, we note that the points of $c$ are precisely the open and bounded valuation rings $V$ of $K$ such that $A_{K}^{+} \subseteq V$. Similarly, the points of $\bar{c}$ are the valuation rings $\bar{V}$ of $\bar{K}$ such that $\bar{A}_{K}^{+} \subseteq \bar{V}$. These two sets agree as each open and bounded valuation ring $V$ of $K$ must contain $K^{\circ \circ}$. As $c$ was arbitrary (and $\pi_{0}(X)=\pi_{0}(\bar{X})$ ) this shows that the specialization map is bijective, and as well specializing. It remains to show that this bijection is a homeomorphism. For this it suffices to see that the specialization map is continuous and quasi-compact, e.g., using [22, Lemma 2.5]. Note that $\bar{X}$ (and $X$ ) satisfy the assumption of Lemma 3.28. Hence the topology of $X$ has a basis given by $U_{\bar{f}, \bar{W}}=\{|\bar{f}| \leq 1\} \cap W$ for $f \in A^{\circ}$ (with reduction $\bar{f} \in A^{\circ} / A^{\circ \circ}$ ) and clopen $\bar{W} \subseteq \bar{X}$. Now, $\operatorname{sp}^{-1}(\{|\bar{f}| \leq 1\})=\{|f| \leq 1\}$ as can be checked by reducing to the case that $\left(A, A^{+}\right)$is an affinoid field, and $\mathrm{sp}^{-1}(\bar{W})=W$ where $W \subseteq X$ is the clopen subset corresponding to $\bar{W}$ under the homeomorphism $\pi_{0}(X) \cong \pi_{0}(\bar{X})$. In particular, we can conclude that $\mathrm{sp}^{-1}\left(U_{\bar{f}, \bar{W}}\right)$ is open and quasi-compact. This finishes the proof.

The proof of Lemma 3.27 made use of the following slightly technical result about the topology on families of Zariski-Riemann spaces:

Lemma 3.28. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be an affinoid adic space such that every connected component is of the form $\operatorname{Spa}\left(K, A_{K}^{+}\right)$, where $K$ is a field and $A_{K}^{+} \subseteq K$ is a subring. Then $|X|$ has a basis of open subsets given by $\{|f| \leq 1\} \cap W$ for varying $f \in A^{\circ}$ and clopen subsets $W \subseteq X$.

Proof. Suppose first that $X$ is connected, so that $A$ is a field. A basis of open subsets of $X$ is given by the rational subsets, i.e. subsets of the form $\left\{\left|f_{1}\right| \leq|g| \neq 0, \ldots,\left|f_{n}\right| \leq|g| \neq 0\right\} \subseteq X$ for certain $f_{1}, \ldots, f_{n}, g \in A$. We can assume that $g \neq 0$, so that $g$ is invertible. Then the above subset agrees with $\left\{\left|\frac{f_{1}}{g}\right| \leq 1, \ldots\left|\frac{f_{n}}{g}\right| \leq 1\right\}$ and hence is an intersection of subsets of the form $\{|f| \leq 1\}$ for varying $f \in A$. Note that if $f$ is not in $A^{0}$ (i.e. is not powerbounded) then $\{|f| \leq 1\}=\varnothing$. Indeed, $A$ is either discrete or non-archimedean. If $A$ is discrete, then $A=A^{\circ}$, and if $A$ is non-archimedean, then $\{|f| \leq 1\} \neq \varnothing$ implies that the point given by the
non-trivial rank 1 valuation on $A$ lies in $\{|\leq| 1\}$, which in turn yields $f \in A^{\circ}$. This finishes the proof in the case that $X$ is connected.

Let now $X$ be general. We argue as in the proof of [16, Lemma 3.6.1]. Fix some open subset $U \subseteq X$. Pick a point $x \in U$ and let $c=\operatorname{Spa}\left(K, A_{K}^{+}\right) \in \pi_{0}(X)$ denote the connected component of $X$ containing $x$. By what we have shown above, there is some $\bar{f} \in A_{K}^{+}$such that $x \in\{|\bar{f}| \leq$ $1\} \subseteq U \cap c$. Note that $c$ is the cofiltered limit of its clopen neighbourhoods in $X$, which implies that $A_{K}^{+}$is the completed filtered colimit of $\mathcal{O}_{X}^{+}(V)$ for the clopen neighbourhoods $W \subseteq X$ of $x$. Therefore, after potentially modifying $\bar{f}$ by some topologically nilpotent element in $A_{K}^{+}$ (which does not affect $\{|\bar{f}| \leq 1\}$ ) we can extend $\bar{f}$ to some $f_{W} \in \mathcal{O}_{X}^{+}(W)$ for some $W$ as before. Then

$$
\bigcap_{c \subseteq W^{\prime} \subseteq W}\left(W^{\prime} \cap X \backslash U \cap\left\{\left|f_{W}\right| \leq 1\right\}\right)=\varnothing \text {, }
$$

where $W^{\prime}$ ranges through all clopen neighbourhoods of $c$ contained in $W$. Each term in the above intersection is closed in the construcible topology on $X$, so by compactness of the constructible topology, already a finite intersection must be empty. This implies that one of the terms must be empty, i.e. there is some clopen neighbourhood $c \subseteq W^{\prime} \subseteq W$ such that $W^{\prime} \cap X \backslash U \cap\left\{\left|f_{W}\right| \leq 1\right\}=\varnothing$. Now let $f \in A^{+}$be the element which restricts to $f_{W}$ on $W^{\prime}$ and to 0 outside of $W^{\prime}$. Let furthermore $f^{\prime} \in A^{+}$be the elment which is 0 on $W^{\prime}$ and 1 outside of $W^{\prime}$. Then $\{|f| \leq 1\} \cap W^{\prime} \subseteq U$. Note that the left-hand side contains $x$ by construction, so we have constructed an open neighbourhood of $x$ inside $U$ which belongs to the claimed basis for the topology. This finishes the proof.

## 4 Overconvergent and nuclear sheaves

In the following we discuss a version of the nuclear $\mathbb{Z}_{\ell}$-sheaves defined in [15] in the case $\ell=p$. In fact, the basic definitions and results all work in the same way (similar to the fact that the basic results on étale $\mathbb{F}_{p}$-shaeves work the same as for $\mathbb{F}_{\ell}$ ), but the theory is lacking the required base-change result (for solid sheaves) to produce a good 6 -functor formalism $\sqrt{13}$ Nevertheless, nuclear $\mathbb{Z}_{p}$-sheaves will play an important role in the proof of descent for +bounded affinoid perfectoid spaces in Theorem 1.1.

## $4.1 \omega_{1}$-solid sheaves

In the following we fix a prime $\ell$, e.g., $\ell=p$, and transport the main definitions and results from [15] to our situation of interest. As in Definition 3.14 resp. [15, Definition 2.1] we say that a spatial diamond $X$ is $\ell$-bounded if there is some integer $d \geq 0$ such that for all static $\mathbb{F}_{\ell}$-modules $\mathcal{M} \in \mathcal{D}_{\text {et }}\left(X, \mathbb{F}_{\ell}\right)$ we have $H^{k}(X, \mathcal{M})=0$ for $k>d$.

From now on we fix a static ring $\Lambda$, which is an adic and profinite $\mathbb{Z}_{\ell}$-algebra. Then $\Lambda \cong \lim _{n} \Lambda / I^{n}$ for finite rings $\Lambda / I^{n}$ and a finitely generated ideal of definition $I \subseteq \Lambda$ containing $\ell$. We will abuse notation and write again $\Lambda$ for the pro-étale sheaf $\varliminf_{\leftarrow} \lim _{n} \Lambda / I^{n}$ with inverse limit formed on the quasi-pro-étale site of a locally spatial diamond.

Before we can define nuclear $\Lambda$-sheaves on $\ell$-bounded spatial diamonds, we need to study solid and $\omega_{1}$-solid $\Lambda$-sheaves as considered in [9, Section VII.1] and [15, Section 2]. In the following, a quasi-pro-étale map $U \rightarrow X$ of spatial diamonds is called basic if it can be written as a cofiltered inverse limit of étale, quasi-compact and separated maps $U_{i} \rightarrow X$ ([15), Definition 2.2]). We note that by [22, Definition 10.1.(i)] quasi-pro-étale maps of diamonds are locally separated by definition, and hence the basic quasi-pro-étale maps form a basic for $X_{\text {qproet }}$.

[^11]Definition 4.1. Let $X$ be a spatial diamond.
(a) For every basic quasi-pro-étale map $U=\lim _{i} U_{i} \rightarrow X$ (see [15, Definition 2.2]) we denote $\Lambda_{\square}[U]:=\lim _{\varliminf_{i}} \Lambda\left[U_{i}\right] \in \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right), 1$
(b) A quasi-pro-étale sheaf $\mathcal{M} \in \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ is called solid if for all basic quasi-pro-étale $U \rightarrow X$ the natural map

$$
\underline{\operatorname{Hom}}\left(\Lambda_{\square}[U], \mathcal{M}\right) \xrightarrow{\sim} \underline{\operatorname{Hom}}(\Lambda[U], \mathcal{M})
$$

is an isomorphism. We denote by $\mathcal{D}_{\square}(X, \Lambda) \subseteq \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ the full subcategory spanned by the solid sheaves.
(c) Assume $X$ is $\ell$-bounded. A solid sheaf $\mathcal{M} \in \mathcal{D}_{\square}(X, \Lambda)$ is called $\omega_{1}$-solid if for every $\omega_{1}$-filtered colimit $U=\lim _{\longleftarrow} U_{i}$ of basic objects in $X_{\text {qproet }}$ the natural map

$$
\underset{i}{\lim } \Gamma\left(U_{i}, \mathcal{M}\right) \xrightarrow{\sim} \Gamma(U, \mathcal{M})
$$

is an isomorphism. We denote by $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ the full subcategory spanned by the $\omega_{1}$-solid objects.
(d) An object $M \in \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ is called complete if it is $I$-adically complete for some finitely generated ideal of definition $I \subseteq \Lambda$.

Proposition 4.2. Let $X$ be a spatial diamond.
(i) The category $\mathcal{D}_{\square}(X, \Lambda)$ is stable under limits and colimits in $\mathcal{D}\left(X_{\mathrm{qproet}}, \Lambda\right)$ and contains all étale sheaves.
(ii) $\mathcal{D}_{\square}(X, \Lambda)$ is compactly generated and a collection of compact generators is given by $\Lambda_{\square}[U]$ for $w$-contractible basic $U \in X_{\text {qproet }}$. Moreover, for each basic quasi-pro-étale $U \rightarrow X$ the object $\Lambda_{\square}[U]$ is static.
(iii) The $t$-structure on $\mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ restricts to at-structure on $\mathcal{D}_{\square}(X, \Lambda)$.
(iv) The composition $\mathcal{D}_{\square}(X, \Lambda) \rightarrow \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right) \rightarrow \mathcal{D}\left(X_{v}, \Lambda\right)$ is fully faithful, and $X \mapsto$ $\mathcal{D}_{\square}(X, \Lambda)$ is a $v$-sheaf of $\infty$-categories.
(v) The inclusion $\mathcal{D}_{\square}(X, \Lambda) \rightarrow \mathcal{D}\left(X_{\mathrm{qproet}}, \Lambda\right)$ admits a left adjoint $(-)_{\square}$ such that $(\Lambda[U])_{\square}=$ $\Lambda_{\square}[U]$ for $U \rightarrow X$ basic quasi-pro-étale.
(vi) There exists a unique closed symmetric monoidal structure $-\otimes_{\Lambda}^{\square}-$ on $\mathcal{D}_{\square}(X, \Lambda)$ such that $(-)_{\square}$ is symmetric monoidal. We have $\Lambda_{\square}[U] \otimes_{\Lambda}^{\square} \Lambda_{\square}\left[U^{\prime}\right] \cong \Lambda_{\square}\left[U \times_{X} U^{\prime}\right]$ for $U, U^{\prime} \rightarrow X$ basic quasi-pro-étale.
(vii) If $M, N \in \mathcal{D}_{\square}^{-}(X, \Lambda)$ are complete, then $M \otimes_{\Lambda}^{\square} N$ is complete.
(viii) For $M \in \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ and $N \in \mathcal{D}_{\square}(X, \Lambda)$ we have $\underline{\operatorname{Hom}}_{\mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)}(M, N) \in \mathcal{D}_{\square}(X, \Lambda)$.
(ix) The natural functor $\mathcal{D}_{\square}(X, \Lambda) \rightarrow \operatorname{Mod}_{\Lambda}\left(\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)\right)$ is an equivalence.

In [9, Defintion VII.1.17], $\mathcal{D}_{\square}(X, \Lambda)$ is defined as $\Lambda$-modules in $\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)$ for any solid $\mathbb{Z}_{\ell^{-}}$ algebra $\Lambda$. If $\Lambda$ is an adic and profinite $\mathbb{Z}_{\ell}$-algebra, then Proposition 4.2 shows that the neat characterization Definition 4.1 holds.

[^12]Proof. We first assume that $\Lambda=\mathbb{Z}_{\ell}$. We first note that our (a priori different) definition Definition 4.1 agrees with [9, Definition VII.1.10]. Indeed, if $M \in \mathcal{D}\left(X_{\text {qproet }}, \mathbb{Z}_{\ell}\right)$ has solid cohomology objects in the sense of [9, Definition VII.1.1] then by [9, VII.1.12] $M$ is solid in the sense of Definition 4.1. Moreover, the full subcategory $\mathcal{C} \subseteq \mathcal{D}\left(X_{\text {qproet }}, \mathbb{Z}_{\ell}\right)$ defined in [9, Definition VII.1.10] is stable under all colimits and limits in $\mathcal{D}\left(X_{\text {qproet }}, \mathbb{Z}_{\ell}\right)$ (by [9, Theorem VII.1.3]) and contains $\mathbb{Z}_{\ell}[U]$ for any basic quasi-pro-étale $U \rightarrow X$. Let $N \in \mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)$. Then $N=\underset{\rightarrow i}{\lim } \Lambda\left[U_{i}\right]$ is a colimit in $\mathcal{D}\left(X, \mathbb{Z}_{\ell}\right)$ for basic quasi-pro-étale morphisms $U_{i} \rightarrow X$ with $U_{i}$ $w$-contractible, as these form a basis of $X_{\text {qproet }}$. This implies that $N$ is a retract of $\lim _{\rightarrow i} \Lambda_{\square}\left[U_{i}\right]$, and thus contained in $\mathcal{C}$. This finishes the claim that $\mathcal{C}=\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)$. Now, the assertions follow from [9, Theorem VII.1.3], [9, Proposition VII.1.13] and [9, Proposition VII.1.14], [9, Proposition VII.1.8], [9, Proposition VII.1.11] and [15, Proposition 2.8] (which does not use the assumption $\ell \neq p)$. This finishes the case that $\Lambda=\mathbb{Z}_{\ell}$.

