EXTENDING TORSORS ON THE PUNCTURED $\mathrm{Spec}(A_{\mathrm{inf}})$

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Abstract. For a parahoric group scheme over the ring of integers of a p-adic field we study the question whether a torsor defined on the punctured spectrum of Fontaine's ring A_{inf} extends to the whole spectrum. We obtain some partial results on this question. Using descent we can extend a similar result of Kisin and Pappas to some cases of wild ramification. Moreover, we treat similarly the case of equal characteristic. As an application of our results we present the construction of a canonical specialization map from the B_{dR}^+-affine Grassmannian to the Witt vector affine Grassmannian.

1. Introduction

Let E be a complete discretely valued field with ring of integers $\mathcal{O}_E$, perfect residue field k of characteristic p and let A be a local $\mathcal{O}_E$-algebra such that the restriction functor

$$\mathrm{Bun}(\mathrm{Spec}(A)) \cong \mathrm{Bun}(U)$$

from vector bundles on $\mathrm{Spec}(A)$ to vector bundles on the punctured spectrum

$$U := \mathrm{Spec}(A) \setminus \{s\},$$

with $s \in \mathrm{Spec}(A)$ the closed point, is an equivalence. Moreover, let

$$\mathcal{G}$$

be a parahoric group scheme over $\mathcal{O}_E$ with (reductive) generic fiber

$$G := \mathcal{G} \otimes_{\mathcal{O}_E} E.$$

In this paper we are interested in the question whether a given $\mathcal{G}$-torsor on U extends to $\mathrm{Spec}(A)$, at least for some specific rings A's. Namely, the situations we are interested in are given by

1) $A = A_E = W(\mathcal{O}_C)\hat{\otimes}_{W(k)}\mathcal{O}_E$ for C some perfect non-archimedean field C with ring of integers $\mathcal{O}_C$ such that $k \subseteq \mathcal{O}_C$ (cf. Lemma 4.1). If E is of mixed characteristic, then A_E is the period ring A_{inf} (associated with C and E) which was considered by Fontaine.
2) $A = R_E = \mathcal{O}_E[[z]]$ a ring of power series over $\mathcal{O}_E$ (cf. Lemma 8.1).

If E has mixed characteristic the case 2) has (basically) been treated in [14] if G splits over some tamely ramified extension of E. If E has equal characteristic the cases 1) and 2) appear in [11] and forthcoming work of Paul Breutmann. If E has mixed characteristic case 1) has applications to mixed characteristic affine Grassmannians (cf. Section 10 and [21, Section 21.2.]).

Unfortnately, we are not able to answer our question in full generality. Let us nevertheless describe our results more precisely, first in the case 1) where $A = A_E$.

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Let $\mathfrak{m}_C \subseteq \mathcal{O}_C$ be the maximal ideal and let $k' := \mathcal{O}_C/\mathfrak{m}_C$ be the residue field of $\mathcal{O}_C$. Let

$$\mathfrak{p}_{\mathrm{crys}} := \mathrm{Ker}(A_E \to W(k')\hat{\otimes}_{W(k)}\mathcal{O}_E$$

and define the "crystalline part" of U by the Zariski localization

$$U_{\mathrm{crys}} := \mathrm{Spec}(A_{E,\mathfrak{p}_{\mathrm{crys}}}).$$

Moreover, set

$$V := \mathrm{Spec}(A_E \otimes_{\mathcal{O}_E} E) \subseteq U.$$

We can now state our main result in the case 1). For this, let $\mathcal{C}$ be the class of pairs (G', E') where E' is a finite separable extension of E and G' a reductive group over $E/prime$ such that for every parahoric model $\mathcal{G}'$ of G' over $\mathcal{O}_{E'}$ every $\mathcal{G}'$-torsor on $U \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$ extends to $\mathrm{Spec}(A_{E'})$ (by Corollary 7.3 this is equivalent to $H^1(G', V \otimes_E E') = \{1\}$).

Theorem 1.1 (cf. Theorem 7.9)**.** 1) *Let $\mathcal{P}$ be a $\mathcal{G}$-torsor on U. Then $\mathcal{P}$ extends (necessarily uniquely) to* $\mathrm{Spec}(A_E)$ *if the restriction $\mathcal{P}_{|U_{\mathrm{crys}}}$ of $\mathcal{P}$ is trivial.*

2) *The class $\mathcal{C}$ is stable under Weil restrictions, central extensions, direct products and contains all pairs (G', E') with either G' split by an unramified extension of E', G' of type A or G' of PEL-type.*

Thus, the cases missing in our description are (essentially) non-trialitarian ramified outer forms of type D if k has characteristic 2, ramified triality groups (in any residue characteristic) and ramified outer forms of type E_6 (again in any characteristic). We note that in particular $\mathcal{C}$ contains all pairs (G', E') where G' is a torus. We prove Theorem 1.1 in several steps. First we reduce to the case that C is algebraically closed and prove the general criterion that $\mathcal{P}$ extends to $\mathrm{Spec}(A_E)$ if and only if $\mathcal{P}_{|V}$ is trivial (cf. Corollary 7.3). This crucially uses the assumption that $\mathcal{G}$ is parahoric. Then we handle the case that G is split by recalling an old argument of Colliot-Thélène and Sansuc (cf. Proposition 6.5). Using the special case $G = \mathrm{PGL}_n$ we can deduce the case of tori (cf. Proposition 7.6). From here it is then easy to deduce that a torsor extends if it is trivial on the crystalline part U_{crys} of U (cf. Theorem 7.9) and that $\mathcal{C}$ is stable under central extensions (cf. Lemma 7.8). The PEL-case of Theorem 1.1 follows from the work of Rapoport and Zink by a concrete description of torsors by lattice chains (cf. [18, Appendix to Chapter 3] and [21, Corollary 21.5.6.]). Finallly, building on the work of Daniel Kirch for even unitary groups (cf. [13]) the case of unitary groups (in arbitrary residue characteristic) can be handled by similarly describing torsors under some affine smooth model $\mathcal{G}_{\mathrm{std},n}$ by vector bundles plus linear algebra data (cf. Theorem 9.10). Here the main novelty is the introduction of a divided discriminant for hermitian quadratic forms of odd rank (cf. 9.4).