Let now $\Lambda$ be a general adic and profinite $\mathbb{Z}_{\ell^{-}}$-algebra. We claim that the natural map

$$
\mathbb{Z}_{\ell, \square}[U] \otimes^{\square} \Lambda \rightarrow \Lambda_{\square}[U]
$$

is an isomorphism for any basic quasi-pro-étale $U \rightarrow X$. Now, $\Lambda=U \rightarrow X$ is represented (as a sheaf on $X_{\text {qproet }}$ ) by a basic quasi-pro-étale morphism over $X$ because $\Lambda$ is adic and profinite. In particular, we see that $\Lambda$ (being solid) is a retract of $\mathbb{Z}_{\ell, \square}[\Lambda]$. We know that

$$
\mathbb{Z}_{\ell, \square}[U] \otimes \mathbb{Z}_{\ell, \square}[\Lambda] \cong \mathbb{Z}_{\ell, \square}\left[U \times_{X} \Lambda\right],
$$

which implies the claim by the passage to a retract (and it implies that $\Lambda_{\square}[U]$ is static). From here it is formal that an object $C \in \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ is solid if and only if its underlying object $C_{\mid \mathbb{Z}_{\ell}} \in \mathcal{D}\left(X_{\text {qproet }}, \mathbb{Z}_{\ell}\right)$ is solid. Indeed, if $C$ is solid, then by the same argument as before $C$ is a retract of a colimit of $\Lambda_{\square}[U]$ 's with $U \rightarrow X$ basic quasi-pro-étale, and each of the $\Lambda_{\square}[U]$ is solid. Conversely, if $C_{\mid \mathbb{Z}_{\ell}}$ is solid, then

$$
\underline{\operatorname{Hom}}_{\Lambda}\left(\Lambda_{\square}[U], C\right) \cong \underline{\operatorname{Hom}}_{\Lambda}\left(\Lambda \otimes \mathbb{Z}_{\ell, \square}[U], C\right) \cong \underline{\operatorname{Hom}}_{\mathbb{Z}_{\ell}}\left(\mathbb{Z}_{\ell, \square}[U], C_{\mid \mathbb{Z}_{\ell}}\right) \cong \Gamma(U, C)
$$

as desired.
We can conclude that $\mathcal{D}_{\square}(X, \Lambda) \subseteq \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ is stable under all colimits and limits, and that it contains all (static) sheaves of $\Lambda$-modules, which are pulled back from $X_{\text {et }}$. Similarly, the claim on the $t$-structure on $\mathcal{D}_{\square}(X, \Lambda)$ follows as well as the existence of the desired left adjoint $(-)_{\square}$. To get the symmetric monoidal structure $-\otimes_{\Lambda}^{\square}-$ it suffices to show that if $C, D \in \mathcal{D}_{\square}(X, \Lambda)$ then $\underline{\operatorname{Hom}}_{\mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)}(C, D)$ is contained in $\mathcal{D}_{\square}(X, \Lambda)$. By stability of $\mathcal{D}_{\square}(X, \Lambda)$ under limits it suffices to handle the case that $C=\Lambda \otimes_{\mathbb{Z}_{\ell}} C^{\prime}$ for some $C^{\prime} \in \mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)$. Then it follows from the same assertion for $\Lambda=\mathbb{Z}_{\ell}$ (implicitly proven in [9, Proposition VII.1.14]).

Assume that $M, N \in \mathcal{D}_{\square}^{-}(X, \Lambda)$ are complete. From the case $\Lambda=\mathbb{Z}_{\ell}$ we can conclude that for each $n \geq 0$ the tensor product $M \otimes_{\mathbb{Z}_{\ell}} \Lambda^{\otimes n} \otimes_{\mathbb{Z}_{\ell}} N$ is $\ell$-adically complete. This implies that $M \otimes_{\Lambda}^{\square} N$ is $\ell$-adically complete as the $\ell$-adic completion commutes with uniformly rightbounded geometric realizations. Using a similar argument for preservation of completedness under geometric realizations, we can reduce first to the case that $\Lambda=\mathbb{F}_{\ell}\left[\left[x_{1}, \ldots, x_{r}\right]\right]$ for some $r \geq 0$, and then to the case that $r=1$. For $\Lambda=\mathbb{F}_{\ell}[[x]]$ the same argument as in [15, Proposition 2.8] applies.

The final claim $\mathcal{D}_{\square}(X, \Lambda) \cong \operatorname{Mod}_{\Lambda}\left(\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)\right)$ follows from the observation that both embed fully faithfully into $\mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ with generators $\mathbb{Z}_{\ell, \square}[U] \otimes^{\square} \Lambda \cong \Lambda_{\square}[U]$. Now, we can conclude that $X \mapsto \mathcal{D}_{\square}(X, \Lambda)$ is a $v$-sheaf of $\infty$-categories on spatial diamonds, e.g., using [12, Corollary 5.2.2.37].

We now turn to the better behaved subcategory of $\omega_{1}$-solid sheaves, which was introduced (for $\ell \neq p$ ) in [15]. We note that the compact generators of $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ are much simpler
than the ones of $\mathcal{D}_{\square}(X, \Lambda)$. In particular, $\Lambda \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is compact, while in general $\Lambda \in \mathcal{D}_{\square}(X, \Lambda)$ is not.

Proposition 4.3. Let $X$ be an $\ell$-bounded spatial diamond.
(i) The category $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is stable under colimits and countable limits in $\mathcal{D}_{\square}(X, \Lambda)$ and contains all étale sheaves.
(ii) $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is compactly generated and a collection of compact generators is given by $\Lambda_{\square}[U]$ for sequential limits $U=\varliminf_{n} U_{n}$ with all $U_{n} \rightarrow X$ being étale, quasicompact and separated.
(iii) The t-structure on $\mathcal{D}_{\square}(X, \Lambda)$ restricts to at-structure on $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
(iv) The tensor product $-\otimes_{\Lambda}^{\square}-$ in $\mathcal{D}_{\square}(X, \Lambda)$ restricts to a closed symmetric monoidal structure on $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
(v) If $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is $\omega_{1}$-compact, and $N \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ compact, then $\underline{\operatorname{Hom}}_{\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}}(M, N)=$ $\underline{\operatorname{Hom}}_{\mathcal{D} \square(X, \Lambda)}(M, N)$.
(vi) $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}} \cong \operatorname{Mod}_{\Lambda}\left(\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)_{\omega_{1}}\right)$.

Proof. If $\Lambda=\mathbb{Z}_{\ell}$ then this - except for the claim on internal Hom's - is proven in [15, Proposition 2.5] (note that the proof of that result uses nowhere that $\ell \neq p$ ). Thus, let $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ be $\omega_{1}$-compact, and let $N \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ be compact. Then $M$ is a countable colimit of compact objects, and by stability of $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ under countable limits we may therefore assume that $M=\Lambda_{\square}[U]$ is compact. As $N$ is compact, we may reduce to the case that it is a countable inverse limit of étale sheaves, and thus we may assume that $N$ is an étale sheaf. In this case, one checks directly that $\underline{\operatorname{Hom}_{\mathcal{D}}^{\square}(X, \Lambda)},(M, N)$ is $\omega_{1}$-solid. This finishes the case $\Lambda=\mathbb{Z}_{\ell}$.

Back in the case that $\Lambda$ is a static adic and profinite $\mathbb{Z}_{\ell}$-algebra we observe that $\Lambda \in$ $\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)_{\omega_{1}}$ and that an object $C \in \mathcal{D}_{\square}(X, \Lambda)$ is $\omega_{1}$-solid if and only if $C_{\mathbb{Z}_{\ell}} \in \mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)$ is $\omega_{1}$-solid. Moreover, $\Lambda_{\square}[U]=\mathbb{Z}_{\ell, \square}[U] \otimes_{\mathbb{Z}_{\ell, \square}} \Lambda$ by the proof of Proposition 4.2. It follows that $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is the same as the category of $\Lambda$-modules in $\mathcal{D}_{\square}\left(X, \mathbb{Z}_{\ell}\right)_{\omega_{1}}$ by Proposition 4.2. All assertions now follow from the case $\Lambda=\mathbb{Z}_{\ell}$.

We note the following stability of $\left(\omega_{1}-\right)$ solid sheaves under pullback and pushforward along maps of spatial diamonds.

Lemma 4.4. Let $f: X^{\prime} \rightarrow X$ be a morphism spatial diamonds, with associated morphisms $f_{\text {qproet }}: X_{\text {qproet }}^{\prime} \rightarrow X_{\text {qproet }}, f_{v}: X_{v}^{\prime} \rightarrow X_{v}$ on sites.
(i) $f_{\text {qproet }}^{*}: \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right) \rightarrow \mathcal{D}\left(X_{\text {qproet }}^{\prime}, \Lambda\right), f_{v}^{*}: \mathcal{D}\left(X_{v}, \Lambda\right) \rightarrow \mathcal{D}\left(X_{v}, \Lambda\right)$ restrict to the same $t$-exact, symmetric monoidal functor

$$
f^{*}: \mathcal{D}_{\square}(X, \Lambda) \rightarrow \mathcal{D}_{\square}\left(X^{\prime}, \Lambda\right)
$$

which preserves $\omega_{1}$-solid objects (if $X^{\prime}, X$ are $\ell$-bounded).
(ii) If $f$ is quasi-pro-étale, then $f_{\text {qproet,* }}: \mathcal{D}\left(X_{\text {qproet }}^{\prime}, \Lambda\right) \rightarrow \mathcal{D}(X, \Lambda)$ restricts to a colimitpreserving functor

$$
f_{*}: \mathcal{D}_{\square}\left(X^{\prime}, \Lambda\right) \rightarrow \mathcal{D}_{\square}(X, \Lambda),
$$

which preserves $\omega_{1}$-solid sheaves (if $X^{\prime}, X$ are $\ell$-bounded).
(iii) If $\ell \neq p$, then $f_{\text {qproet }, *}: \mathcal{D}\left(X_{\text {qproet }}^{\prime}, \Lambda\right) \rightarrow \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right), f_{v, *}: \mathcal{D}\left(X_{v}^{\prime}, \Lambda\right) \rightarrow \mathcal{D}\left(X_{v}, \Lambda\right)$ restrict to the same colimit-preserving functor

$$
f_{v, *}: \mathcal{D}_{\square}\left(X^{\prime}, \Lambda\right) \rightarrow \mathcal{D}_{\square}(X, \Lambda),
$$

which preserves $\omega_{1}$-solid sheaves (if $X^{\prime}, X$ are $\ell$-bounded).
Proof. The first assertion follows from [9, Proposition VII.1.8] (and the proof of [15, Proposition 2.6] for the preservation of $\omega_{1}$-solid sheaves). The third assertion (for $f_{v, *}$ ) is [9, Proposition VII.2.1]. Using [22, Corollary 16.8], [22, Corollary 16.9] and [22, Corollary 16.10] the same argument implies the second assertion and the case $f_{\text {qproet,* }}$ in the third assertion. The preservation of $\omega_{1}$-solid sheaves follows as in [15, Proposition 2.6] (replacing the use of [22, Corollary 16.8.(ii)] by [22, Corollary 16.8.(i)] for the second part).

Corollary 4.5. (i) The functors $X \mapsto \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}, X \mapsto \mathcal{D}_{\square}(X, \Lambda)$ are hypercomplete quasi-pro-étale sheaves on the site of $\ell$-bounded spatial diamonds. 15
(ii) If $\ell \neq p$, then $X \mapsto \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}, X \mapsto \mathcal{D}_{\square}(X, \Lambda)$ are hypercomplete $v$-sheaves on the site of $\ell$-bounded spatial diamonds.

Proof. The case for $\mathcal{D}_{\square}(X, \Lambda)$ is clear by [9, Proposition VII.1.8]. Thanks to Lemma 4.4 the proof of [15, Corollary 2.7] implies both assertions for $\omega_{1}$-solid sheaves.

### 4.2 Overconvergent solid sheaves

As in Section 4.1 we fix a prime $\ell$ and a static adic and profinite $\mathbb{Z}_{\ell}$-algebra $\Lambda$.
We now discuss the definition of overconvergent objects in $\mathcal{D}_{\square}(X, \Lambda)$ following [15, Section 6], although with some differences. Most notably, we don't require that overconvergent sheaves are nuclear (in the sense of [15, Definition 3.1]). Note that if $T$ is a profinite set, then we can consider its quasi-pro-étale site $T_{\text {qproet }}{ }^{16}$, and the full subcategories $\mathcal{D}_{\square}(T, \Lambda)_{\omega_{1}} \subseteq$ $\mathcal{D}_{\square}(T, \Lambda) \subseteq \mathcal{D}\left(T_{\text {qproet }}, \Lambda\right)$ similarly defined as in Definition 4.1. These categories satisfy the same properties as in Proposition 4.2 and Proposition 4.3 (this can be checked, e.g., by taking the product of $T$ with $\operatorname{Spa}\left(C, \mathcal{O}_{C}\right)$ for $C$ a perfectoid algebraically closed field in characteristic $p$ ). In particular, we can call a morphism $S \rightarrow T$ of profinite sets "basic quasi-pro-étale" if $S=\lim _{i} S_{i} \rightarrow T$ with $S_{i} \rightarrow T$ a local isomorphism for $S_{i}$ a profinite set.

We recall that if $X$ is a spatial diamond, then there exists a natural morphism $\pi=$ $\pi_{X}: X_{\text {qproet }} \rightarrow \pi_{0}(X)_{\text {qproet }}$ of sites such that $\pi^{-1}(T)=T \times_{\pi_{0}(X)} X$ ([17, Definition 5.5]).

Lemma 4.6. Let $X$ be a spatial diamond with morphism of sites $\pi: X_{\text {qproet }} \rightarrow \pi_{0}(X)_{\mathrm{qproet}}$.
(i) The functor $\pi^{-1}: \overline{\pi_{0}(X)_{\text {qproet }}} \rightarrow \overline{X_{\text {qproet }}}$ commutes with limits, and $\pi_{*} \circ \pi^{-1}=\operatorname{Id}_{\pi_{0}(X)_{\text {qproet }}}$. In fact, if $V \in X_{\text {qproet }}$ is qcqs, then $\pi^{-1}(\mathcal{F})(V)=\mathcal{F}\left(\pi_{0}(V)\right)$ for any sheaf of sets $\mathcal{F}$ on $\pi_{0}(X)$.
(ii) The functor $\pi^{-1}: \overline{\pi_{0}(X)_{\text {qproet }}} \rightarrow \overline{X_{\text {qproet }}}$ admits a left adjoint $\pi_{\natural}: \overline{X_{\text {qproet }}} \rightarrow \overline{\pi_{0}(X)_{\text {qproet }}}$ such that $\pi_{\natural}(V \rightarrow X)=\left(\pi_{0}(V) \rightarrow \pi_{0}(X)\right)$ if $V \in X_{\text {qproet }}$ is qcqs.

Proof. The first assertion is [17, Lemma 5.7], and the second is implied by [17, Lemma 5.4].

In the strictly totally disconnected case stronger properties hold true (generalizing the results in [15, Lemma 6.7]).

[^13]Lemma 4.7. Let $X$ be a strictly totally disconnected perfectoid space.
(i) If $X$ is strictly totally disconnected, then $\pi^{*}: \mathcal{D}\left(\pi_{0}(X)_{\text {qproet }}, \Lambda\right) \rightarrow \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ is fully faithful, and preserves ( $\omega_{1^{-}}$)solid sheaves and all colimits and limits. Its restriction, $\mathcal{D}_{\square}\left(\pi_{0}(X)_{\text {qproet }}, \Lambda\right) \rightarrow \mathcal{D}_{\square}\left(X_{\text {qproet }}, \Lambda\right)$ is symmetric monoidal.
(ii) The functor $\pi_{*}: \mathcal{D}\left(X_{\text {qproet }}, \Lambda\right) \rightarrow \mathcal{D}\left(\pi_{0}(X)_{\mathrm{qproet}}, \Lambda\right)$ is $t$-exact and preserves ( $\omega_{1}$ - $)$ solid sheaves.
(iii) The functor $\pi^{*}: \mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right) \rightarrow \mathcal{D}_{\square}(X, \Lambda)$ admits a left adjoint $\pi_{\natural}$, which is symmetric monoidal and preserves $\omega_{1}$-solid sheaves. Moreover, if $U \rightarrow X$ is basic quasi-pro-étale, then $\pi_{\natural}\left(\Lambda_{\square}[U]\right) \cong \Lambda_{\square}\left[\pi_{0}(U)\right]$.
(iv) For $M \in \mathcal{D}_{\square}(X, \Lambda)$ and $N \in \mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right)$ the natural map

$$
\underline{\operatorname{Hom}}_{\mathcal{D}_{\square}(X, \Lambda)}\left(M, \pi^{*} N\right) \rightarrow \pi^{*} \underline{\operatorname{Hom}}_{\mathcal{D} \square}\left(\pi_{0}(X), \Lambda\right)\left(\pi_{\natural} M, N\right)
$$

is an isomorphism. Similarly, with $\mathcal{D}_{\square}(-, \Lambda)$ replaced by $\mathcal{D}_{\square}(-, \Lambda)_{\omega_{1}}$.
Proof. Note that Lemma 4.6 implies that $\Lambda$ is pulled back from $\pi_{0}(X)_{\text {qproet }}$, and thus $\pi^{*}=$ $\pi^{-1}$ (on static sheaves of $\Lambda$-modules). Hence, for fully faithfulness in the first assertion it suffices (by Lemma 4.6) to see that $\pi_{*}$ is exact on static sheaves of $\Lambda$-modules. This can be checked on extremally disconnected profinite sets $T$ over $\pi_{0}(X)$. For such $T$ the pullback $Z:=\pi^{-1}(T)=T \times_{\pi_{0}(X)} X$ is $w$-contractible in $X_{\text {qproet }}$, i.e., each pro-étale cover of $Z$ splits. Preservation of $\left(\omega_{1^{-}}\right)$solid sheaves by $\pi^{*}$ follows from commutation of $\pi^{-1}$ with limits and colimits by checking on the generators of $\mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right)$ resp. $\mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right)_{\omega_{1}}$ provided by Proposition 4.2 resp. Proposition 4.3 (translated to $\pi_{0}(X)$ ).
That $\pi^{*}$ is symmetric monoidal can be checked on generators where it follows by Proposition 4.2 from the fact that

$$
\left(T \rightarrow \pi_{0}(X)\right) \mapsto T \times_{\pi_{0}(X)} X
$$

commutes with fiber products. Here, $T$ is assumed to be profinite.
For (ii) we have to check that $\pi_{*}$ preserves $\left(\omega_{1^{-}}\right)$solid sheaves. Let $N \in \mathcal{D}_{\square}(X, \Lambda)$ and $T \rightarrow \pi_{0}(X)$ profinite, written as $T=\lim _{i} T_{i}$ with $T_{i}$ profinite and $T_{i} \rightarrow \pi_{0}(X)$ a local isomorphism. Set $\Lambda_{\square}[T]:=\lim _{\varliminf_{i}} \Lambda\left[T_{i}\right]$ on $\pi_{0}(X)_{\text {qproet. }}$. Note that $\pi^{*}\left(\Lambda_{\square}[T]\right)=\Lambda_{\square}\left[\pi^{-1}(T)\right]$ as $\pi^{-1}$ preserves limits. Then we can conclude

$$
\begin{aligned}
& \underline{\operatorname{Hom}}\left(\Lambda_{\square}[T], \pi_{*} N\right) \\
= & \left.\pi_{*}\left(\underline{\operatorname{Hom}}\left(\Lambda_{\square}\left[\pi^{-1} T\right]\right), N\right)\right) \\
= & \pi_{*}\left(\underline{\operatorname{Hom}}\left(\Lambda\left[\pi^{-1} T\right], N\right)\right) \\
= & \pi_{*}\left(\underline{\operatorname{Hom}}\left(\pi^{-1}(\Lambda[T]), N\right)\right) \\
= & \underline{\operatorname{Hom}}\left(\Lambda[T], \pi_{*}(N)\right)
\end{aligned}
$$

using that $N$ is solid and the formal identity $\pi^{-1}(\Lambda[T]) \cong \Lambda\left[\pi^{-1} T\right]$. For for any cofiltered limit of maps $T_{j} \rightarrow \pi_{0}(X)$ with $T_{j}$ profinite we have
as $\pi^{-1}$ preserves limits. From here it follows that $\pi_{*}$ preserves $\omega_{1}$-solid objects.
The existence of $\pi_{\natural}$ follows as $\pi^{-1}$ preserves limits (or from [17, Lemma 5.4]). It follows formally from the formula $\pi^{-1} \mathcal{F}(V)=\mathcal{F}\left(\pi_{0}(V)\right)$ that $\pi_{\natural}\left(\Lambda_{\square}[U]\right)=\Lambda_{\square}\left[\pi_{0}(U)\right]$ for $U \rightarrow X$ basic quasi-pro-étale. Using [15, Lemma 5.8] we can conclude that $\pi_{0}\left(U \times{ }_{X} V\right)=\pi_{0}(U) \times \pi_{0}(X)$
$\pi_{0}(V)$ for $U, V$ basic quasi-pro-étale over $X$. This implies that $\pi_{\natural}$ is symmetric monoidal by Proposition 4.2, We need to see that $\pi_{\natural}$ preserves $\omega_{1}$-solid sheaves. The explicit formula for $\pi_{\natural} \Lambda_{\square}[U]$ and the fact that $\pi_{0}\left(\lim _{i} U_{i}\right) \cong \varliminf_{\varliminf_{i}} \pi_{0}\left(U_{i}\right)$ for étale, separated maps $U_{i} \rightarrow X$ in a cofiltered system $\left\{U_{i}\right\}_{i}$ shows that it suffices to show that $\pi_{\natural}$ maps $\Lambda_{\square}[U]$ to an $\omega_{1}$-solid sheaf of $U \rightarrow X$ is étale quasi-compact and separated. This proven in Lemma 4.8. The assertion (iv) follows formally from symmetric monoidality of $\pi_{\natural}$. In fact, the left hand side is right adjoint as a functor in $N$ to $\pi_{\natural}\left(M \otimes_{\Lambda}^{\square}(-)\right)$ while the right hand side is right adjoint to $\pi_{\natural}(M) \otimes_{\Lambda} \pi_{\natural}(-)$.

The proof of Lemma 4.7 made use of following result about $\omega_{1}$-solidness of $\pi_{\natural} \Lambda[U]$. Be aware that even if $U \hookrightarrow X$ is an open immersion, the map $\pi_{0}(U) \rightarrow \pi_{0}(X)$ is in general quite far from being an open immersion itself, which makes the $\omega_{1}$-solidness of $\pi_{\natural} \Lambda[U]=\Lambda_{\square}\left[\pi_{0}(U)\right]$ more subtle than one might think at first.

Lemma 4.8. Let $j: U \rightarrow X$ be a quasi-compact, separated, étale morphism to a strictly totally disconnected space $X$. Then $\pi_{\natural} \Lambda[U]=\Lambda_{\square}\left[\pi_{0}(U)\right] \in \mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right)$ is $\omega_{1}$-solid.

Proof. Write $X=\operatorname{Spa}\left(A, A^{+}\right)$. As $X$ is strictly totally disconnected, each point $u \in U$ has a quasi-compact open neighborhood $V_{u}$, such that the map $V_{u} \rightarrow X$ is an open immersion. Indeed, this follows from the definition of an étale morphism ([22, Definition 6.2]) using that each finite étale map onto some quasi-compact open in $X$ will split. Hence, without loss of generality $j: U \rightarrow X$ can be assumed to be a quasi-compact open immersion (using that $\omega_{1}$-solid sheaves are stable under finite colimits). By [22, Lemma 7.6] $U$ is the intersection of subsets $\{|f| \leq 1\}$ for varying $f \in A$. As $U \subseteq X$ is quasi-compact open, $X \backslash U$ is quasi-compact in the constructible topology. As any $\{|f| \leq 1\}$ for $f \in A$ is closed in the constructible topology, this implies that $U$ is the finite intersection of subsets $U_{1}=\left\{\left|f_{1}\right| \leq 1\right\}, \ldots, U_{r}=$ $\left\{\left|f_{r}\right| \leq 1\right\}$ with $f_{1}, \ldots, f_{r} \in A$. By [17, Lemma 5.8] $\pi_{0}(U)=\pi_{0}\left(U_{1}\right) \times_{\pi_{0}(X)} \cdots \times_{\pi_{0}(X)} \pi_{0}\left(U_{r}\right)$ and thus Proposition 4.2, Proposition 4.3 imply that we may assume that $U=U_{1}=\{|f| \leq 1\}$ for $f:=f_{1}$. Let $\pi \in A$ be a pseudo-uniformizer and set

$$
W_{n}:=\left\{|f|<\left|\pi^{-1 / p^{n}}\right|\right\}
$$

for $n \in \mathbb{N}$. Each $W_{n}$ is a spectral space, being closed in the constructible topology ( $[27,0902]$ ), and hence $\pi_{0}\left(W_{n}\right)$ is a profinite set. We claim:

1) $\pi_{0}(U), \pi_{0}\left(W_{n}\right), n \in \mathbb{N}$, are subspaces of $\pi_{0}(X)$,
2) $\bigcap_{n \in \mathbb{N}} W_{n}$ is the closure of $U$,
3) $\pi_{0}(U)=\bigcap_{n \in \mathbb{N}} \pi_{0}\left(W_{n}\right)$,
4) for each $n \in \mathbb{N}$ the subspace $\pi_{0}(U)$ lies in the interior of $\pi_{0}\left(W_{n}\right) \subseteq \pi_{0}(X)$.

The claims imply that $\Lambda_{\square}\left[\pi_{0}(U)\right]$ is $\omega_{1}$-solid. Indeed, we may inductively construct quasicompact open subsets $S_{n} \subseteq \pi_{0}(X)$ with $\pi_{0}(U) \subseteq S_{n} \subseteq \pi_{0}\left(W_{n}\right)$, and $S_{n+1} \rightarrow S_{n}$. Then $\Lambda_{\square}\left[\pi_{0}(U)\right]=\lim _{n \in \mathbb{N}} \Lambda\left[S_{n}\right]$ is $\omega_{1}$-solid by Proposition 4.3.
To show 1) it suffices (because the $\pi_{0}$ are profinite and the maps continuous) to show that $\pi_{0}(U) \rightarrow \pi_{0}(X), \pi_{0}\left(W_{n}\right) \rightarrow \pi_{0}(X)$ are injective. Now, $U$ is open, and thus stable under generalization while the $W_{n}$ are closed, hence stable under specializations. This implies that if $Z \subseteq X$ is a connected component, then $Z \cap U, Z \cap W_{n}$ are connected (if non-empty) because $Z$ is a linearly ordered chain of points. This in turn implies injectivity.

As each $W_{n}$ is closed, and $U \subseteq W_{n}$ we see $\bar{U} \subseteq \bigcap_{n \in \mathbb{N}} W_{n}$. The reverse implication may be checked after intersection with each connected component $Z$ of $X$ (because $\bar{U}$ is the set of specializations of $U$ by [27, 0903]). If $Z \cap U \neq$, then $\bar{U} \cap Z=Z$ and similarly for the intersection with the $W_{n}$ 's. Let $z$ be the generic point of $Z$ and assume that $Z \cap \bar{U}=\varnothing$ or equivalently $Z \cap U=\varnothing$. Thus, $|f(z)|>1$ and as $z$ is a rank 1 valuation this implies that $z \notin\left\{|f| \leq\left|\pi^{-1 / p^{n}}\right|\right\}$ for some $n \in \mathbb{N}$. This implies that $Z \cap W_{n}=\varnothing$ as the set $\left\{|f| \leq\left|\pi^{-1 / p^{n}}\right|\right\}$ is stable under generalizations. Thus $Z \cap \bigcap_{n \in \mathbb{N}} W_{n}=\varnothing$ as well.

Assertion 3) follows from 2) because $\pi_{0}(U)=\pi_{0}(\bar{U})=\bigcap_{n \in \mathbb{N}} \pi_{0}\left(W_{n}\right)$.
We are left with assertion 4). Fix a connected component $Z \subseteq X$ with $Z \cap U \neq \varnothing$, i.e., the point $z \in \pi_{0}(X)$ defined by $Z$ lies in $\pi_{0}(U)$. We can write $Z=\bigcap_{i \in I} f^{-1}\left(T_{i}\right)$ with $T_{i} \subseteq \pi_{0}(X)$ running through the quasi-compact open neighborhoods of $z$ and $f:|X| \rightarrow \pi_{0}(X)$ the natural quotient map. Note that the $f^{-1}\left(T_{i}\right)$ are closed and open in $X$. Moreover, let us fix $n \in \mathbb{N}$. We know that $Z \subseteq W_{n}$ as $\bar{U} \subseteq W_{n}$ and the generic point of $Z$ lies in $U$. Furthermore, $X \backslash W_{n}=\left\{\left|\pi^{-1 / p^{n}}\right| \leq|f|\right\}$ is a rational open subset of $X$. In particular, $X \backslash W_{n}$ is compact in the constructible topology. As $X \backslash W_{n} \cap \bigcap_{i \in I} f^{-1}\left(T_{i}\right)=X \backslash W_{n} \cap Z=\varnothing$, we can conclude that there exists some $i_{0}$ such that $X \backslash W_{n} \cap f^{-1}\left(T_{i_{0}}\right)=\varnothing$, i.e., $f^{-1}\left(T_{i_{0}}\right) \subseteq W_{n}$. This implies that $z \in T_{i_{0}} \subseteq \pi_{0}\left(W_{n}\right)$. In particular, $z$ lies in the interior of $\pi_{0}\left(W_{n}\right)$. This finishes the proof.

With the previous results at hand, we can now introduce a well-behaved (see Lemma 4.10) notion of overconvergent sheaves on general spatial diamonds:

Definition 4.9. (a) Let $X$ be a strictly totally disconnected space with morphism of sites $\pi: X_{\text {qproet }} \rightarrow \pi_{0}(X)_{\text {qproet }}$. An object $M \in \mathcal{D}_{\square}(X, \Lambda)$ is called overconvergent if $M \cong$ $\pi^{*} N$ for some $N \in \mathcal{D}_{\square}\left(\pi_{0}(X)_{\text {proet }}, \Lambda\right)$.
(b) Let $X$ be a spatial diamond. An object $M \in \mathcal{D}_{\square}(X, \Lambda)$ is called overconvergent if $f^{*} M$ is overconvergent for any quasi-pro-étale morphism $f: X^{\prime} \rightarrow X$ with $X^{\prime}$ strictly totally disconnected.
We let $\mathcal{D}_{\square}(X, \Lambda)^{\mathrm{oc}} \subseteq \mathcal{D}_{\square}(X, \Lambda)$ (resp. $\left.\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\text {oc }} \subseteq \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}\right)$ denote the full subcategories spanned by the overconvergent objects (resp. the overconvergent and $\omega_{1}$-solid objects).

Lemma 4.10. Let $f: X^{\prime} \rightarrow X$ be a morphism of strictly totally disconnected spaces.
(i) If $M \in \mathcal{D}_{\square}(X, \Lambda)$ is overconvergent, then so is $f^{*} M$.
(ii) Conversely, if $f$ is a quasi-pro-étale cover and $f^{*} M$ is overconvergent, then so is $M$.

Proof. The first claim follows by naturality in $X$ of the morphism of sites $\pi: X_{\text {qproet }} \rightarrow$ $\pi_{0}(X)_{\text {qproet }}$. For the converse we note that (using [17, Lemma 5.8]) it suffices to see that $X \mapsto \mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right)$ is a hypercomplete quasi-pro-étale sheaf on strictly totally disconnected spaces. Now Corollary 4.5) implies that $T \mapsto \mathcal{D}_{\square}(T, \Lambda)$ is a quasi-pro-étale sheaf on profinite sets, in other words a condensed sheaf of $\infty$-categories on $*_{\text {qproet }}$. As $X \mapsto \pi_{0}(X)$ sends quasi-pro-étle covers, even $v$-covers, to quasi-pro-étale covers the claim follows.

The next result summarizes the very good properties of the full subcategory of overconvergent objects.

Lemma 4.11. Let $X$ be an $\ell$-bounded spatial diamond.
(i) Overconvergent objects satisfy quasi-pro-étale hyperdescent on the big site of $\ell$-bounded spatial diamonds.
(ii) The inclusion $\mathcal{D}_{\square}(X, \Lambda)^{\mathrm{oc}} \subseteq \mathcal{D}_{\square}(X, \Lambda)$ admits a symmetric monoidal left adjoint $M \mapsto$ $M_{\mathrm{oc}}$ and a right adjoint $M \mapsto M^{\mathrm{oc}}$. In particular, $\mathcal{D}_{\square}(X, \Lambda)^{\text {oc }}$ is stable under all colimits and limits in $\mathcal{D}_{\square}(X, \Lambda)$.
(iii) The inclusion $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\mathrm{oc}} \subseteq \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ admits a symmetric monoidal left adjoint $M \mapsto$ $M_{\mathrm{oc}}$ and a right adjoint $M \mapsto M^{\mathrm{oc}}$. In particular, $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\mathrm{oc}}$ is stable under all colimits and limits in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
(iv) If $M, N \in \mathcal{D}_{\square}(X, \Lambda)$ and $N$ is overconvergent, then $\underline{\operatorname{Hom}}_{\mathcal{D}_{\square}(X, \Lambda)}(M, N)$ is overconvergent and naturally isomorphic to $\underline{\operatorname{Hom}}_{\mathcal{D}_{\square}(X, \Lambda)}\left(M_{\mathrm{oc}}, N\right)$. Similarly, with $\mathcal{D}_{\square}(X, \Lambda)$ replaced by $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
(v) If $U=\lim _{\overparen{i} \in} U_{i} \rightarrow X$ with $U_{i} \rightarrow X$ étale, quasi-compact and separated, then the natural map $\Lambda_{\square}[U] \rightarrow \Lambda_{\square}\left[\bar{U}^{/ X}\right]$ induces an isomorphism $\Lambda_{\square}[U]_{\mathrm{oc}} \cong \Lambda_{\square}\left[\bar{U}^{/ X}\right]$. In particular, the functor $\mathcal{D}_{\square}(X, \Lambda) \rightarrow \mathcal{D}_{\square}(X, \Lambda), M \mapsto M_{\mathrm{oc}}$ preserves compact objects.
(vi) The functor $\mathcal{D}_{\square}(X, \Lambda) \rightarrow \mathcal{D}_{\square}(X, \Lambda), M \mapsto M^{\text {oc }}$ preserves colimits and right bounded objects.