In the second case $A = R_E$ we (mostly) deduce our results from the case $A = A_E$ by descent (cf. Lemma 8.2 and Lemma 6.3). Let $\mathcal{D}$ be the class of pairs $(\mathcal{G}', E')$ where E'/E is a finite extension and $\mathcal{G}'$ is a parahoric group scheme over $\mathcal{O}_{E'}$ such that every $\mathcal{G}'$-torsor on the punctured spectrum $U_{E'} = \mathrm{Spec}(R_{E'}) \setminus \{s\}$ extends to $\mathrm{Spec}(R_{E'})$. For $\mathcal{G}'$ let

$$G' := \mathcal{G}' \otimes_{\mathcal{O}_{E'}} E'$$

be its reductive generic fiber. Our main theorem in this case is the following.

Theorem 1.2 (cf. Theorem 8.4). 1) *Let $\mathcal{P}$ be a $\mathcal{G}$-torsor on U. Then $\mathcal{P}$ extends (necessarily uniquely) to $\mathrm{Spec}(R_E)$ if the restriction $\mathcal{P}_{|\mathrm{Spec}(\mathrm{Frac}(R_E))}$ of $\mathcal{P}$ is trivial.*

2) *The class $\mathcal{D}$ is stable under Weil restrictions, direct products and contains all pairs $(\mathcal{G}', E')$ with either G' split by some tamely ramified extension of E', G' of type A, G' a torus or G' simply-connected.*

The case where G' is split by some tamely ramified extension (not containing factors of type E_8) has been handled in [14] (at least if E is of mixed characteristic) and thus our new contributions here are unitary groups in residue characteristic 2, tori and wildly ramified simply connected groups.

As a final application of our result we present in Section 10 the construction of a specialization map between mixed characteristic affine Grassmannianns.

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2. Notations

We fix the notation used throughout the paper. Let E be a discretely valued field with ring of integers $\mathcal{O}_E$, perfect residue field k of characteristic $p > 0$ and let $\pi \in \mathcal{O}_E$ be a uniformizer. We denote by $\mathcal{G}$ a parahoric group scheme over $\mathcal{O}_E$ and set

$$G := \mathcal{G} \otimes_{\mathcal{O}_E} E$$

for its (connected) reductive fiber over E. Let C be a perfect, complete non-archimedean extension such that $k \subseteq \mathcal{O}_C$ where $\mathcal{O}_C$ is the ring of integers of C. Let $\mathfrak{m}_C \subseteq \mathcal{O}_C$ be the maximal ideal of $\mathcal{O}_C$ and let $k' := \mathcal{O}_C/\mathfrak{m}_C$ be the residue field of $\mathcal{O}_C$. For a perfect ring S we denote its ring of Witt vectors by $W(S)$. Set

$$A_E := W(\mathcal{O}_C)\hat{\otimes}_{W(k)}\mathcal{O}_E.$$

Then

$$A_E \cong \begin{cases} \mathcal{O}_C[[\pi]] & \text{if } \mathrm{char}(E) > 0 \\ A_{\mathrm{inf}} \otimes_{W(k)} \mathcal{O}_E & \text{if } \mathrm{char}(E) = 0 \end{cases}$$

with $A_{\mathrm{inf}} = W(\mathcal{O}_C)$ Fontaine's ring associated with C. Let

$$[\cdot]\colon \mathcal{O}_C \to A_E$$

be the Teichmüller lift. Then every element $a \in A_E$ can be uniquely written as

$$a = \sum_{i=0}^{\infty} [a_i]\pi^i$$

with $a_i \in \mathcal{O}_C$. Let

$$s := s_{A_E} \in \mathrm{Spec}(A_E)$$

be the unique closed point given by the unique maximal ideal

$$\mathfrak{m} := \{ \sum_{n\geq 0}^{\infty} [a_n]\pi^n \mid a_0 \in \mathfrak{m}_C\}$$

of A_E. Let

$$U := U_{A_E} := \mathrm{Spec}(A_E) \setminus \{s\}$$

be the punctured spectrum of A_E. Moreover, set

$$V := V_{A_E} := \mathrm{Spec}(A_E[1/\pi]) \subseteq U.$$

Finally, define the "crystalline point"

$$\mathfrak{p}_{\mathrm{crys}} := \{\sum_{n\geq 0}^{\infty} [a_n]\pi^n \mid a_n \in \mathfrak{m}_C \text{ for all } n\}$$

and the "crystalline part"

$$U_{\mathrm{crys}} := \mathrm{Spec}(A_{E,\mathfrak{p}_{\mathrm{crys}}}) \subseteq U.$$

For a finite field extension E'/E we denote by $\mathcal{O}_{E'}$ its ring of integers. Note that

$$A_{E'} \cong A_E \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$$

and

$$U_{A'_E} = U_{A_E} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}, V_{A'_E} = V_{A_E} \otimes_E E', \text{ etc.}$$

If not stated explicitly otherwise, H^* will always mean étale cohomology.

3. The spectrum of A_E

We use the notation from Section 2. Moreover, we assume that C is algebraically closed. Recall that an element

$$\xi \in A_E$$

is called distinguished (or primitive) of degree 1 (cf. [8, Définition 2.2.1.]) if

$$\xi = u(\pi - [\varpi])$$

for some unit $u \in A_E^\times$ and some $\varpi \in \mathfrak{m}_C$.

For E of equal characteristic the next lemma can be found in [11, Lemma 8.3] as well.

Lemma 3.1. *The spectrum* $\mathrm{Spec}(A_E)$ *of* A_E *is given as*

$$\mathrm{Spec}(A_E) = U_{\mathrm{crys}} \cup \{\mathfrak{m}\} \cup \bigcup_{\substack{\xi \in A_E \\ \textit{distinguished of degree } 1}} \{(\xi)\}.$$