Proof. Assertion (i) follows from Lemma 4.10 and Corollary 4.5. For assertion (ii) it suffices by the adjoint functor theorem to show that $\mathcal{D}_{\square}(X, \Lambda)^{\text {oc }}$ is stable under all colimits and limits in $\mathcal{D}_{\square}(X, \Lambda)$. By Proposition 4.2 and (i) we may then assume that $X$ is a strictly totally disconnected perfectoid space. Then the assertion follows from Lemma 4.7. We note that in this case we have $M_{\mathrm{oc}}=\pi^{*} \pi_{\natural}(M)$ and $M^{\mathrm{oc}}=\pi^{*} \pi_{*}(M)$ for $M \in \mathcal{D}_{\square}(X, \Lambda)$. As in Lemma 4.7 symmetric monoidality of $(-)_{\text {oc }}$ implies assertion (iv).

For assertion (iii) we may again reduce to the case that $X$ is strictly totally disconnected. Indeed, for the existence of $(-)^{\circ c}$ this follows by stability of overconvergent sheaves under colimits (which can be checked on a strictly totally disconnected cover), and the existence of $(-)_{\text {oc }}$ can be descended from the strictly totally disconnected case as there $(-)_{\text {oc }}$ commutes with pullback as will be proved in Lemma 4.12. Then the assertion follows again from Lemma 4.7 as $\pi_{\natural}, \pi^{*}, \pi_{*}$ all preserve $\omega_{1}$-solid sheaves.

Let us note that if $U \rightarrow X$ is quasi-pro-étale, then $\bar{U}^{/ X} \rightarrow X=\bar{X}^{/ X}$ is quasi-pro-étale by [22, Corollary 18.8.(vii)]. If $X$ is strictly totally disconnected and $U$ qcqs, then the natural map $\bar{U}^{/ X} \rightarrow \pi_{0}(U) \times \pi_{0}(X) X$ is an isomorphism as it is qcqs and bijective if $X$ is connected. To check (v) we may (by Lemma 4.12 below) reduce to the case that $X$ is strictly totally disconnected. Indeed, in this case $\Lambda_{\square}\left[\bar{U}^{/ X}\right] \cong \Lambda_{\square}\left[\pi_{0}(U) \times_{\pi_{0}(X)} X\right]$ is overconvergent, which implies the same for a general $\ell$-bounded spatial diamond. This produces the natural map $\Lambda_{\square}[U]_{\mathrm{oc}} \rightarrow \Lambda\left[\bar{U}^{/ X}\right]$, which then can be checked to be an isomorphism again under the assumption that $X$ is strictly totally disconnected. Here, it follows from Lemma 4.7 as $(-)_{\mathrm{oc}} \cong \pi^{*} \pi_{\natural}(-)$. It follows from the explicit description that the functor $(-)_{\text {oc }}$ maps compact objects in $\mathcal{D}_{\square}(X, \Lambda)$ (resp. $\left.\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}\right)$ to compact objects in $\mathcal{D}_{\square}(X, \Lambda)$ (resp. $\left.\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}\right)$. This implies that the inclusion of overconvergent into solid sheaves preserves compact objects, which implies that the right adjoint $(-)^{\text {oc }}$ (for solid or $\omega_{1}$-solid objects) preserves colimits. Now pick a right bounded object $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$. As $X$ is $\ell$-bounded can conclude that there exists some $a \in \mathbb{Z}$ such that for each compact object $N \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$, which is supported to the right of $a$ and has terms given by $\Lambda_{\square}[U]$ for $U=\underset{{ }_{n}}{\lim } U_{n} \rightarrow X$ with $U_{n}$ étale quasi-compact and separated, we have $\operatorname{Hom}(N, M)=0$. This implies

$$
\operatorname{Hom}\left(N_{\mathrm{oc}}, M\right)=\operatorname{Hom}\left(N_{\mathrm{oc}}, M^{\mathrm{oc}}\right)=0
$$

for any such $N$ by the explict description of $(-)_{\mathrm{oc}}$ in (v). This shows that $M^{\mathrm{oc}}$ is again right bounded.

The left adjoint $(-)_{\text {oc }}$ commutes with base change (while the right adjoint $(-)^{\text {oc }}$ does not, even on strictly totally disconnected spaces).

Lemma 4.12. Let $f: X^{\prime} \rightarrow X$ be a morphism of $\ell$-bounded spatial diamonds.
(i) For $M \in \mathcal{D}_{\square}(X, \Lambda)$ the natural map

$$
\left(f^{*} M\right)_{\mathrm{oc}} \rightarrow f^{*}\left(M_{\mathrm{oc}}\right)
$$

is an isomorphism.
(ii) If $f$ is quasi-pro-étale, then $f_{*}\left(\mathcal{D}_{\square}\left(X^{\prime}, \Lambda\right)^{\mathrm{oc}}\right) \subseteq \mathcal{D}_{\square}(X, \Lambda)^{\mathrm{oc}}$ and $f_{*}\left(\mathcal{D}_{\square}\left(X^{\prime}, \Lambda\right)_{\omega_{1}}^{\mathrm{oc}}\right) \subseteq$ $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\text {oc }}$

Proof. By quasi-pro-étale hyperdescent it suffices to treat the case that $X, X^{\prime}$ are strictly totally disconnected. Let $\pi_{X}: X_{\text {qproet }} \rightarrow \pi_{0}(X)_{\text {qproet }}, \pi_{X^{\prime}}: X_{\text {qproet }}^{\prime} \rightarrow \pi_{0}\left(X^{\prime}\right)_{\text {qproet }}$ be the morphisms of sites. It suffices to check that the natural map $\pi_{X^{\prime}, \mathrm{t}} f^{*} M \rightarrow \pi_{0}(f)^{*} \pi_{X, 4}(M)$ is an isomorphism for $M \in \mathcal{D}_{\square}(X, \Lambda)$. Passing to right adjoints, it suffices to see that for $N \in \mathcal{D}_{\square}\left(\pi_{0}\left(X^{\prime}\right), \Lambda\right)$

$$
f_{*} \pi_{X^{\prime}}^{*} N \cong \pi_{X}^{*} \pi_{0}(f)_{*}(N)
$$

via the natural map. This follows from [17, Corollary 5.9]. This formula (together with Lemma (4.4) imply the second assertion as well if $X$ is strictly totally disconnected. The case for general $X$ follows by quasi-pro-étale descent (for which $f_{*}$ satisfies base change by Lemma 4.4).

### 4.3 Nuclear sheaves

In this section we will study nuclear objects in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ and show that they are overconvergent. Critically, we will moreover show in Lemma 4.20 that the full subcategory of nuclear objects is generated under colimits by right bounded complete objects. In order to achieve this, we need a better understanding of the compact objects of $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\mathrm{oc}}$.
Let us recall from [6, Lecture VIII] that a morphism $f: P \rightarrow Q$ in a closed symmetric monoidal $\infty$-category $\mathcal{C}$ with compact unit, is called of trace class if it lies in the image of the natural morphism

$$
\pi_{0}((\underline{\operatorname{Hom}}(P, 1) \otimes Q)(*)) \rightarrow \pi_{0}(\operatorname{Hom}(P, Q)) .
$$

Here, $\otimes, \underline{\operatorname{Hom}}, 1$ refer to the tensor product, internal Hom, unit in $\mathcal{C}$, while $(-)(*):=\operatorname{Hom}(1,-)$. We will use critically the following property of trace class maps: If $P_{0} \rightarrow P_{1} \rightarrow \ldots$ is a sequential diagram with all morphisms $P_{n} \rightarrow P_{n+1}$ of trace class, then for any $Q \in \mathcal{C}$ the natural map
is an isomorphism. Indeed, each witness in $\left(\underline{\operatorname{Hom}}\left(P_{n}, 1\right) \otimes P_{n+1}\right)(*)$ for $P_{n} \rightarrow P_{n+1}$ being trace class yields a factorization

$$
\operatorname{Hom}\left(P_{n+1}, Q\right) \rightarrow\left(\underline{\operatorname{Hom}}\left(P_{n}, 1\right) \otimes Q\right)(*) \rightarrow \operatorname{Hom}\left(P_{n}, Q\right) .
$$

Similarly, if $\ldots \rightarrow P_{1} \rightarrow P_{0}$ is a sequential diagram with all $P_{n+1} \rightarrow P_{n}$ of trace class, then the natural map

$$
\begin{equation*}
\left.\underset{n}{\lim }\left(\underline{\operatorname{Hom}}\left(P_{n}, 1\right) \otimes Q\right)(*)\right) \rightarrow \underset{\vec{n}}{\lim } \operatorname{Hom}\left(P_{n}, Q\right) \tag{4.12.1}
\end{equation*}
$$

is an isomorphism.
Following [6, Definition 8.5] we make the following definition.

Definition 4.13. Let $X$ be an $\ell$-bounded spatial diamond and $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
(a) $M$ is called basic nuclear if $M=\underset{\longrightarrow}{\lim }\left(M_{0} \rightarrow M_{1} \rightarrow \ldots\right)$ with all $M_{n} \rightarrow M_{n+1}$ of trace class.
(b) $M$ is called basic conuclear if $M=\underset{\varliminf}{\lim }\left(\ldots \rightarrow M_{1} \rightarrow M_{0}\right)$ with all $M_{n+1} \rightarrow M_{n}$ of trace class.
(c) $M$ is called nuclear if for all compact objects $P \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$, the map

$$
\left(\underline{\operatorname{Hom}}_{\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}}(P, \Lambda) \otimes_{\Lambda}^{\square} M\right)(*) \rightarrow \operatorname{Hom}_{\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}}(P, M)
$$

is an equivalence.
We let $\mathcal{D}_{\text {nuc }}(X, \Lambda) \subseteq \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ be the full subcategory of nuclear objects.
Remark 4.14. We warn the reader that the abstract notion of nuclearity from [6, Lecture VIII] that we employ here is different from the more subtle "geometric nuclearity" in [15, Definition 3.1] as noted in [15, Warning 3.3]. In Lemma 4.17] below we will prove that the abstract notion of nuclearity singles out the overconvergent nuclear sheaves from [15, Definition 6.8] (thereby providing details to [15, Warning 3.3]).

The basic observation for the overconvergence of the nuclear objects in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is the overconvergence of $\underline{\operatorname{Hom}}_{\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}}\left(\Lambda_{\square}[U], \Lambda\right)$.

Lemma 4.15. Let $X$ be an $\ell$-bounded spatial diamond.
(i) Each trace class map $P \rightarrow Q$ in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ factors as $P \rightarrow P_{\mathrm{oc}} \rightarrow Q$ with $P_{\mathrm{oc}} \rightarrow Q$ of trace class.
(ii) Each nuclear object in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is overconvergent.
(iii) Each basic conuclear object $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is overconvergent and complete. In fact, $M \cong \lim _{n} M_{n}$ with $M_{n}$ étale and compact, and $M_{n+1} \rightarrow M_{n}$ trace class for each $n \geq 0$. If the $M_{n}$ are uniformly bounded, then $M$ is compact.
(iv) Basic conuclear objects are stable under countable limits.

In particular, $\mathcal{D}_{\text {nuc }}(X, \Lambda)_{\omega_{1}}$ is equivalent to the category $\operatorname{Nuc}\left(\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\text {oc }}\right)$ of nuclear objects in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\mathrm{oc}}$.

Proof. By [6, Theorem 8.6] each nuclear object is a colimit of basic nuclear object. By Lemma 4.11 the subcategory of overconvergent objects is stable under limits and colimits. Hence, it suffices to prove the first assertion to show overconvergence. Let $P \rightarrow Q$ be a trace class morphism in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$. As $\Lambda$ is overconvergent, we can conclude from Lemma 4.11 that

$$
\underline{\operatorname{Hom}}(P, \Lambda) \cong \underline{\operatorname{Hom}}\left(P_{\mathrm{oc}}, \Lambda\right)
$$

as $(-)_{\text {oc }}$ is symmetric monoidal. This implies the desired factorization.
In order to see that each basic conuclear object $M=\underset{n \in \mathbb{N}}{\lim _{\overparen{N}}} M_{n}$ is complete it suffices to note that each of the trace class morphisms $M_{n+1} \rightarrow M_{n}$ factors over a compact object ([6, Lemma 8.4]), and that compact objects in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ are complete (Proposition 4.3) as is therefore their limit $M$. To prove the final assertion, we may assume that the $M_{n}$ are compact. Let $I \subseteq \Lambda$ by an ideal of definition. Then by completeness of $M$ we get $M \cong \underset{{ }_{n}}{\underset{\sim}{\underset{n}{2}}} M_{n} / I^{k_{n}}$ for each sequence $k_{n}$ going to $\infty$ if $n \rightarrow \infty$. Moreover, the $M_{n} / I^{k_{n}}$ are compact, while all morphisms
$M_{n+1} / I^{k_{n+1}} \rightarrow M_{n} / I^{k_{n}}$ are still of trace class, at least after reindexing the $n$. Indeed, this follows from the fact that a composition of two cones of trace class morphism is again trace class ([24, Lecture XIII.Proposition 13.13]). Moreover, we may write each $M_{n} / I^{k_{n}}$ as a countable limit of compact, étale sheaves $M_{n, i}$ (this follows from the proof of [15, Proposition 2.5], which works for general $\Lambda$ ). By étaleness and compactness of the $M_{n, i}$ each morphism $M_{n+1} / I^{k n+1} \rightarrow M_{n} / I^{k_{n}}$ arises from a compatible system of morphisms $M_{n+1, i} \rightarrow M_{n, i}$ (up to reindexing). Then $M \cong \underset{n}{\lim _{n}} M_{n, n}$ using that the diagonal $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is cofinal. If the $M_{n, n}$ can be chosen to be uniformly bounded, then it follows from [22, Theorem VII.1.3] that $M$ is compact, even in $\mathcal{D}_{\square}(X, \Lambda)$. Moreover, after potentially reindexing the $n$, the morphisms $M_{n+1, n+1} \rightarrow M_{n, n}$ can be assumed to be of trace class by the same argument as above. With this presentation for basic conuclear objects, the proof of [24, Proposition 13.13], [6, Theorem 8.6] dualizes and shows that basic conuclear objects are stable under countable limits.

In the strictly totally disconnected case, we can be more explicit and identify nuclear sheaves with nuclear modules over the ring of continuous functions:

Lemma 4.16. Let $X$ be a strictly totally disconnected perfectoid space and let $T$ be a profinite set.
(i) The functors $\pi_{*}, \pi^{*}$ induce a symmetric monoidal equivalence $\mathcal{D}_{\mathrm{nuc}}(X, \Lambda) \cong \mathcal{D}_{\mathrm{nuc}}\left(\pi_{0}(X), \Lambda\right)$.
(ii) Let $\alpha: T_{\text {qproet }} \rightarrow *_{\text {qproet }}$ be the natural morphism of sites. Then $\alpha_{*}$ induces an equivalence

$$
\mathcal{D}_{\text {nuc }}\left(T_{\text {qproet }}, \Lambda\right) \cong \mathcal{D}_{\text {nuc }}(C(T, \Lambda))
$$

where $C(T, \Lambda) \cong \alpha_{*}(\Lambda)$ is the nuclear $\Lambda$-algebra of continuous functions $T \rightarrow \Lambda$.
In particular, $\mathcal{D}_{\text {nuc }}(X, \Lambda) \cong \mathcal{D}_{\text {nuc }}(C(X, \Lambda))$.
We recall that here "nuclear" refers to Definition 4.13 (and not [15, Definition 3.1]). However, for $\pi_{0}(X)$ both notions agree (by Lemma4.16] and [15, Lemma 6.7.(iv)] or by [15, Remark 3.10] if $\Lambda=\mathbb{Z}_{\ell}$ ).