Proof. Let $\mathfrak{p} \subseteq A_E$ be an arbitrary prime ideal. If $\mathfrak{p}$ contains a distinguished element ξ of degree 1 (or equivalently some power), then $\mathfrak{p}$ lies in the subset

$$\mathrm{Spec}(A_E/(\xi)) \subseteq \mathrm{Spec}(A_E).$$

But (ξ) being distinguished of degree 1 implies that $A_E/(\xi)$ is isomorphic to the ring of integers $\mathcal{O}_{C^\#}$ for some non-archimedean field $C^\#$, possibly $C^\# \cong C$ (cf. [8, Corollaire 2.2.23]). The ring $\mathcal{O}_{C^\#}$ contains exactly two prime ideals, namely (0) and $\mathfrak{m}/(\xi)$. In particular, $\mathfrak{p} = (\xi)$ or $\mathfrak{p} = \mathfrak{m}$. Now assume that $\mathfrak{p}$ does not contain a distinguished element $\xi \in A_E$. We want to prove that

$$\mathfrak{p} \subseteq \mathfrak{p}_{\mathrm{cris}} = \{\sum_{n\geq 0} [x_i]\pi^i \mid x_i \in \mathfrak{m}_C \}.$$

Assume the contrary. Then

$$0 \neq (\mathfrak{p} + \mathfrak{p}_{\mathrm{cris}})/\mathfrak{p}_{\mathrm{cris}} \subseteq A_E/\mathfrak{p}_{\mathrm{cris}}$$

is non-zero and there exists an element $a \in \mathfrak{p}$ not mapping to zero in $A_E/\mathfrak{p}_{\mathrm{cris}}$. Write

$$a = \sum_{i=0}^{\infty} [x_i]\pi^i$$

with $x_i \in \mathcal{O}_C$. As $\pi \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime we can assume $x_0 \neq 0$ after dividing possibly by some power of π. Moreover, one x_i must be a unit in $\mathcal{O}_C$ as a does not map to 0 in $A_E/\mathfrak{p}_{\mathrm{cris}}$. In other words, a is primitive in the sense of [8, Définition 2.2.1.]. By [8, Théorème 2.4.1.] resp. [16] the element a can be written as a product

$$a = \prod_{i=1}^{n} a_i$$

for some distinguished elements a_i of degree 1 as we assumed that C is algebraically closed. As $\mathfrak{p}$ is a prime ideal, one of these a_i must lie in $\mathfrak{p}$ which is the contradiction we were looking for. This finishes the proof. □

We can record the following corollary.

Corollary 3.2. *Let $\xi \in A_E$ be a distinguished element of degree 1. Then the local ring*

$$A_{E,(\xi)}$$

is a discrete valuation ring.

Proof. Without knowing that the local ring $A_{E,(\xi)}$ is a discrete valuation ring it is known (by [8, Définition 2.7.1.]) that the ξ-adic completion of $A_{E,(\xi)}$ is a discrete valuation ring described (at least for $(\xi) \neq (\pi)$, but the remaining case $(\pi) = (\xi)$ is clear). Let $\mathfrak{p} \subseteq A_E$ be a prime ideal contained in (ξ), i.e., $\mathfrak{p}$ lies in the spectrum

$$\mathrm{Spec}(A_{E,(\xi)}) \subseteq \mathrm{Spec}(A_E)$$

of the localisation $A_{E,(\xi)}$. Every $a \in \mathfrak{p}$ can be written as

$$a = b\xi$$

for some $b \in A_E$. If we assume $\xi \notin \mathfrak{p}$, then

$$b = \frac{a}{\xi} \in \mathfrak{p},$$

i.e., $\xi\mathfrak{p} = \mathfrak{p}$. But A_E injects into the ξ-adic completion

$$R := (A_{E,(\xi)})^{\wedge}_{\xi}$$

which is a discrete valuation ring with uniformizer $\xi \in R$. Let

$$\mathfrak{q} = \mathfrak{p}R.$$

Then $\xi\mathfrak{q} = \mathfrak{q}$ which implies

$$\mathfrak{q} = 0$$

as R is a discrete valuation ring. But then $\mathfrak{p} = 0$ as well. In other words, we have proven that the spectrum

$$\mathrm{Spec}(A_{E,(\xi)}) = \{(\xi), (0)\}$$

contains exactly two prime ideals, both of which are finitely generated. By [10, Chapitre 0, Proposition (6.4.7.)] this implies that $A_{E,(\xi)}$ is noetherian and then more precisely a discrete valuation ring. □

We remark that the subset

$$U_{\mathrm{crys}} \subseteq \mathrm{Spec}(A_E)$$

remains mysterious. For example it contains the non-closed prime ideal

$$\bigcup_{\varpi \in \mathfrak{m}_C} [\varpi] A_E \subsetneq \mathfrak{p}_{\mathrm{cris}}$$

(cf. [8, Section 1.10.4.]). In particular, the Krull dimension of A_E is at least 3. 1

4. Some commutative algebra over A_E

We shortly want to mention some results on commutative algebra over A_E generalizing those in [3, Chapter 4]. Recall that $\pi \in \mathcal{O}_E$ is a uniformizer, that $s \in \mathrm{Spec}(A_E)$ denotes the unique closed point of $\mathrm{Spec}(A_E)$ and that $U = \mathrm{Spec}(A_E) \backslash \{s\}$ is the punctured spectrum of A_E. The proof of [3, Lemma 4.6.] generalizes to the following lemma.

Lemma 4.1. *The restriction of vector bundles induces an equivalence of categories between vector bundles on* $\mathrm{Spec}(A_E)$ *and vector bundles on* U*. In particular, all vector bundles on* U *are free.*

Proof. Replacing p by π in [3, Lemma 4.6.] the same proof works and we refer the reader to its proof. □

The next corollary is (nearly) [3, Corollary 4.12.].

Corollary 4.2. *Let* N *be a finite projective* $A_E[1/\pi]$*-module. Then* N *is free.*

Proof. Let $M \subset N$ be a finitely generated $A_E[1/\pi]$-submodule such that $M[1/\pi] = N$. The localisation

$$A_{E,(\pi)}$$

of A_E at the prime ideal (π) for $\pi \in \mathcal{O}_E$ a uniformizer is a discrete valuation ring (cf. Corollary 3.2 or [3, Lemma 4.10]). As M is finitely generated and π-torsion free the localized module $M \otimes_{A_E} A_{E,(\pi)}$ is finite free. Using Beauville-Laszlo (cf. [2]) (and that $\mathrm{Spec}(A_E/p) \cong \mathrm{Spec}(\mathcal{O}_C)$ has exactly two points) the quasi-coherent sheaf $\mathcal{M}$ on $\mathrm{Spec}(A_E)$ defined by M restricts thus to a vector bundle on the punctured spectrum U. By Lemma 4.1 this vector bundle is trivial which implies that N is already free. □

In particular, we can conclude that every line bundle on U resp. V is trivial, i.e.,

$$H^1(U, \mathbb{G}_m) = H^1(V, \mathbb{G}_m) = \{1\}.$$

In Theorem 7.9 we have need for the following lemma.

Lemma 4.3. *Let* M *be a finitely presented* A_E*-module of projective dimension* 1*, i.e., there exists an exact sequence* $0 \to F_1 \to F_2 \to M \to 0$ *with* F_1, F_2 *finite free over* A_E*, such that* $\pi M = 0$*. Then* M *is free over* $A_E/\pi \cong \mathcal{O}_C$*.*

Proof. As $\mathcal{O}_C$ is a valuation ring it suffices to show that M is $[\varpi]$-torsion free, where $\varpi \in \mathfrak{m}_C \setminus \{0\}$ is a pseudo-uniformizer. The sequence $(\pi, [\varpi])$ on A_E is regular, thus the Koszul complex

$$0 \to A_E \xrightarrow{(-[\varpi],\pi)} A_E \oplus A_E \xrightarrow{(\pi,[\varpi])} A_E$$

is a resolution of $A_E/(\pi,[\varpi])$. In particular, we see

$$\mathrm{Ext}^i_{A_E}(A_E/(\pi,[\varpi]),F)=0$$

for every finite free A_E-module F and $i=0,1$. Assume that $m\in M$ is $[\varpi]$-torsion. Then m is in the image of some homomorphism

$$f\colon A_E/(\pi,[\varpi])\to M$$

as M is killed by π. Taking $\mathrm{Ext}^*_{A_E}(A_E/(\pi),-)$ of the short exact sequence

$$0\to F_1\to F_2\to M\to 0$$

yields an exact sequence

$$\mathrm{Hom}_{A_E}(A_E/(\pi,[\varpi]),F_2)\to\mathrm{Hom}_{A_E}(A_E/(\pi,[\varpi]),M)\to\mathrm{Ext}^1_{A_E}(A_E/(\pi,[\varpi]),F_1)$$

where the outer terms are trivial as was shown above. In particular, f and hence m are zero. □

5. Generalities on torsors

In this section we collect some general facts about torsors we will use later. The following theorem of Steinberg will be very important for us.

Theorem 5.1. *Let K be a field of characteristic p, such that K is of dimension ≤ 1, i.e., for every finite field extension K'/K the Brauer group $\mathrm{Br}(K')$ vanishes. Then for every (connected) reductive group G/K the cohomology set $H^1(K,G)=\{1\}$ is trivial.*

Proof. This is [22, Chapitrie III.2.3, Théorème 1′] (noting [22, Remarques 1)] following it). □

For example, fields complete under a discrete valuation whose residue field is algebraically closed are of dimension 1 (cf. [22, Chapitre II.3.3.c)]).

We now want to discuss shortly the Beauville-Laszlo glueing for torsors. Thus we consider the following situation.

Let A be a ring and let $f\in A$ be a non-zero divisor. Let A_f be the localisation of A at f and let $\widehat{A}$ be the f-adic completion of A. Moreover, let

$$\mathcal{G}\to\mathrm{Spec}(A)$$

be an affine, flat group scheme over A. Then we have the following immediate consequence of the Beauville-Laszlo glueing lemma (cf. [2]).

In the following, "torsor" means "torsor for the fpqc-topology".

Lemma 5.2. *Sending a $\mathcal{G}$-torsor $\mathcal{P}$ on $\mathrm{Spec}(A)$ to*

$$(\mathcal{P}_1:=\mathcal{P}_{|\mathrm{Spec}(A_f)},\mathcal{P}_2:=\mathcal{P}_{|\mathrm{Spec}(\widehat{A})},\alpha\colon\mathcal{P}_{1|\mathrm{Spec}(\widehat{A}[1/f])}\cong\mathcal{P}_{|\mathrm{Spec}(\widehat{A}[1/f])}\cong\mathcal{P}_{2|\mathrm{Spec}(\widehat{A}[1/f]})$$

defines an equivalence between the groupoid of $\mathcal{G}$-torsors on $\mathrm{Spec}(A)$ and the category of triples

$$(\mathcal{P}_1,\mathcal{P}_2,\alpha)$$

with $\mathcal{P}_1$ a $\mathcal{G}_{|\mathrm{Spec}(A_f)}$-torsor on $\mathrm{Spec}(A_f)$, $\mathcal{P}_2$ a $\mathcal{G}_{|\mathrm{Spec}(\widehat{A})}$-torsor on $\mathrm{Spec}(\widehat{A})$ and $\alpha\colon\mathcal{P}_{1|\mathrm{Spec}(\widehat{A}[1/f])}\cong\mathcal{P}_{2|\mathrm{Spec}(\widehat{A}[1/f])}$ an isomorphism.

Proof. From [2] one can conclude that the category of flat A-modules M is equivalent to the category of triples

$$(M_f, \widehat{M}, \alpha)$$

with M_f a flat A_f-module, $\widehat{M}$ a flat $\widehat{A}$-module and $\alpha\colon M_f \otimes_{A_f} \widehat{A}[1/f] \cong \widehat{M}[1/f]$ an isomorphism. This equivalence respects tensor products and hence induces an equivalence on algebra/coalgebra objects. Moreover, a faithfully flat affine scheme X over $\mathrm{Spec}(A)$ with an action by $\mathcal{G}$ is a $\mathcal{G}$-torsor for the fpqc-topology if and only if the canonical morphism

$$\mathcal{G} \times_{\mathrm{Spec}(A)} X \to X \times_{\mathrm{Spec}(A)} X,\ (g, x) \mapsto (gx, x)$$

is an isomorphism. This condition can be phrased in terms of coordinate rings and hence we obtain the lemma. □

If $\mathcal{G}$ is smooth over $\mathrm{Spec}(A)$, then every fpqc-torsor is actually trivial for the étale topology and we obtain Lemma 5.2 with "fpqc" replaced by "étale". In fact, if $\mathcal{G}$ is smooth, then every $\mathcal{G}$-torsor $\mathcal{P}$ is smooth and thus admits sections étale locally.

For $A = A_E$ (equipped with the $(\pi, [\varpi])$-adic topology for some pseudo-uniformizer C) we shortly discuss a comparison for "algebraic" and "adic torsors". We recommend [21, Appendix to lecture XIX] for a discussion of torsors over adic spaces.