Proof. As $\pi^{*}$ is symmetric monoidal, $\pi^{*}$ preserves nuclear objects. Moreover, the unit $M \rightarrow$ $\pi_{*} \pi^{*} M$ is an isomorphism for any $M \in \mathcal{D}_{\text {nuc }}\left(\pi_{0}(X), \Lambda\right)$ by Lemma 4.6. Let $N \in \mathcal{D}_{\text {nuc }}(X, \Lambda)$ be nuclear. We need to see that the counit $\pi^{*} \pi_{*} N \rightarrow N$ is an isomorphism. This follows from Lemma 4.15,

Let us proof the second assertion. Set $A:=C(T, \Lambda)$. As $A$ is a nuclear $\Lambda$-algebra, e.g., by Lemma 2.18.(i), $\mathcal{D}_{\text {nuc }}(A)$ is equivalent to $\operatorname{Mod}_{A}\left(\mathcal{D}_{\text {nuc }}(\Lambda)\right)$, cf. [6, Corollary 8.20]. The existence of a left adjoint $\alpha^{*}: \mathcal{D}_{\square}\left(*_{\text {qproet }}, \Lambda\right)_{\omega_{1}} \rightarrow \mathcal{D}_{\square}(T, \Lambda)_{\omega_{1}}$ to $\alpha_{*}$ follows as in the proof of [15, Lemma 6.7.(iv)]. Similarly, it follows (using Lemma 2.18.(i) that $\alpha^{*}$ preserves nuclearity, and that $\alpha_{*} \alpha^{*} M \rightarrow M$ is an isomorphism for $M \in \mathcal{D}_{\text {nuc }}(A)$. It remains to see that $\alpha^{*}$ is essentially surjective. Here, we a priori have to offer a different proof as in [15, Lemma 6.7.(iv)] as we use a different notion of nuclearity. However, [15, Remark 3.10] shows that both notions agree in this case as $T$ is profinite. Thus, the assertion follows from [15, Lemma 6.7].

We note that $\alpha_{*}: \mathcal{D}_{\square}(T, \Lambda) \rightarrow \operatorname{Mod}_{C(T, \Lambda)} \mathcal{D}_{\square}(\Lambda)$ is in general not an equivalence (if $T$ is not finite), even if $\Lambda=\mathbb{F}_{\ell}$. Indeed, for each $t \in T$ the morphism $\{t\} \rightarrow T$ is pro-étale, and thus $\Lambda[\{t\}] \in \mathcal{D}_{\square}(T, \Lambda)$ is compact. But if $\Lambda$ is discrete, then $\alpha_{*}(\Lambda[\{t\}]) \cong \Lambda$ viewed as a $C(T, \Lambda)$-module via evaluation at $t$. If $t \in T$ is not an open point, then this implies that $\alpha_{*}(\Lambda[\{t\}])$ is not a (discrete) finitely presented $C(T, \Lambda)$-module, and hence not compact in $\operatorname{Mod}_{C(T, \Lambda)} \mathcal{D}_{\square}(\Lambda)$. Nuclear objects satisfy quasi-pro-étale descent:

Lemma 4.17. The functor $X \mapsto \mathcal{D}_{\text {nuc }}(X, \Lambda)$ is a hypercomplete quasi-pro-étale sheaf on the big quasi-pro-étale site of $\ell$-bounded spatial diamonds.

Proof. This is formally implied by Corollary 4.5, Pullback of nuclear objects are nuclear (by symmetric monoidality of pullbacks) and if $f_{\bullet}: X_{\bullet} \rightarrow X$ is a quasi-pro-étale hypercover and $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ with $f_{n}^{*} M$ nuclear for each $n \in \Delta$, then for $P \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ compact we have

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}}(P, M) & =\underbrace{\lim _{n}}_{n \in \Delta} \operatorname{Hom}_{\mathcal{D}_{\square}\left(X_{n}, \Lambda\right)_{\omega_{1}}}\left(f_{n}^{*} P, f_{n}^{*} M\right) \\
& =\underbrace{\lim _{m}}_{n \in \Delta} \operatorname{Hom}_{\mathcal{D}_{\square}\left(X_{n}, \Lambda\right)_{\omega_{1}}}\left(\Lambda, \underline{\left.\operatorname{Hom}_{X_{n}^{\prime}}\left(f_{n}^{*} P, \Lambda\right) \otimes_{\Lambda}^{\square} f_{n}^{*} M\right)}\right. \\
& =\varliminf_{n \in \Delta}^{\lim _{n}} \operatorname{Hom}_{\mathcal{D}_{\square}\left(X_{n}, \Lambda\right)_{\omega_{1}}}\left(\Lambda, f_{n}^{*}\left(\underline{\operatorname{Hom}_{X}}(P, \Lambda) \otimes_{\Lambda}^{\square} M\right)\right) \\
& =\operatorname{Hom}_{\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}}\left(\Lambda, \underline{\operatorname{Hom}}(P, \Lambda) \otimes_{\Lambda}^{\square} M\right)
\end{aligned}
$$

using (in this order) the description of Hom-spaces in limits of categories, nuclearity of all the $f_{n}^{*} M$, the fact that $f_{n}^{*}$ is the restriction for some slice (plus Proposition 4.3 to control the Hom using that $f_{n}^{*} P$ is compact), and again the description of Hom-spaces in limits of categories.

We now analyze complete objects in $\mathcal{D}_{\text {nuc }}(X, \Lambda)$. More precisely, fix an ideal of definition $I \subseteq \Lambda$. In the following, we adhere to the conventions in Section 1.2 (and in particular implicitly fix a finite set of generators of $I$ ). Morever, we use the terminology "complete" instead of " $I$-adically complete", whenever convenient.

Lemma 4.18. Let $X$ be an $\ell$-bounded spatial diamond and $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ with $I$-adic completion $\widehat{M} \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
(i) If $M$ is nuclear, then $\widehat{M}$ is nuclear.
(ii) If $M \in \mathcal{D}_{\text {nuc }}(X, \Lambda)$ is I-adically complete in $\mathcal{D}_{\text {nuc }}(X, \Lambda)$, then the natural map $M \rightarrow \widehat{M}$ is an isomorphism.

We note that a priori completeness in $\mathcal{D}_{\text {nuc }}(X, \Lambda)$ and $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ are different.
Proof. We start with (i). By Lemma 4.17 and Lemma 4.4, specifically the commutation of pullbacks with limits, the claim is quasi-pro-étale local on $X$ and hence we may assume that $X$ is strictly totally disconnected. By Lemma 4.6 and Lemma 4.16 the claim reduces to the case that $X$ is a profinite set (realized as a qcqs perfectoid space over some $\operatorname{Spa}\left(C, \mathcal{O}_{C}\right)$ with $C$ algebraically closed and perfectoid). More precisely, we use that $\pi^{*}: \mathcal{D}_{\square}\left(\pi_{0}(X), \Lambda\right)_{\omega_{1}} \rightarrow$ $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ commutes with limits to see that $\widehat{M}=\pi^{*} \pi_{*}(\widehat{M})$ is the pullback along $\pi^{*}$ of the completion of $\pi_{*}(M)$. Using that $\mathcal{D}_{\text {nuc }}(X, \Lambda)=\mathcal{D}_{\text {nuc }}(C(X, \Lambda))$ by Lemma 4.6, it suffices by Lemma 2.18.(i) to see that $\widehat{M}$ is discrete modulo $I$. But $M / I \cong \widehat{M} / I$, and the claim follows.

Now, (ii) is a consequence of (i): By (i) $M \rightarrow \widehat{M}$ is a morphism of nuclear objects and both are $I$-adically complete in $\mathcal{D}_{\text {nuc }}(X, \Lambda)$. Indeed, for $\widehat{M}$ this follows as the nuclearization $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}} \rightarrow \mathcal{D}_{\text {nuc }}(X, \Lambda)$, i.e., the right adjoint to the inclusion, preserves limits. Now, $M / I \cong \widetilde{M} / I$ is an isomorphism, which implies the claim.

In the next lemma we will use the notation $M_{\text {et }}$ for the "étalification" of an object $M \in$ $\mathcal{D}\left(X_{\text {qproet }}, \Lambda\right)$ if $X$ is an $\ell$-bounded spatial diamond, i.e., $M_{\text {et }}:=\varepsilon^{*} \varepsilon_{*} M$ for $\varepsilon: X_{\text {qproet }} \rightarrow X_{\mathrm{et}}$ the natural morphism of sites.

Lemma 4.19. Let $X$ be an $\ell$-bounded spatial diamond. Assume that $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\text {oc }}$ is generated under colimits by basic conuclear objects, which are compact.
(i) The right adjoint $(-)_{\mathrm{nuc}}: \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}} \rightarrow \mathcal{D}_{\mathrm{nuc}}(X, \Lambda)$ commutes with colimits.
(ii) If $Q \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\text {oc }}$ is compact, then the natural morphism $\left(Q_{\mathrm{et}}\right)^{\text {oc }} \rightarrow Q$ induces an isomorphism $\widehat{\left(\left(Q_{\mathrm{et}}\right)^{\text {oc }}\right)} \rightarrow Q_{\mathrm{nuc}}$ with $\widehat{(-)}$ refering to I-adic completion for an ideal of definition in $\Lambda$.
(iii) $\mathcal{D}_{\text {nuc }}(X, \Lambda)$ is generated under colimits by right bounded, complete objects.

Proof. Statements (1), (2) imply (3): Indeed, if $Q \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ is compact, then (2) with Lemma 4.11 and the assumption that $X$ is $\ell$-bounded imply that $Q_{\text {nuc }}$ is complete and right bounded (note that by Lemma 4.18 the notion of completedness for nuclear objects is unambiguous). Given now any nuclear object $N \in \mathcal{D}_{\text {nuc }}(X, \Lambda)_{\omega_{1}}$, we can write $N$ as a colimit of a diagram of compact objects $Q_{i}, i \in I$. Then $N \cong N_{\text {nuc }}$ is the colimit of the $Q_{i, \text { nuc }}, i \in I$, by (1).

To show (1),(2) we follow the proof of [15, Proposition 3.12.]. We consider the colimit preserving functor

$$
F: \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}} \rightarrow \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\circ \mathrm{c}},
$$

which sends a compact object $Q$ to $F(Q):=\widehat{\left(Q_{\mathrm{et}}\right)^{\text {oc }}} \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\mathrm{oc}}$. We first check that $F(Q)$ is a nuclear object in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\text {oc }}$ for each compact object $Q \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$. By assumption it suffices to check that

$$
(\underline{\operatorname{Hom}}(P, \Lambda) \otimes F(Q))(*) \cong \operatorname{Hom}(P, F(Q))
$$

for each compact, basic conuclear object $P$. Note that by Proposition 4.2 the left hand side is complete. As the same holds true for the right hand side, we may assume that $Q$ is an étale sheaf by replacing $Q$ with $Q / I$ for some ideal of definition $I \subseteq \Lambda$ ). Then we may in fact assume that $\Lambda$ is discrete. Let us write $P=\lim _{n \in \mathbb{N}} P_{n}$ with all morphism $P_{n+1} \rightarrow P_{n}$ of trace class and (without loss of generality) $P_{n}$ compact. Now we can calculate

$$
\begin{aligned}
& \operatorname{Hom}(P, F(Q)) \\
= & \operatorname{Hom}\left(P,\left(Q_{\mathrm{et}}\right)^{\mathrm{oc}}\right) \\
= & \operatorname{Hom}\left(P, Q_{\mathrm{et}}\right) \\
= & \underset{n}{\lim } \operatorname{Hom}\left(P_{n}, Q_{\mathrm{et}}\right) \\
= & \underset{\vec{\longrightarrow}}{\lim } \operatorname{Hom}\left(P_{n, \mathrm{oc}}, Q_{\mathrm{et}}\right) \\
= & \underset{\vec{n}}{\lim } \operatorname{Hom}\left(P_{n}, Q_{\mathrm{et}}^{\mathrm{oc}}\right) \\
= & \underset{\vec{n}}{\lim }\left(\underline{\operatorname{Hom}}\left(P_{n}, \Lambda\right) \otimes Q_{\mathrm{et}}^{\mathrm{oc}}\right)(*) \\
= & \left.\underline{\operatorname{Hom}}(P, \Lambda) \otimes Q_{\mathrm{et}}^{\mathrm{oc}}\right)(*) \\
= & \underline{\operatorname{Hom}}(P, \Lambda) \otimes F(Q))(*)
\end{aligned}
$$

(by definition of $F$ )
(as $P$ is overconvergent by Lemma 4.15) (as $Q_{\text {et }}$ is étale and the $P_{n}$ compact)
(using Lemma 4.15)
(using adjunction for $(-)_{\mathrm{oc}}$ and $(-)^{\mathrm{oc}}$ )
(by (Eq. (4.12.1)))
(using that $\Lambda$ is discrete)
(by definition).

We now have established that $F(M)$ is nuclear for any $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$. The natural transformation $F(-) \rightarrow(-)$ induces therefore a natural transformation $F(-) \rightarrow(-)_{\text {nuc }}$, which we want to show to be an isomorphism. For this we fix $M \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$ and a basic nuclear object $N$ written as a sequential colimit $N_{1} \rightarrow N_{2} \rightarrow \ldots$ along trace class maps. Define the colimit preserving functor

$$
F^{\prime}: \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}} \rightarrow \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}
$$

sending a compact object $Q$ to $\widehat{Q_{\text {et }}}$ (this agrees if $\Lambda=\mathbb{Z}_{\ell}$ for $\ell \neq p$ with the nuclearization considered in [15, Proposition 3.12]). By Lemma4.11 we can conclude that $F(-) \cong\left(F^{\prime}(-)\right)^{\text {oc }}$ as $(-)^{\text {oc }}$ commutes with colimits. Moreover, we know

$$
\operatorname{Hom}(N, F(M))=\operatorname{Hom}\left(N, F^{\prime}(M)\right)
$$

as $N$ is overconvergent, and we have to see that this agrees with

$$
\operatorname{Hom}\left(N, M_{\mathrm{nuc}}\right)=\operatorname{Hom}(N, M) .
$$

We can now follow the proof of [15, Proposition 3.12]. More precisely,
a) The definition of $\operatorname{Hom}^{\operatorname{tr}}(-,-)$ with its natural transformation $\operatorname{Hom}^{\operatorname{tr}}(-,-) \rightarrow \operatorname{Hom}(-,-)$ from [15, Definition 3.4] literally adapts from $\mathbb{Z}_{\ell}, \ell \neq p$ to general $\Lambda$. [15, Lemma 3.6] remains valid as well.
b) There exists a natural transformation $(\underline{\operatorname{Hom}}(P, \Lambda) \otimes(-))(*) \rightarrow \operatorname{Hom}^{\operatorname{tr}}(P,-)$ for each compact object $P \in \mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$. In particular, any trace class map in the sense of [6, Definition 8.1] is trace class in the sense of [15, Definition 3.4].
c) With $N=\underset{n}{\lim } N_{n}$ basic nuclear and $M$ as above the natural morphisms

$$
\left.{\underset{\zeta}{n}}_{\lim }^{(\operatorname{Hom}}\left(N_{n}, \Lambda\right) \otimes M\right)(*) \rightarrow \underset{{ }_{n}}{\lim _{n}} \operatorname{Hom}^{\operatorname{tr}}\left(N_{n}, M\right) \rightarrow \underset{{ }_{n}}{\lim } \operatorname{Hom}\left(N_{n}, M\right)
$$

are isomorphisms.
We these adjustments the proof in [15, Proposition 3.12] goes through. Let us write $N_{n}=$ ${\underset{j}{l}}_{\underset{j}{\lim }} N_{n, j}$ with $N_{n, j}$ compact and étale (this is possible by [15, Proposition 2.5.(ii)] whose prove works for $\mathbb{Z}_{\ell}, \ell \neq p$ replaced by $\Lambda$ as well $)$, and $M=\underset{i}{\lim } M_{i}$ with $M_{i}$ compact. Then:

$$
\begin{aligned}
& \operatorname{Hom}\left(N, F^{\prime}(M)\right) \\
& =\lim _{n} \operatorname{Hom}\left(N_{n}, F^{\prime}(M)\right) \quad \text { (by pulling out the colimit)) } \\
& =\lim _{n} \underset{i}{\lim _{\vec{i}}} \operatorname{Hom}\left(N_{n}, \widehat{\left(M_{i}\right)_{\mathrm{et}}}\right) \quad \text { (using the definition of } F \text { and compactness of } N_{n} \text { ) } \\
& =\lim _{n} \underset{i}{\lim _{i}}\left(\operatorname{Hom}\left(N_{n},\left(M_{i}\right)_{\text {et }}\right)^{\wedge}\right) \quad \text { (by pulling out the completion) } \\
& =\lim _{n} \underset{i}{\lim _{\overparen{l}}}\left(\underset{\underset{j}{l}}{\lim } \operatorname{Hom}\left(N_{n, j},\left(M_{i}\right)_{\mathrm{et}}\right)^{\wedge}\right) \quad \text { (using that }\left(M_{i}\right)_{\mathrm{et}} \text { is étale and the } N_{n, j} \text { compact) } \\
& =\lim _{n} \underset{i}{\lim }\left(\underset{j}{\lim } \operatorname{Hom}\left(N_{n, j}, M_{i}\right)^{\wedge}\right) \quad \text { (using that the } N_{n, j} \text { are étale) } \\
& \left.=\lim _{n} \underset{i}{\lim } \operatorname{Hom}^{\operatorname{tr}}\left(N_{n}, M_{i}\right) \quad \quad \text { (the definition of } \operatorname{Hom}^{\operatorname{tr}}(-,-)\right) \\
& =\underset{{ }_{n}}{\lim _{n}} \operatorname{Hom}^{\operatorname{tr}}\left(N_{n}, M\right) \quad \text { (again the definition of } \operatorname{Hom}^{\operatorname{tr}}(-,-) \text { ) } \\
& =\lim _{n} \operatorname{Hom}\left(N_{n}, M\right) \quad \text { (using that the above properties of } \operatorname{Hom}^{\operatorname{tr}}(-,-) \text { ) } \\
& =\operatorname{Hom}(N, M) \quad \text { (by pulling in the inverse limit) }
\end{aligned}
$$

This finished the proof that $F(-) \cong(-)_{\text {nuc }}$. By construction, (1) and (2) follow.