Proposition 5.3. *Let $s \in \mathrm{Spec}(A_E)$ resp. $s' \in \mathrm{Spa}(A_E, A_E)$ be the closed point, where $\mathrm{Spa}(A_E, A_E)$ denotes the adic spectrum of A_E. Then for every affine smooth group scheme $\mathcal{G}/\mathcal{O}_E$ the groupoids of $\mathcal{G}$-torsors on $U := \mathrm{Spec}(A_E) \setminus \{s\}$ and $\mathcal{G}^{\mathrm{adic}}$-torsors on $\mathcal{U} := \mathrm{Spa}(A_E, A_E) \setminus \{s'\}$ are canonically equivalent.*

Proof. By [20, Theorem 14.1.2] (resp. [21, Theorem 14.2.1]) and [20, Lemma 14.2.1] (resp. [21, Lemma 14.2.3.] , cf. Lemma 4.1) there are natural equivalences

$$\mathrm{Bun}(\mathcal{U}) \cong \mathrm{Bun}(\mathrm{Spec}(A_E))$$

and

$$\mathrm{Bun}(U) \cong \mathrm{Bun}(\mathrm{Spec}(A_E))$$

(the same proof works if E is of equal characteristic). Using [21, Theorem 19.5.2.] we therefore have to prove that

$$\mathrm{Bun}(\mathcal{U}) \cong \mathrm{Bun}(U)$$

as *exact* categories, namely by [21, Theorem 19.5.2.] the groupoid of $\mathcal{G}$-torsors identifies with the groupoid of fiber functors on $\mathrm{Rep}_{\mathcal{O}_E}(\mathcal{G})$ over U resp. $\mathcal{U}$. Let $u := [\varpi] \in A_E$ be the Teichmüller lift of some $\varpi \in \mathfrak{m}_C \setminus \{0\}$. If

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

is an exact sequence of vector bundles on U, then setting $M_i = H^0(U, \mathcal{M}_i)$ we obtain an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to Q \to 0$$

of A_E-modules where the finitely presented Q is killed by some power of the ideal (π, u). For every affinoid $\mathrm{Spa}(B, B^+) \subseteq \mathcal{U}$ either π or u is invertible on B. In particular,

$$\mathrm{Tor}_i^{A_E}(B, Q) = 0$$

for $i \geq 0$, which shows that

$$0 \to B \otimes_{A_E} M_1 \to B \otimes_{A_E} M_2 \to B \otimes_{A_E} M_3 \to 0$$

is exact. This proves that the functor

$$\mathrm{Bun}(U) \to \mathrm{Bun}(\mathcal{U})$$

is exact. Conversely, assume that

$$0 \to \mathcal{N}_1 \to \mathcal{N}_2 \to \mathcal{N}_3 \to 0$$

is an exact sequence of vector bundles on $\mathcal{U}$. Let $N_i = H^0(\mathcal{U}, \mathcal{N}_i)$ be the associated finite free A_E-modules under the equivalence $\mathrm{Bun}(\mathcal{U}) \cong \mathrm{Bun}(\mathrm{Spec}(A_E))$ from [20, Theorem 14.1.2] (resp. [21, Theorem 14.2.1.]). Set

$$\begin{gathered} \mathcal{U}_1 := \{\ |\pi| \leq |u| \neq 0\} \subseteq \mathcal{U} \\ \mathcal{U}_2 := \{\ |u| \leq |\pi| \neq 0\} \subseteq \mathcal{U} \\ \mathcal{U}_{12} := \mathcal{U}_1 \cap \mathcal{U}_2. \end{gathered}$$

By definition we obtain a diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & N_3 \arrow[d] & \\
0 \arrow[r] & \begin{smallmatrix} H^0(\mathcal{U}_1,\mathcal{N}_1) \\ \oplus H^0(\mathcal{U}_2,\mathcal{N}_1) \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} H^0(\mathcal{U}_1,\mathcal{N}_2) \\ \oplus H^0(\mathcal{U}_2,\mathcal{N}_2) \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} H^0(\mathcal{U}_1,\mathcal{N}_3) \\ \oplus H^0(\mathcal{U}_2,\mathcal{N}_3) \end{smallmatrix} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathcal{U}_{12}, \mathcal{N}_1) \arrow[r] \arrow[d] & H^0(\mathcal{U}_{12}, \mathcal{N}_2) \arrow[r] \arrow[d] & H^0(\mathcal{U}_{12}, \mathcal{N}_3) \arrow[r] \arrow[d] & 0 \\
 & H^1(\mathcal{U}, \mathcal{N}_1) \arrow[r] \arrow[d] & H^1(\mathcal{U}, \mathcal{N}_2) \arrow[r] \arrow[d] & H^1(\mathcal{U}, \mathcal{N}_3) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

After inverting π the groups $H^1(\mathcal{U}, \mathcal{N}_i)$ vanish. In fact, they are (as in the proof of [20, Theorem 14.1.2.] resp. [21, Theorem 14.2.1.]) given by $H^1(\mathrm{Spa}(\tilde{A}[1/\pi]), \tilde{\mathcal{N}}_i)$ with $\tilde{\mathcal{N}}_i$ a vector bundle on the affinoid adic space

$$\mathrm{Spa}(\tilde{A}[1/\pi]) = \{\ x \in \mathrm{Spa}(\tilde{A}, \tilde{A})\ |\ |\pi(x)| \neq 0\},$$

where $\tilde{A} = A_E$ but equipped with the π-adic topology. Namely, consider the spaces

$$\tilde{\mathcal{U}}_1 := \{|\pi| \leq |u| \neq 0\} \subseteq \mathrm{Spa}(\tilde{A}[1/\pi])$$

and

$$\tilde{\mathcal{U}}_2 := \{|u| \leq |\pi| \neq 0\} \subseteq \mathrm{Spa}(\tilde{A}[1/\pi]).$$

Then

$$\tilde{\mathcal{U}}_2 \cong \mathcal{U}_2$$

and

$$H^0(\tilde{\mathcal{U}}_1, \mathcal{O}_{\tilde{\mathcal{U}}_1}) \cong H^0(\mathcal{U}_1, \mathcal{O}_{\mathcal{U}_1})[1/\pi]$$

by [21, Lemma 14.3.1.]. Moreover,

$$\tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2 \cong \mathcal{U}_1 \cap \mathcal{U}_2.$$

Thus the modules $H^0(\mathcal{U}_1, \mathcal{N}_i)[1/\pi]$ and $H^0(\mathcal{U}_2, \mathcal{N}_i)$ glue to a vector bundle $\tilde{\mathcal{N}}_i$ on $\mathrm{Spa}(\tilde{A}[1/\pi])$ as

$$\mathrm{Spa}(\tilde{A}[1/\pi])$$

is sheafy (cf. [21, Proof of Proposition 13.1.1.]). By sheafiness we can thus conclude that

$$H^1(\mathcal{U}, \mathcal{N}_i)[1/\pi] \cong H^1(\mathrm{Spa}(\tilde{A}[1/\pi]), \tilde{\mathcal{N}}_i) = 0$$

(cf. [21, Theorem 5.2.6.]). This implies that the cokernel Q of $N_2 \to N_3$ is π-torsion. Hence, to show that it vanishes on U it suffices to prove that

$$Q \otimes_{A_E} R = 0$$

where R is the π-adic completion of $A_E[1/u]$. But R is flat over A_E and by assumption the sequence

$$0 \to N_1 \otimes_{A_E} R \to N_2 \otimes_{A_E} R \to N_3 \otimes_{A_E} R \to 0$$

is exact it identifies with the π-adic completion of the stalk at $(\pi) \in \mathcal{U}$ of the exact sequence

$$0 \to \mathcal{N}_1 \to \mathcal{N}_2 \to \mathcal{N}_3 \to 0.$$

This finishes the proof. □

6. Generalities on extending torsors

We want to draw some consequences of Lemma 4.1 in a more abstract setup. For this let A be any ring and let $U \subseteq \mathrm{Spec}(A)$ be a quasi-compact open subset. We assume that the restriction functor for vector bundles

$$\mathrm{Bun}(\mathrm{Spec}(A)) \xrightarrow{\cong} \mathrm{Bun}(U)$$

is an equivalence.

For example, A can be A_E (cf. Lemma 4.1) or a two-dimensional regular local ring (cf. Lemma 8.1). Even under this general assumption, we can conclude that the functor inverse to restriction must send a vector bundle $\mathcal{V}$ on U to the quasi-coherent module on $\mathrm{Spec}(A)$ associated with the A-module

$$H^0(U, \mathcal{V})$$

because if $\mathcal{V}'$ denotes a preimage of $\mathcal{V}$, i.e., $\mathcal{V}'$ is a vector bundle on $\mathrm{Spec}(A)$ and $\mathcal{V}'_{|U} \cong \mathcal{V}$, then

$$H^0(U, \mathcal{V}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{V}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}(A)}}(\mathcal{O}_{\mathrm{Spec}(A)}, \mathcal{V}'(\mathrm{Spec}(A))) \cong \mathcal{V}'(\mathrm{Spec}(A)).$$

In particular, we obtain that for $\mathcal{V}$ a vector bundle on U the global sections $H^0(U, \mathcal{V})$ are finite locally free over A.

Corollary 6.1. *Let A be as above and let $f\colon X \to \mathrm{Spec}(A)$ be an affine morphism and let $g\colon U \to X_U := X \times_{\mathrm{Spec}(A)} U$ be a section of X over U. Then g extends uniquely to a section $g'\colon \mathrm{Spec}(A) \to X$.*

Proof. This is a consequence of the adjunction for the Spec-functor. Namely write $X = \mathrm{Spec}(B)$ for some A-algebra B. Then

$$\begin{array}{rl} & \mathrm{Hom}_{\mathrm{Spec}(A)}(U, X) \\ \cong & \mathrm{Hom}_A(B, \Gamma(U, \mathcal{O}_U)) \\ \cong & \mathrm{Hom}_A(B, A) \\ \cong & \mathrm{Hom}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A), \mathrm{Spec}(B)) \end{array}$$

because $\Gamma(U, \mathcal{O}_U) \cong A$. □

Proposition 6.2. *Let A be as above. Then the base change functor*

$$\begin{array}{ccc} \{X \to \mathrm{Spec}(A) \text{ affine and flat }\} & \to & \{Y \to U \text{ affine and flat }\} \\ X & \mapsto & X \times_{\mathrm{Spec}(A)} U \end{array}$$

is fully faithful and its essential image consists of all affine and flat morphisms $g\colon Y \to U$ such that $H^0(Y, \mathcal{O}_Y)$ is a flat $A = H^0(U, \mathcal{O}_U)$-module.

Proof. We prove that the base change $X \mapsto X \times_{\mathrm{Spec}(A)} U$ induces an equivalence of the categories

$$\mathcal{C} := \{X \to \mathrm{Spec}(A) \text{ affine and flat}\}$$

and

$$\mathcal{D} := \{g\colon Y \to U \text{ affine and flat such that } H^0(Y, \mathcal{O}_Y) \text{ is flat over } A\}.$$

In fact, for $g\colon Y \to U$ in $\mathcal{D}$ we set

$$X := \mathrm{Spec}(H^0(Y, \mathcal{O}_Y))$$

for the flat A-algebra $H^0(Y, \mathcal{O}_Y)$, which defines a functor from $\mathcal{D}$ to $\mathcal{C}$. For

$$f\colon X \to \mathrm{Spec}(A)$$

affine and flat with base change $g\colon X \times_{\mathrm{Spec}(A)} U \to U$ let us write

$$f_* \mathcal{O}_X = \varinjlim \mathcal{V}_i$$

for vector bundles $\mathcal{V}_i$ on $\mathrm{Spec}(A)$ (using Lazard's theorem [23, Tag 058G]) and compute

$$H^0(U, g_*(\mathcal{O}_X)) = H^0(U, \varinjlim \mathcal{V}_{i|U}) = \varinjlim \mathcal{V}_i(\mathrm{Spec}(A)) = \mathcal{O}_X(X)$$

where we used our assumption on A and that U and g are quasi-compact (and quasi-separated) to commute global sections resp. the direct image and filtered colimits. On the other hand, if we start with