We now provide examples for $\ell$-bounded spatial diamonds $X$ satisfying the assumptions of Lemma 4.19.

Lemma 4.20. Let $X$ be an $\ell$-bounded affinoid perfectoid space. Then:
(i) $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}^{\circ \mathrm{c}}$ is generated by basic conuclear objects, which are compact.
(ii) $\mathcal{D}_{\text {nuc }}(X, \Lambda)$ is generated by right bounded, complete objects.

Proof. By Lemma 4.19 the first assertion implies the second. To prove the first, we claim that for each $U \rightarrow X$ étale, quasi-compact and separated, the object

$$
\Lambda_{\square}[U]_{\mathrm{oc}}=\Lambda_{\square}\left[\bar{U}^{/ X}\right]
$$

is basic conuclear (and compact). This implies the first claim by stability of basic conuclear objects under countable limits (Definition 4.13). Let $I \subset \Lambda$ be an ideal of definition. Then a trace class map over $\Lambda / I$ is a trace class map over $\Lambda$. ${ }^{17}$ Hence, we may assume that $\Lambda$ is discrete (again using Definition 4.13), but potentially derived. By stability of basic conuclear objects under finite colimits we may shrink $U$. Hence, we may assume that there exists a commutative diagram

with cartesian squares, and $g$ finite étale, while $V \rightarrow X, W \rightarrow X$ are the inclusions of rational open subspaces. Indeed, we can first construct $f: U \rightarrow V$ finite étale over some rational open subspace $V \subseteq X$. Then $\bar{f}: \bar{U}^{/ X} \rightarrow \bar{V}^{/ X}$ is finite étale, and spreads out to some rational open neighborhood $W$ of $\bar{V}^{/ X}$ (by [22, Proposition 6.4.(i)]). Write $X=\operatorname{Spa}\left(A, A^{+}\right)$and $V=\left\{\left|f_{1}\right|, \ldots,\left|f_{r}\right| \leq|g| \neq 0\right\}$ for $f_{1}, \ldots, f_{r}, g \in A$ generating $A$. Let $\pi \in A$ be a uniformizer. We can set

$$
V_{n}:=\left\{\left|f_{1}\right|, \ldots,\left|f_{r}\right| \leq\left|g \pi^{-1 / p^{n}}\right| \neq 0\right\} .
$$

Then $\bar{V}^{/ X}=\bigcap_{n \in \mathbb{N}} V_{n}$ as can be checked as in Lemma 4.8, There exists an $n_{0}$ such that $V_{n} \subseteq W$ for every $n \geq n_{0}$. For $n \geq n_{0}$ set $U_{n}:=V_{n} \times_{W} \widetilde{W}$. Note that $U_{n+1} \subseteq \overline{U_{n+1}} / X \subseteq U_{n}$ for $n \geq n_{0}$. By Lemma 4.21 we can conclude that $\Lambda_{\square}\left[U_{n+1}\right] \rightarrow \Lambda_{\square}\left[U_{n}\right]$ is trace class. Hence, it suffices to show that the natural map

$$
\left(\Lambda_{\square}[U]\right)_{\mathrm{oc}} \rightarrow \lim _{n \geq n_{0}} \Lambda_{\square}\left[U_{n}\right]
$$

is an isomorphism. This can be checked after pullback to a strictly totally disconnected $X^{\prime} \rightarrow X$ covering $X$. Hence, without loss of generality $X$ is strictly totally disconnected. In this case, $\left(\Lambda_{\square}[U]\right)_{\text {oc }}=\Lambda_{\square}\left[\pi_{0}(U) \times_{\pi_{0}(X)} X\right]$ by Lemma 4.7. The same holds for the right hand side as

$$
\pi_{0}(U) \times_{\pi_{0}(X)} X=\pi_{0}\left(\bar{U}^{/ X}\right) \times \times_{\pi_{0}(X)} X=\underset{\underset{n}{\mid}}{\lim _{n} \bar{U}_{n} / X}=\lim _{\underset{n}{ }} U_{n}
$$

using that $U_{n+1} \subseteq{\overline{U_{n}}}^{/ X} \subseteq U_{n}$.

[^14]In the proof of the previous result we made use of the following geometric source of trace class maps:

Lemma 4.21. Let $X$ be an $\ell$-bounded spatial diamond and $f: U \rightarrow V, g: V \rightarrow X$ étale, quasi-compact and separated maps. Assume that $\Lambda$ is discrete (but potentially derived). If $f: U \rightarrow V$ extends to a map $h: \bar{U}^{/ X} \rightarrow V$, then $\Lambda[U] \rightarrow \Lambda[V]$ is trace class in $\mathcal{D}_{\square}(X, \Lambda)_{\omega_{1}}$.
Proof. As symmetric monoidal functors map trace class maps to trace class maps we may assume that $\Lambda=\mathbb{Z}_{\ell} / \ell^{m}$ for some $m \geq 0$. We can calculate $\underline{\operatorname{Hom}(\Lambda[U], \Lambda) \otimes \Lambda[V]=(g \circ}$ $f)_{*}\left(\Lambda_{U}\right) \otimes g!\left(\Lambda_{V}\right) \cong g!\left(f_{*}\left(\Lambda_{U}\right)\right)$ (here $g!$ denotes the usual étale exceptional pushforward for an étale morphism). The existence of $h$ implies that the natural map $g_{!}\left(f_{*}\left(\Lambda_{U}\right)\right) \rightarrow(g \circ f)_{*}\left(\Lambda_{U}\right)$ is an isomorphism. Indeed, one can check this on stalks over $X$, i.e., if $X=\operatorname{Spa}\left(C, C^{+}\right)$for $C$ an algebraically closed perfectoid field, and $C^{+}$an open and bounded valuation subring. In this case the existence of $h$ implies that $U \rightarrow V$ factors over some $V_{0} \subseteq V$, which is finite étale over $X$. Then the statement is clear. Thus,

$$
(\underline{\operatorname{Hom}}(\Lambda[U], \Lambda) \otimes \Lambda[V])(*) \cong(g \circ f)_{*}\left(\Lambda_{U}\right)(*)
$$

admits the canonical element $1 \in \Gamma(U, \Lambda)=\Gamma\left(X,(g \circ f)_{*}\left(\Lambda_{U}\right)\right)$, which can be checked to witness that the morphism $\Lambda[U] \rightarrow \Lambda[V]$ is trace class. Indeed, this statement only involves morphisms between static sheaves and hence can again be checked after pullback to $\mathrm{Spa}\left(C, C^{+}\right)$for $C$ an algebraically closed perfectoid field over $X$. This reduces the assertion easily to the cases $U=\varnothing$ or $V=X$, where it is clear.

## 4.4 $\mathcal{C}$-valued nuclear sheaves

In order to prove strong descent results for $\mathcal{O}^{+}$-modules on perfectoid spaces, we need a slightly exotic version of nuclear $\Lambda$-modules, where we introduce an additional "algebraic" solid structure. In fact, the following definitions and results work more generally for certain $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear categories $\mathcal{C}$.

We continue with the notation from Section [4.2, in particular $\Lambda$ is an adic and profinite $\mathbb{Z}_{\ell}$-algebra.
Remark 4.22. The category $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$ of presentable $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear categories has some favourable properties due to the rigidity of $\mathcal{D}_{\text {nuc }}(\Lambda)$ as a presentably, symmetric monoidal $\infty$-category ${ }^{18}$ These properties are the following:

1. A $\mathcal{D}_{\text {nuc }}(\Lambda)$-module $\mathcal{C}$ in $\operatorname{Pr}_{\mathrm{Sp}}^{L}$, i.e., an object of $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$, is dualizable in the symmetric monoidal category $\left(\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L},-\otimes_{\mathcal{D}_{\text {nuc }}(\Lambda)}-\right)$ if and only it is dualizable in $\left(\operatorname{Pr}_{\mathrm{Sp}}^{L},-\otimes-\right)$ ([10, Proposition 9.4.4]).
2. Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a morphism in $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$, and assume that the right adjoint $G: \mathcal{N} \rightarrow$ $\mathcal{M}$ commutes with colimits. Then for $A \in \mathcal{D}_{\text {nuc }}(\Lambda)$ and $N \in \mathcal{N}$ the natural morphism ${ }^{19}$

$$
A \otimes G(N) \rightarrow G(A \otimes N)
$$

is an isomorphism ([10, Lemma 9.3.6]). In other words, $G$ is $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear.
Moreover, any morphism $\mathcal{D}_{\text {nuc }}(\Lambda) \rightarrow \mathcal{R}$ in $\operatorname{CAlg}\left(\operatorname{Pr}_{\mathrm{Sp}}^{L}\right)$, where the symmetric monoidal unit $1_{\mathcal{R}} \in \mathcal{R}$ is compact, automatically has a colimit-preserving right adjoint, i.e., is strongly continuous.

[^15]The $\mathcal{D}_{\text {nuc }}(\Lambda)$-modules that we will interested in have particular properties. In the following we use the short hand notation "complete" instead of " $I$-adically complete for some ideal of definition $I \subseteq \Lambda$ ".

Definition 4.23. We say that a $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear presentable category $\mathcal{C}$ is nicely generated if there is a small family of objects $\left(P_{i}\right)_{i \in I}$ in $\mathcal{C}$ satisfying the following properties:
(a) For every bounded complete $M \in \mathcal{D}_{\text {nuc }}(\Lambda)$ and all $i \in I$ the object $M \otimes P_{i} \in \mathcal{C}$ is complete.
(b) For every $i \in I$ the functor $\operatorname{Hom}^{\mathcal{D}_{\text {nuc }}(\Lambda)}\left(P_{i},-\right): \mathcal{C} \rightarrow \mathcal{D}_{\text {nuc }}(\Lambda)$ preserves all small colimits.
(c) The family of functors $\operatorname{Hom}^{\mathcal{D}_{\text {nuc }}(\Lambda)}\left(P_{i},-\right)$ from (ii) is conservative.

Here, $\operatorname{Hom}^{\mathcal{D}_{\text {nuc }}(\Lambda)}(X,-)$ for $X \in \mathcal{C}$ is the functor right adjoint to the functor $(-) \otimes_{\Lambda}$ $X: \mathcal{D}_{\text {nuc }}(\Lambda) \rightarrow \mathcal{C}, M \mapsto M \otimes_{\Lambda} X$. In particular,

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{D}_{\text {nuc }}(\Lambda)}\left(\Lambda, \operatorname{Hom}^{\mathcal{D}_{\text {nuc }}(\Lambda)}(X, Y)\right) \cong \operatorname{Hom}_{\mathcal{C}}(X, Y) \tag{4.23.1}
\end{equation*}
$$

for $X, Y \in \mathcal{C}$.
Lemma 4.24. Let $\mathcal{C}$ be a nicely generated $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear presentable category.
(i) The category $\mathcal{C}$ is compactly generated, and dualizable as an object in $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$.
(ii) For every compact $P \in \mathcal{C}$ the functor $\operatorname{Hom}^{\mathcal{D}_{\text {nuc }}(\Lambda)}(P,-): \mathcal{C} \rightarrow \mathcal{D}_{\text {nuc }}(\Lambda)$ is $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear.

Proof. As $\Lambda \in \mathcal{D}_{\text {nuc }}(\Lambda)$ is compact, each $P_{i}$ from Definition 4.23 is compact, hence $\mathcal{C}$ is compactly generated. By [14, Proposition D.7.2.3] the category $\mathcal{C}$ is therefore dualizable in $\operatorname{Pr}_{\mathrm{Sp}}^{L}$, and hence in $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$ by Remark 4.22, If $P \in \mathcal{C}$ is compact, then the $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear functor $\mathcal{D}_{\text {nuc }}(\Lambda) \rightarrow \mathcal{C}, A \mapsto A \otimes P$ has the colimit-preserving right adjoint $\operatorname{Hom}^{\mathcal{D}_{\text {nuc }}(\Lambda)}(P,-): \mathcal{C} \rightarrow$ $\mathcal{D}_{\text {nuc }}(\Lambda)$, which is therefore $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear by Remark 4.22.,

Recall that in Definition 4.13 we have introduced the $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear category $\mathcal{D}_{\text {nuc }}(X, \Lambda)$ for an $\ell$-bounded spatial diamond $X$. Using this category, we now give the following definition.

Definition 4.25. Let $\mathcal{C}$ be a nicely generated $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear presentable category. Then for every $\ell$-bounded spatial diamond $X$ we denote

$$
\mathcal{D}_{\mathrm{nuc}}(X, \mathcal{C}):=\mathcal{D}_{\mathrm{nuc}}(X, \Lambda) \otimes_{\mathcal{D}_{\mathrm{nuc}}(\Lambda)} \mathcal{C}
$$

and call it the category of $\mathcal{C}$-valued nuclear (or nuclear overconvergent) sheaves on $X$.
Proposition 4.26. Let $\mathcal{C}$ be a nicely generated $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear presentable category. Then:
(i) The assignment $X \mapsto \mathcal{D}_{\text {nuc }}(X, \mathcal{C})$ defines a hypercomplete sheaf on the big quasi-pro-étale site of $\ell$-bounded spatial diamonds.
(ii) For every $\ell$-bounded affinoid perfectoid space $X$ the category $\mathcal{D}_{\text {nuc }}(X, \mathcal{C})$ is generated under colimits by $\pi$-adically complete objects.
(iii) If $X$ is a strictly totally disconnected space then $\mathcal{D}_{\mathrm{nuc}}(X, \mathcal{C})$ naturally identifies with the category of $C\left(\pi_{0}(X), \Lambda\right)$-modules in $\mathcal{C}$.

Proof. Using [12, Theorem 4.8.4.6] claim (iii) follows from the fact that

$$
\mathcal{D}_{\mathrm{nuc}}(X, \Lambda) \cong \mathcal{D}_{\mathrm{nuc}}\left(C\left(\pi_{0}(X), \Lambda\right)\right) \cong \operatorname{Mod}_{C\left(\pi_{0}(X), \Lambda\right)}\left(\mathcal{D}_{\mathrm{nuc}}(\Lambda)\right)
$$

if $X$ is strictly totally disconnected (see Lemma 4.16 and its proof).
Let us prove claim (i) and take a quasi-pro-étale hypercover $X_{\bullet} \rightarrow X$ of $\ell$-bounded spatial diamonds. We need to see that the natural functor

$$
F: \mathcal{D}_{\mathrm{nuc}}(X, \mathcal{C}) \rightarrow \lim _{\underset{n \in \Delta}{ }} \mathcal{D}_{\mathrm{nuc}}\left(X_{n}, \mathcal{C}\right)
$$

is an equivalence. By Lemma 4.17 this holds if $\mathcal{C}=\mathcal{D}_{\text {nuc }}(\Lambda)$, and then in $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$, i.e., all transition functors are $\mathcal{D}_{\text {nuc }}(\Lambda)$-linear (as they are induced by pullback functors for $\left.\mathcal{D}_{\text {nuc }}(-, \Lambda)\right)$. Therefore the claim follows by dualizability of $\mathcal{C}$ in $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$ as established in Lemma 4.24, Indeed, dualizability implies that $\mathcal{C} \otimes_{\mathcal{D}_{\text {nuc }}(\Lambda)}$ - commutes with limits in $\operatorname{Pr}_{\mathcal{D}_{\text {nuc }}(\Lambda)}^{L}$.