$$g\colon Y \to U$$

in $\mathcal{D}$, then we obtain a canonical morphism

$$Y \to \mathrm{Spec}(H^0(U, g_*(\mathcal{O}_Y)))$$

over $\mathrm{Spec}(A)$ which restricts to the isomorphism (as $g\colon Y \to U$ is affine)

$$Y \cong \underline{\mathrm{Spec}}(g_*(\mathcal{O}_Y))$$

over U. This finishes the proof. □

Of course, Proposition 6.2 fails without flatness. An example of an affine flat morphism $g\colon Y \to U$ which does not extend to an affine and flat scheme over $\mathrm{Spec}(A)$ can be given as follows (more examples are provided by Example 7.4). Let

$$A' = k[x, y]$$

be the polynomial ring and consider the A'-algebra

$$B' := k[u, v]$$

with $u = x$ and $v = \frac{y}{x}$ (an affine chart of the blow-up of A' in (x, y)). Set A as the localisation of A' in (x, y) and $B := A \otimes_{A'} B'$. Then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat when restricted to $U = \mathrm{Spec}(A) \setminus \{(x, y)\}$ but B is not flat over A.

Using Proposition 6.2 we arrive at the following descent criterion for extending some affine and flat morphism $Y \to U$ to $\mathrm{Spec}(A)$.

Now let $A \to A'$ be some morphism such that the restriction functor

$$\mathrm{Bun}(\mathrm{Spec}(A')) \xrightarrow{\cong} \mathrm{Bun}(U')$$

of vector bundles from $\mathrm{Spec}(A)$ to $U' := U \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ is an equivalence.

Lemma 6.3. *With the notations from above assume furthermore that A' is faithfully flat over A. Let $Y \to U$ be an affine and faithfully flat morphism and assume that*

$$Y \times_U U' \to U'$$

extends to some affine faithfully flat $\mathrm{Spec}(A')$*-scheme. Then Y extends to an affine faithfully flat* $\mathrm{Spec}(A)$*-scheme.*

Proof. Indeed, by Proposition 6.2 we have to check whether the global sections

$$H^0(Y, \mathcal{O}_Y)$$

are faithfully flat over A. But this can be checked after the faithfully flat base change $A \to A'$ and over A' it holds true after assumption and Proposition 6.2. □

Now let $\mathcal{G}/\mathrm{Spec}(A)$ be an affine flat group scheme over $\mathrm{Spec}(A)$.

Lemma 6.4. *The restriction functor from $\mathcal{G}$-torsors on* $\mathrm{Spec}(A)$ *to $\mathcal{G}_{|U}$-torsors on U is fully faithful with essential image given by $\mathcal{G}_{|U}$-torsors $\mathcal{P}$ on U such that the global sections $H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}})$ are faithfully flat over A.*

Proof. Each $\mathcal{G}$-torsor on $\mathrm{Spec}(A)$ is represented by some affine, faithfully flat scheme over $\mathrm{Spec}(A)$. Therefore we may apply Proposition 6.2 to conclude fully faithfulness of restrictions of $\mathcal{G}$-torsors. Let $\mathcal{P}$ be a $\mathcal{G}_{|U}$-torsor on U such that the global sections of $\mathcal{P}$ are faithfully flat over A. By Proposition 6.2 the underlying scheme of $\mathcal{P}$ extends to a faithfully flat scheme $\mathcal{P}'$ over $\mathrm{Spec}(A)$. By fully faithfulness of restrictions the $\mathcal{G}_{|U}$-action extends to $\mathcal{P}'$. That $\mathcal{P}'$ is a $\mathcal{G}$-torsors can then again be checked after restricting to U. This finishes the proof. □

In the reductive case $\mathcal{G}$-torsors on U will automatically extend to $\mathrm{Spec}(A)$. We recall the argument of Colliot-Thél'ene and Sansuc (cf. [7, Théorème 6.13]) (at least if A is local).

Proposition 6.5. *Assume that A is local and that $\mathcal{G}$ is a reductive group scheme over A. Let $\mathcal{P}$ be a $\mathcal{G}_{|U}$-torsor on U. Then $\mathcal{P}$-extends (uniquely) to* $\mathrm{Spec}(A)$.

Proof. By [1, Corollary 9.7.7.] there exists an embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_n$ such that the quotient $\mathrm{GL}_n/\mathcal{G}$ is affine. Now consider the exact sequence (of cohomology sets taken for the étale topology)

$$H^0(U, \mathrm{GL}_n/\mathcal{G}) \xrightarrow{\delta} H^1(U, \mathcal{G}) \to H^1(U, \mathrm{GL}_n).$$

As we assumed that A is local $H^1(U, \mathrm{GL}_n) = \{1\}$ and thus the class of $\mathcal{P} \in H^1(U, \mathcal{G})$ lies in the image of δ. But as $\mathrm{GL}_n/\mathcal{G}$ is affine

$$H^0(U, \mathrm{GL}_n/\mathcal{G}) = H^0(\mathrm{Spec}(A), \mathrm{GL}_n/\mathcal{G})$$

by Corollary 6.1 and thus the morphism δ factors as desired over $H^1(\mathrm{Spec}(A), \mathcal{G})$ by naturality of the connecting morphism. □

7. Extending torsors to $\mathrm{Spec}(A_E)$

In this section we want to prove the main theorem Theorem 1.1 (resp. Theorem 7.9) from the introduction on extending torsors which are defined on the punctured spectrum

$$U_E = \mathrm{Spec}(A_E) \setminus \{s\}$$

of A_E. We continue to use the notation from Section 2. In particular, we use the notation $\mathcal{G}/\mathcal{O}_E$ for a parahoric group scheme over $\mathcal{O}_E$ with reductive generic fiber G/E.

From our general discussion of extending torsors and Lemma 4.1 we can conclude that the restriction functor

$$\{\mathcal{G} - \text{torsors on } \mathrm{Spec}(A_E)\} \to \{\mathcal{G} - \text{torsors on } U\}$$

is fully faithful and an equivalence if $\mathcal{G}$ is reductive (cf. Lemma 6.4 and Proposition 6.5).

For the moment let us shortly denote A_E by $A_{E,C}$ and fix an extension C'/C of perfect non-archimedean fields over k. Set

$$A_{E,C'} := W(\mathcal{O}_{C'})\hat{\otimes}_{W(k)}\mathcal{O}_E.$$

We obtain the following descent statement.