It remains to prove (ii), so let $X$ be a given $\ell$-bounded affinoid perfectoid space. Then $\mathcal{D}_{\text {nuc }}(X, \mathcal{C})$ is generated under colimits by objects $\mathcal{M} \boxtimes P$, where $\mathcal{M} \in \mathcal{D}_{\text {nuc }}(X, \Lambda)$ and $P \in \mathcal{C}$ is compact (and hence also complete by our assumption that $\mathcal{C}$ is nicely generated). By Lemma 4.20 we may assume that $M$ is complete, and additionally bounded above.

It is therefore enough to show that each of these $\mathcal{M} \boxtimes P$ is complete. By (i) we can check this after pullback to any quasi-pro-étale hypercover (using that limits of complete objects are again complete), so we can assume that $X$ is strictly totally disconnected. Then the claim follows from (iii) and the fact that complete objects in $\mathcal{C}$ are stable under the action of complete objects in $\mathcal{D}_{\text {nuc }}(\Lambda)$ by assumption. More precisely, completeness of $M \boxtimes P$ can be checked after applying the forgetful functor to $\mathcal{C}$, and there $M \boxtimes P$ is given by $|M| \otimes P$ with $|M| \in \mathcal{D}_{\text {nuc }}(\Lambda)$ the underlying object of $M \in \mathcal{D}_{\text {nuc }}(X, \Lambda) \cong \mathcal{D}_{\text {nuc }}\left(C\left(\pi_{0}(X), \Lambda\right)\right)$ and $\otimes$ the given $\mathcal{D}_{\text {nuc }}(\Lambda)$-action on $\mathcal{C}$.

## 5 Main theorem: $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$on +-bounded affinoid perfectoid spaces

In this section we will prove the remaining assertion of Theorem 1.1, i.e., that for $X=$ $\operatorname{Spa}\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}+$-bounded the natural map

$$
\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)
$$

is an equivalence. For this we will use the criterion of Lemma 3.22 by using the (overconvergent) nuclear $\mathcal{C}$-valued sheaves from Section 4.4. The strategy can roughly be described as realizing the category $\mathcal{D}_{\hat{仓}}^{a}\left(\mathcal{O}_{X}^{+}\right)$as a category of $\mathcal{O}^{+}$-modules in $\mathcal{D}_{\text {nuc }}(X, \Lambda)$, where $\Lambda=\mathbb{Z}_{p}[[\pi]]$ with $\pi$ mapping to a pseudo-uniformizer in $A^{+}$.

In order to make this precise, we need to first introduce the following variant for nuclear $\mathcal{C}$-valued sheaves.

Definition 5.1. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be an affinoid perfectoid space with pseudouniformizer $\pi$. Then for every $p$-bounded spatial diamond $Y$ we denote

$$
\mathcal{D}_{\mathrm{nuc}}\left(Y,\left(A^{+}\right)_{\hat{\imath}}^{a}\right):=\mathcal{D}_{\mathrm{nuc}}\left(Y, \mathbb{Z}_{p}[[\pi]]\right) \otimes_{\mathcal{D}_{\mathrm{nuc}}\left(\mathbb{Z}_{p}[[\pi]]\right)} \mathcal{D}_{\hat{\varrho}}^{a}\left(A^{+}\right)
$$

with $\mathcal{D}_{\text {nuc }}\left(Y, \mathbb{Z}_{p}[[\pi]]\right)$ as defined in Definition 4.13.
Note that the category $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$is not compactly generated, and hence not a nicely generated $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear category in the sense of Definition4.23, Nevertheless we get following assertions analogous to Proposition 4.26

Lemma 5.2. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be an affinoid perfectoid space with pseudouniformizer $\pi$. Then:
(i) The assignment $Y \mapsto \mathcal{D}_{\text {nuc }}\left(Y,\left(A^{+}\right)_{\hat{\circ}}^{a}\right)$ defines a hypercomplete sheaf of categories on the big quasi-pro-étale site of p-bounded spatial diamonds.
(ii) For every p-bounded affinoid perfectoid space $Y$ the category $\mathcal{D}_{\text {nuc }}\left(Y,\left(A^{+}\right)_{\hat{\hat{\circ}}}^{a}\right)$ is generated under colimits by complete objects.
(iii) If $Y$ is a strictly totally disconnected space then $\mathcal{D}_{\text {nuc }}\left(Y,\left(A^{+}\right)_{\hat{\circ}}^{a}\right)=\mathcal{D}_{\hat{\emptyset}}^{a}\left(C\left(\pi_{0}(Y), A^{+}\right)\right)$.

Proof. Note that $\mathcal{D}_{\hat{\mathrm{n}}}\left(A^{+}\right)$is a nicely generated $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear category. To prove this, let $\left(X_{i}\right)_{i \in I}$ be the family of compact generators $A_{\hat{0}}^{+}[S]$ for profinite sets $S$, i.e., $A_{\hat{\square}}^{+}[S]=$ $\alpha^{*}\left(A^{+}[S]\right)$ with $\alpha^{*}$ as in Lemma [2.5, We show that they satisfy properties (i), (ii) and (iii) from Definition 4.23. Part (iii) is obvious by (Eq. (4.23.1)). For (ii) we note that the functor $\operatorname{Hom}\left(A_{\hat{\square}}^{+}[S],-\right): \mathcal{D}_{\hat{\imath}}\left(A^{+}\right) \rightarrow \mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$ factors as the composition of functors

$$
\mathcal{D}_{\hat{\emptyset}}\left(A^{+}\right) \xrightarrow{\text { Hom }\left(A_{\emptyset}^{+}[S],-\right)} \mathcal{D}_{\hat{\imath}}\left(A^{+}\right) \xrightarrow{\text { forget }} \mathcal{D}_{\square}\left(\mathbb{Z}_{p}[[\pi]]\right) \xrightarrow{(-)_{\mathrm{nuc}}} \mathcal{D}_{\mathrm{nuc}}\left(\mathbb{Z}_{p}[[\pi]]\right)
$$

The first functor preserves colimits because compact objects in $\mathcal{D}_{\hat{\imath}}\left(A^{+}\right)$are stable under tensor products. The second functor obviously preserves colimits, while Proposition 2.19 shows that the third functor preserves colimits. This proves that property (ii) from Definition 4.23 is satisfied. For property (i) we note that for every right bounded and complete $M \in \mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$ we have

$$
M \otimes A_{\emptyset}^{+}[S]=\left(M \otimes_{\mathbb{Z}_{p}[[\pi]]_{\square}} \mathbb{Z}_{p}[[\pi]]_{\square}[S]\right) \otimes_{\mathbb{Z}_{p}[[\pi]]_{\square}} A_{\hat{\square}}^{+}
$$

so we conclude by Lemma 2.12.(iv). We have finally shown that $\mathcal{D}_{\hat{\bullet}}\left(A^{+}\right)$is a nicely generated $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear category. Thus by Proposition4.26 all claims are true for $\mathcal{D}_{\text {nuc }}\left(-, A_{\hat{0}}^{+}\right):=$ $\mathcal{D}_{\text {nuc }}\left(-, \mathbb{Z}_{p}[[\pi]]\right) \otimes_{\mathcal{D}_{\text {nuc }}}\left(\mathbb{Z}_{p}[[\pi]]\right) \mathcal{D}_{\hat{\imath}}\left(A^{+}\right)$in place of $\mathcal{D}_{\text {nuc }}\left(-,\left(A^{+}\right)_{\hat{\hat{a}}}^{a}\right)$. The almostification functor $(-)^{a}: \mathcal{D}_{\hat{\wedge}}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\hat{}}}^{a}\left(A^{+}\right)$(with left adjoint $\left.(-)_{!}\right)$realizes $\mathcal{D}_{\hat{\hat{\circ}}}^{a}\left(A^{+}\right)$as a retract of $\mathcal{D}_{\hat{\wedge}}\left(A^{+}\right)$, which shows that $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)$is dualizable as a $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear category (by Remark 4.22). This implies (i) by the argument in Proposition 4.26. Furthermore, part (ii) and (iii) follow from Proposition 4.26 by Lemma 2.35.

We note that $\mathcal{D}_{\text {nuc }}\left(Y,\left(A^{+}\right)_{\hat{0}}^{a}\right)$ is naturally a symmetric monoidal $\infty$-category for a $p$-bounded spatial diamond. Indeed, [6, Theorem 8.6] implies that the category $\mathcal{D}_{\text {nuc }}\left(Y, \mathbb{Z}_{p}[[\pi]]\right)$ is stable under tensor products in $\mathcal{D}_{\square}\left(Y, \mathbb{Z}_{p}[[\pi]]\right)_{\omega_{1}}$.
Definition 5.3. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be an affinoid perfectoid space and $Y$ a $p$-bounded spatial diamond over $X^{b}$. By Lemma 5.2 and Theorem 3.7 there is a unique algebra object $\mathcal{O}^{+a} \in \mathcal{D}_{\text {nuc }}\left(Y,\left(A^{+}\right)_{\hat{D}}^{a}\right)$ such that for every strictly totally disconnected space $Z$ over $Y$ with untilt $Z^{\sharp}=\operatorname{Spa}\left(B, B^{+}\right)$over $X$, the pullback of $\mathcal{O}^{+a}$ to $Z$ is the object

$$
B^{+a} \in \mathcal{D}_{\hat{\emptyset}}^{a}\left(C\left(\pi_{0}(Z), A^{+}\right)\right) \cong \mathcal{D}_{\mathrm{nuc}}\left(Z,\left(A^{+}\right)_{\hat{\llcorner }}^{a}\right)
$$

with the last isomorphism supplied by Lemma 5.2.(iii). We denote

$$
\mathcal{D}_{\mathrm{nuc}}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\mathrm{a}}}^{a}\right):=\operatorname{Mod}_{\mathcal{O}^{+a}}\left(\mathcal{D}_{\mathrm{nuc}}\left(Y,\left(A^{+}\right)_{\hat{\mathrm{a}}}^{a}\right)\right)
$$

The category $\mathcal{D}_{\text {nuc }}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{n}}^{a}\right)$ is a precise version of the heuristic category of " $\mathcal{O}^{+}$ modules in $\mathcal{D}_{\text {nuc }}\left(Y, \mathbb{Z}_{p}[[\pi]]\right)$ ". The extra factor $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)$accomodates the almost mathematics.

Lemma 5.4. Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be a p-bounded affinoid perfectoid space with pseudouniformizer $\pi$. Then:
（i）The assignment $Y \mapsto \mathcal{D}_{\text {nuc }}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\circ}}^{a}\right)$ defines a hypercomplete sheaf of categories on $X_{\text {qproet }}^{b}$ ．
（ii）For every affinoid perfectoid space $Y \in X_{\mathrm{qproet}}^{b}$ the category $\mathcal{D}_{\text {nuc }}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right){ }_{\hat{\circ}}^{a}\right)$ is gen－ erated under colimits by complete objects．
（iii）If $Y=\operatorname{Spa}\left(B, B^{+}\right) \in X_{\text {qproet }}$ is strictly totally disconnected，then

$$
\mathcal{D}_{\mathrm{nuc}}\left(Y^{\mathrm{b}},\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\emptyset}}^{a}\right)=\mathcal{D}_{\hat{\llcorner }}^{a}\left(B^{+}, A^{+}\right)
$$

（iv）For every $Y \in X_{\mathrm{qproet}}^{\mathrm{b}}$ we have

$$
\mathcal{D}_{\mathrm{nuc}}\left(\bar{Y}^{/ X^{b}},\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\Delta}}^{a}\right)=\mathcal{D}_{\mathrm{nuc}}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\Delta}}^{a}\right) .
$$

Proof．First note that the $p$－boundedness of $X$ implies that every $Y \in X_{\text {qproet }}^{b}$ is $p$－bounded by Lemma 3．17．Part（i）follows from Lemma 5．2．（i）］as in the proof of Lemma［2．40，we note that the pullback functors for $\mathcal{D}_{\text {nuc }}\left(-,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\circ}}^{a}\right)$ along quasi－pro－étale maps are the same as before on underlying objects in $\mathcal{D}_{\text {nuc }}\left(-,\left(A^{+}\right)_{\hat{\circ}}^{a}\right)$（because $\mathcal{O}^{+a}$ is preserved under these pull－ backs by definition）．Part（ii）follows from Lemma 5．2．（ii）by noting that $\mathcal{D}_{\text {nuc }}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\circ}}^{a}\right)$ is generated under colimits by objects of the form $\mathcal{O}^{+a} \otimes \mathcal{M}$ for $\mathcal{M} \in \mathcal{D}_{\text {nuc }}\left(Y,\left(A^{+}\right)_{0}^{a}\right)$ running through a set of generator．Part（iii）follows from Lemma 5．2．（iii）and the definition of $\mathcal{O}^{+a}$ ． Indeed，

$$
\begin{aligned}
& \operatorname{Mod}_{B^{+a}}\left(\mathcal{D}_{乞}^{a}\left(C\left(\pi_{0}(Z), A^{+}\right)\right)\right) \cong \operatorname{Mod}_{B^{+a}}\left(\operatorname{Mod}_{C\left(\pi_{0}(Z), A^{+}\right)}\left(\mathcal{D}_{乞}^{a}\left(A^{+}\right)\right)\right) \cong \operatorname{Mod}_{B^{+a}}\left(\mathcal{D}_{乞}^{a}\left(A^{+}\right)\right) \\
& \quad \cong \mathcal{D}_{乞}^{a}\left(B^{+}, A^{+}\right)
\end{aligned}
$$

using in the first isomorphism that $A^{+} \rightarrow C\left(\pi_{0}(Z), A^{+}\right)$is integral mod $\pi$（and hence induces the analytic ring structure）．Part（iv）follows from $\mathcal{D}_{\text {nuc }}\left(\bar{Y}^{/ X^{b}}, \mathbb{Z}_{p}[[\pi]]\right)=\mathcal{D}_{\text {nuc }}\left(Y, \mathbb{Z}_{p}[[\pi]]\right)$ （and the overconvergence of $\mathcal{O}^{+a}$ ）．Indeed，the claim can（by Lemma 4．17］be checked in the case that $X$ is strictly totally disconnected．Then $\bar{Y}^{/ X^{b}}, Y$ are strictly totally disconnected as they are quasi－pro－étale over $X$ ．Then the claim follows from Lemma 4．16 as $\pi_{0}\left(\bar{Y}^{/ X^{b}}\right)=$ $\pi_{0}(Y)$ ．

Corollary 5．5．Let $X=\operatorname{Spa}\left(A, A^{+}\right)$be a p－bounded affinoid perfectoid space with pseudouni－ formizer $\pi$ such that $X^{b}$ admits a map of finite dim．trg to some totally disconnected space． Then there is a natural equivalence

$$
\mathcal{D}_{\hat{\ominus}}^{a}\left(\mathcal{O}_{X}^{+}\right)=\mathcal{D}_{\mathrm{nuc}}\left(X,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\circ}}^{a}\right) .
$$

In particular，the following is true：
（i） $\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)$is generated under colimits by right bounded，complete objects．
（ii）The natural functor $\Gamma(X,-): \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)$preserves all small colimits．
Proof．We first construct a natural map of hypercomplete quasi－pro－étale sheaves

$$
\alpha(-): \mathcal{D}_{\mathrm{nuc}}\left(-,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\Delta}}^{a}\right) \rightarrow \mathcal{D}_{\stackrel{1}{a}}^{a}\left(\mathcal{O}_{(-)^{\sharp}}^{+}\right)
$$

on $X_{\text {qproet }}^{b}$ ，where $(-)^{\sharp}: X_{\text {qproet }}^{b} \rightarrow X_{\text {qproet }}$ is inverse to $(-)^{b}$ ．It is enough to construct this map on strictly totally disconnected spaces $Y$ with until $Y^{\sharp}=\operatorname{Spa}\left(B, B^{+}\right)$over $X$ ，where by Theorem 3.7 and Lemma 5．4．（iii）it boils down to the natural map $\alpha(Y): \mathcal{D}_{\hat{\circ}}^{a}\left(B^{+}, A^{+}\right) \rightarrow$ $\mathcal{D}_{\hat{\circ}}^{a}\left(B^{+}\right)$（which is is compatible with pullback in $Y$ ）．

In the following, we will identify $X_{\mathrm{qproet}} \cong X_{\mathrm{qproet}}^{\mathrm{b}}$ and view both sides as sheaves on $X_{\mathrm{qproet}}$. Now fix a strictly totally disconnected space $Y=\operatorname{Spa}\left(B, B^{+}\right) \in X_{\text {qproet }}$ and let $Z:=\bar{Y}^{/ X}$. We claim that $\alpha(Z)$ is an equivalence. By Lemmas 5.4.(iii) and 5.4.(iv) we have

$$
\mathcal{D}_{\mathrm{nuc}}\left(Z,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\llcorner }}^{a}\right) \cong \mathcal{D}_{\mathrm{nuc}}\left(Y,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\llcorner }}^{a}\right) \cong \mathcal{D}_{\hat{\wedge}}^{a}\left(B^{+}, A^{+}\right)
$$

On the other hand, writing $Z=\mathrm{Spa}\left(C, C^{+}\right)$we obtain from Theorem 3.23 that

$$
\mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{Z}^{+}\right)=\mathcal{D}_{\hat{\emptyset}}^{a}\left(C^{+}\right)=\mathcal{D}_{\hat{\emptyset}}^{a}\left(B^{+}, A^{+}\right)
$$

By going through the natural identifications one checks that $\alpha(Z)$ is the obvious equivalence.
Now pick a hypercover $Y_{\bullet} \rightarrow X$ by strictly totally disconnected spaces $Y_{n} \in X_{\text {qproet }}$ and let $Z_{\bullet}=\bar{Y}_{\bullet}^{/ X}$. Then $Z_{\bullet} \rightarrow X$ is a hypercover in $X_{\text {qproet }}$ and by the above we know that $\alpha\left(Z_{n}\right)$ is an equivalence for all $n$. By descent we deduce that $\alpha(X)$ is an isomorphism, i.e. we obtain the desired identity $\mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)=\mathcal{D}_{\text {nuc }}\left(X,\left(\mathcal{O}^{+}, A^{+}\right)_{\square}^{a}\right)$.

It remains to prove the "in particular" claims. Part (i) is an immediate consequence of Lemma 5.4.(ii) (noting that the right boundedness holds for the constructed complete generators). It remains to prove (ii). We first observe that $\Gamma(X,-)$ is the right adjoint of the natural functor $\mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\hat{\emptyset}}^{a}\left(\mathcal{O}_{X}^{+}\right)$obtained from sheafification. Under the equivalence $\alpha(Z)$ the latter functor can be written as the composition

$$
\mathcal{D}_{\hat{\llcorner }}^{a}\left(A^{+}\right) \rightarrow \mathcal{D}_{\mathrm{nuc}}\left(X,\left(A^{+}\right)_{\hat{\llcorner }}^{a}\right) \xrightarrow{-\otimes \mathcal{O}^{+a}} \mathcal{D}_{\mathrm{nuc}}\left(X,\left(\mathcal{O}^{+}, A^{+}\right)_{\hat{\square}}^{a}\right),
$$

where the first functor is induced (by tensoring over $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$ with $\mathcal{D}_{\hat{\hat{0}}}^{a}\left(A^{+}\right)$) from the pullback $\rho^{*}: \mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right) \rightarrow \mathcal{D}_{\text {nuc }}\left(X, \mathbb{Z}_{p}[[\pi]]\right)$ for the morphism of sites $\rho: X_{\text {qproet }} \rightarrow *_{\text {qproet }}$. Thus $\Gamma(X,-)$ can be written as the composition of the right adjoints of the above two functors, so it is enough to show that each of these right adjoints preserves all small colimits. The right adjoint of $-\otimes \mathcal{O}^{+a}$ is the forgetful functor, which clearly preserves all small colimits. The right adjoint $G$ of the first functor can be obtained from the global sections functor $\rho_{*}: \mathcal{D}_{\text {nuc }}\left(X, \mathbb{Z}_{p}[[\pi]]\right) \rightarrow \mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$ by tensoring with $\mathcal{D}_{\hat{\hat{O}}}^{a}\left(A^{+}\right)$. Indeed, $\rho_{*}$ commutes with colimits and is $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear as $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$ is rigid, its left adjoint $\rho^{*}$ is symmetric monoidal and the unit in $\left.\mathcal{D}_{\text {nuc }}(X), \mathbb{Z}_{p}[[\pi]]\right)$ is compact (Remark 4.22). This implies that $\rho_{*} \otimes_{\mathcal{D}_{\mathrm{nuc}}\left(\mathbb{Z}_{p}[[\pi]]\right)} \mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$defines the right adjoint $G$ of $\mathcal{D}_{\hat{\circ}}^{a}\left(A^{+}\right)$by functoriality in the 2-category of $\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)$-linear presentable categories 20 This proves the claim.

We can finally show the main descent result of this paper, generalizing [16, Theorem 3.5.21] and finishing the proof of Theorem 1.1.

Theorem 5.6. Let $X=\operatorname{Spa}\left(A, A^{+}\right) \in \operatorname{Perfd}^{\text {aff }}$ be + -bounded. Then

$$
\mathcal{D}_{\hat{\circ}}^{a}\left(\mathcal{O}_{X}^{+}\right)=\mathcal{D}_{\hat{0}}^{a}\left(A^{+}\right) .
$$

Proof. By Theorem $3.18 X$ is $p$-bounded. Hence, we may apply Lemma 3.22 thanks to Corollary 5.5 (and its proof to see the right boundedness of $\Gamma(X, \mathcal{N})$ for the constructed complete generators in Corollary 5.5).

## References

[1] G. Andreychev. Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces. 2021. Preprint. Available at https://arxiv.org/abs/2105.12591.

[^16][2] J. Anschütz, A.-C. Le Bras, and L. Mann. A 6-functor formalism for solid quasi-coherent sheaves on the Fargues-Fontaine curve. 2024. in preparation.
[3] B. Bhatt and P. Scholze. The pro-étale topology for schemes. Astérisque, (369):99-201, 2015.
[4] N. Bourbaki. Algèbre commutative: Chapitres 5 à 7. Springer Science \& Business Media, 2007.
[5] D. Clausen. Lectures on Efimov $k$-theory. 2023. IHES Summer School - Recent Advances in Algebraic K-theory.
[6] D. Clausen and P. Scholze. Lectures on Complex Geometry, 2022. https://people.mpim-bonn.mpg.de/scholze/Complex.pdf.
[7] P. Colmez. Fonctions d'une variable p-adique. Astérisque, 330(13-59):9-29, 2010.
[8] A. I. Efimov. K-theory of large categories. In Proc. Int. Cong. Math, volume 3, pages 2212-2226, 2022.
[9] L. Fargues and P. Scholze. Geometrization of the local Langlands correspondence. 2021. https://arxiv.org/abs/2102.13459.
[10] D. Gaitsgory and N. Rozenblyum. A Study in Derived Algebraic Geometry: Volume I: Correspondences and Duality. Mathematical Surveys and Monographs. American Mathematical Society, 2019.
[11] Y. Liu and W. Zheng. Enhanced six operations and base change theorem for higher Artin stacks. 2017. Preprint. Available at https://arxiv.org/abs/1211.5948.
[12] J. Lurie. Higher Algebra. Available at https://people.math.harvard.edu/~lurie/papers/HA.pdf.
[13] J. Lurie. Higher Topos Theory (AM-170). Princeton University Press, 2009.
[14] J. Lurie. Spectral algebraic geometry. Available at https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf, 2018.
[15] L. Mann. The 6-Functor Formalism for $\mathbb{Z}_{\ell}$-and $\mathbb{Q}_{\ell}$-sheaves on Diamonds. arXiv preprint arXiv:2209.08135, 2022.
[16] L. Mann. A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry. 2022. Preprint. Available at https://arxiv.org/pdf/2206.02022.pdf.
[17] L. Mann and A. Werner. Local Systems on Diamonds and p-Adic Vector Bundles. International Mathematics Research Notices, July 2022.
[18] A. Mathew. The Galois group of a stable homotopy theory. Advances in Mathematics, 291, 2016.
[19] Y. Mikami. Fppf-descent for condensed animated rings, 2023.
[20] J. Neukirch, A. Schmidt, and K. Wingberg. Cohomology of Number Fields. Grundlehren der mathematischen Wissenschaften. Springer Berlin, Heidelberg, 2008.
[21] P. Scholze. Perfectoid spaces. Publ. Math. Inst. Hautes Études Sci., 116:245-313, 2012.
[22] P. Scholze. Étale cohomology of diamonds. https://www.math.uni-bonn.de/people/scholze/EtCoh 2018.
[23] P. Scholze. Lectures on Condensed Mathematics, 2019. https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.
[24] P. Scholze. Lectures on Analytic Geometry, 2020. https://www.math.uni-bonn.de/people/scholze/Analytic.pdf.
[25] P. Scholze and J. Weinstein. Berkeley lectures on p-adic geometry. Annals of Math Studies, 207, 2020.
[26] M. Temkin. Relative Riemann-Zariski spaces. Israel Journal of Mathematics, 185:1-42, 2011.
[27] The Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu, 2018.


[^0]:    ${ }^{1}$ By definition, an element $a \in \pi_{0}(A)(*)$ is topologically nilpotent if the associated map $\mathbb{Z}[T] \rightarrow A$ factors over $\mathbb{Z}[[T]]$ as a map of rings. As $A$ is solid and $\mathbb{Z}[[T]] \otimes_{(\mathbb{Z}[T], \mathbb{Z})_{\square}} \mathbb{Z}[[T]] \cong \mathbb{Z}[[T]]$ this factorization is unique if it exists.

[^1]:    ${ }^{2}$ This follows from the fact that $\left.\underline{\operatorname{Hom}(M, N}\right)$ is adically complete whenever $N \in \mathcal{D}(A)$ is adically complete.

[^2]:    ${ }^{3}$ We show in Lemma 2.15 that $\mathcal{D}_{\text {cpl }}(\mathcal{A})$ and $\mathcal{D}_{\text {cpl }}(\hat{\mathcal{A}})$ agree.

[^3]:    ${ }^{4}$ Here, we are using that $\mathcal{A}$ is living over $\mathbb{Z}_{\square}$, and that the compact projective generators for $\mathbb{Z}_{\square}$ are internally projective to obtain the right-boundedness of $P^{\vee}$.

[^4]:    ${ }^{5}$ Let $b \in J$. Then $\pi_{0}(B[x])$ must contain some element $x$ whose coefficient in front of $x^{n}$ is $b^{n}$ as follows by the universal case of the ring $\mathbb{Z}[[b]]$. This forces $b$ to be nilpotent, and thus $J$ as well because $J$ is finitely generated.

[^5]:    ${ }^{6}$ I.e., $f$ lies in the smallest subcategory of adic morphisms of adic analytic rings, which contains the adically descendable maps and is stable under $\omega_{1}$-filtered colimits.

[^6]:    ${ }^{7}$ Let $E \rightarrow \Delta^{\triangleleft}, E^{\prime} \rightarrow \Delta^{\triangleleft}$ be the cocartesian fibrations classified by $\chi, \chi^{\prime}$. The natural transformation $\sigma$ unstraightens to a functor $\kappa: E^{\prime} \rightarrow E$, which preserves cocartesian arrows. Using [12, 7.3.2.6.] the functor $\kappa$ admits an adjoint $\lambda: E \rightarrow E^{\prime}$ (relative to $\Delta^{\triangleleft}$ ) as this is true over each $[n] \in \Delta^{\triangleleft}$. Now, steadiness of $f$ implies that $\lambda$ preserves cocartesian arrows, and thus defines a natural transformation $\rho: \chi \rightarrow \chi^{\prime}$. By construction of $\lambda, \rho_{[n]}: \chi([n]) \rightarrow \chi^{\prime}([n])$ is given by the pushforward $g_{n, *}$ as desired.

[^7]:    ${ }^{8}$ Given any countable diagram $j_{k}, k \in K$, in $J$, we can write the category $K$ as a filtered union of finite subgraphs $K_{n}, n \in \mathbb{N}$, and set $j^{\prime}(n) \in J_{n}$ as an object with a map from each $j_{k}(l), k \in K_{n}$ and $l \leq n$ such that these morphisms to $j^{\prime}(n)$ equalize all morphisms $j_{k}(l) \rightarrow j_{k^{\prime}}(l)$ induced by a morphism in $K_{n}$. Inductively, we can even construct compatible morphisms $j^{\prime}(n) \rightarrow j^{\prime}(n+1)$, and set $j_{k} \rightarrow j^{\prime}$ for $k \in K_{n}$ as the morphism, which for $m \geq n$ is the chosen morphism $j_{k}(m) \rightarrow j^{\prime}(m)$.

[^8]:    ${ }^{9}$ More precisely, we as well use that descent of $\mathcal{D}_{\hat{\wedge}}(-)$ for $\left(A^{+}\right)_{\square} \rightarrow\left(B^{+}\right)_{\square}$ implies descent of $\mathcal{D}_{\hat{\Delta}}^{a}(-)$ by embedding the almost category (via j!) into $\mathcal{D}_{\hat{\mathrm{人}}}(-)$ and using Remark 2.36 to see that this embedding is compatible with pullback.
    ${ }^{10}$ We note that a qcqs perfectoid space $Y$ with a quasi-pro-étale map $Y \rightarrow X$ to a strictly totally disconnected perfectoid space $X$ is itself again strictly totally disconnected. This need not be true if $X=\operatorname{Spa}\left(K, K^{+}\right)$ is connected and totally disconnected as the valuation ring $K^{+}$need not be henselian.

[^9]:    ${ }^{11}$ It follows by [13, Proposition 5.5.3.13] that $\mathcal{D}_{\hat{\rho}}^{a}\left(\mathcal{O}_{Z}^{+}\right)$is presentable for any $Z \in$ Perfd. By reduction to the totally disconnected case it follows that $f^{*}$ preserves colimits, and hence admits a right adjoint.

[^10]:    ${ }^{12}$ As the proof shows it is sufficient that $f$ has $\operatorname{dim} \cdot \operatorname{trg}(f)<\infty$.

[^11]:    

[^12]:    ${ }^{14}$ The quasi-pro-étale site is defined in [22, Definition 14.1.(ii)].

[^13]:    ${ }^{15}$ By Lemma 3.17 being $\ell$-bounded ascents along under quasi-compact, separated quasi-pro-étale maps
    ${ }^{16}$ For example, defined as the slice $*_{\text {proet }} / T$ with $*_{\text {proet }}$ the pro-étale site of a point, [3, Section 4.3].

[^14]:    ${ }^{17}$ This follows by an easy diagram chase from the fact that the morphism $\underline{H o m}_{\Lambda}(M, \Lambda) \otimes_{\Lambda} \Lambda / I \rightarrow$ $\underline{\operatorname{Hom}}_{\Lambda}(M, \Lambda / I)$ is an isomorphism as $\Lambda / I$ denotes the derived quotient for an implicit choice of generators of $I$. For underived quotients this statement is not true: If $\Lambda=\mathbb{F}_{\ell}[x] / x^{2}$, then the identity on $\mathbb{F}_{\ell}$ is trace class over $\mathbb{F}_{\ell}$, but not over $\Lambda$ as $\mathbb{F}_{\ell}$ is not a perfect complex over $\Lambda$.

[^15]:    ${ }^{18}$ This result is announced by Efimov, and follows from [8, Theorem 3.5] and the general theory of rigid categories as, e.g., presented in 5].
    ${ }^{19}$ This morphism is adjoint to $F(A \otimes G(N)) \cong A \otimes F \circ G(N) \rightarrow A \otimes N$, with the first isomorphism using $\mathcal{D}_{\text {nuc }}(\Lambda)$-linearity of $F$.

[^16]:    ${ }^{20}$ For this argument, we don't need any theory of $(\infty, 2)$-categories as the necessary natural transformation can be constructed directly by functoriality of $(-) \otimes_{\mathcal{D}_{\text {nuc }}\left(\mathbb{Z}_{p}[[\pi]]\right)} \mathcal{D}_{\hat{\emptyset}}^{a}\left(A^{+}\right)$.