Lemma 7.1. *A $\mathcal{G}$-torsor $\mathcal{P}$ on the punctured spectrum of* $\mathrm{Spec}(A_{E,C})$ *extends to* $\mathrm{Spec}(A_E)$ *if the base change of $\mathcal{P}$ to the punctured spectrum of* $\mathrm{Spec}(A_{E,C'})$ *does.*

Proof. By [21, Theorem 19.5.1.] the torsor $\mathcal{P}$ defines an exact tensor functor

$$\omega\colon \mathrm{Rep}_{\mathcal{O}_E}(\mathcal{G}) \to \mathrm{Bun}(U)$$

from the category $\mathrm{Rep}_{\mathcal{O}_E}(\mathcal{G})$ of representations of $\mathcal{G}$ on finite free $\mathcal{O}_E$-modules to the category of vector bundles on U. By Lemma 4.1

$$\mathrm{Bun}(U) \cong \mathrm{Bun}(\mathrm{Spec}(A_{E,C}))$$

and thus (by [21, Theorem 19.5.1.] again) it suffices to show that the functor

$$\omega'\colon \mathrm{Rep}_{\mathcal{O}_E}(\mathcal{G}) \to \mathrm{Bun}(\mathrm{Spec}(A_{E,C}))$$

induced by ω is exact. As the functor

$$\mathrm{Bun}(U) \to \mathrm{Bun}(\mathrm{Spec}(A_{E,C})),\ \mathcal{V} \mapsto H^0(U, \mathcal{V})$$

is left exact it suffices to prove right exactness of ω'. Let U' be the punctured spectrum of $A_{E,C'}$. Then the diagram

$$\begin{tikzcd} \mathrm{Bun}(U) \arrow[r, "H^0(U,-)"] \arrow[d] & \mathrm{Bun}(\mathrm{Spec}(A_{E,C}) \arrow[d] \\ \mathrm{Bun}(U') \arrow[r, "H^0(U',-)"] & \mathrm{Bun}(\mathrm{Spec}(A_{E,C'}) \end{tikzcd}$$

commutes because it does when $H^0(U,-)$ resp. $H^0(U',-)$ are replaced by their inverses (which are restriction of vector bundles to U resp. U'). By the assumption that the base change of $\mathcal{P}$ to U' extends to $\mathrm{Spec}(A_{E,C'})$ we can conclude that the composition

$$\omega' \otimes_{A_{E,C}} A_{E,C'} \colon \mathrm{Rep}_{\mathcal{O}_E}(\mathcal{G}) \to \mathrm{Bun}(\mathrm{Spec}(A_{E,C'}))$$

is exact. Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence in $\mathrm{Rep}_{\mathcal{O}_E}(\mathcal{G})$ and let Q be the cokernel of $\omega'(V_2) \to \omega'(V_3)$. It suffices to show that $Q = 0$. As Q is finitely generated it suffices to show that

$$Q/\pi Q = Q \otimes_{A_{E,C}} \mathcal{O}_C = 0$$

by Nakayama's lemma. But $\mathcal{O}_C \to \mathcal{O}_{C'}$ is faithfully flat and

$$(Q \otimes_{A_{E,C}} \mathcal{O}_C) \otimes_{\mathcal{O}_C} \mathcal{O}_{C'} = (Q \otimes_{A_{E,C}} A_{E,C'}) \otimes_{A_{E,C'}} \mathcal{O}_{C'} = 0$$

as we know that $Q \otimes_{A_{E,C}} A_{E,C'} = 0$. This finishes the proof. □

Thus from now on we may (and do) assume that C is algebraically closed. By [5, 4.6.20.Remarques] the base change of a parahoric group scheme along an étale extension $\mathcal{O}_E \to \mathcal{O}_{E'}$ is again parahoric. The same holds for passing to the completion. Thus from now on we may further assume that $\mathcal{O}_E$ is π-adically complete and strictly henselian, i.e. that k is algebraically closed. Under these assumptions E is of dimension 1 (in fact C1, cf. [15]) and thus every reductive group over E is automatically quasi-split by Steinberg's theorem (applied to the adjoint quotient). Moreover, the π-adic completion of $A_E[1/[\varpi]]$ (for $\varpi \in \mathfrak{m}_C$ non-zero) will be the complete discrete valuation ring

$$\mathcal{O}_{\mathcal{E}} := W(C)\hat{\otimes}_{W(k)}\mathcal{O}_E$$

with algebraically closed residue field C. Hence, its fraction field

$$\mathcal{E} := W(C)\hat{\otimes}_{W(k)}E$$

will again be of dimension 1. This observation will be crucial as by Steinberg's theorem it implies that every G-torsor on $\mathcal{E}$ is trivial (cf. Theorem 5.1). As C is algebraically closed the ring A_E is moreover strictly henselian. Hence we can conclude that a $\mathcal{G}$-torsor over U extends to $\mathrm{Spec}(A_E)$ if and only if it is trivial. Let us start the question on extending torsors by clarifying the assumption that $\mathcal{G}$ is parahoric and not some arbitrary affine smooth model of G.

Proposition 7.2. *The double coset space*

$$\mathcal{G}(\mathcal{O}_{\mathcal{E}})\backslash G(\mathcal{E})/G(A_E[1/\pi]) = \{1\}$$

is trivial.

Proof. The argument in [14, Proposition 1.4.3. Step 3] works in our situation, however using affine Grassmannians we can give a simpler and more conceptual argument. For this consider the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ of $\mathcal{G}$, i.e., the (étale) sheafification of the presheaf

$$R \mapsto G(W(R)\hat{\otimes}_{W(k)}E)/\mathcal{G}(W(R)\hat{\otimes}_{W(k)}\mathcal{O}_E)$$

on the category of perfect k-algebras. As $\mathcal{G}$ is parahoric the sheaf $\mathrm{Gr}_{\mathcal{G}}$ is represented by an ind-perfectly proper (strict) ind-scheme (cf. [24, (1.4.2.)] resp. [19, Corollary 1.3]). In particular it satisfies the valuative criterion for properness. Moreover, as C and $\mathcal{O}_C$ are strictly henselian

$$\mathrm{Gr}_{\mathcal{G}}(C) = G(\mathcal{E})/\mathcal{G}(\mathcal{O}_{\mathcal{E}})$$

resp.

$$\mathrm{Gr}_{\mathcal{G}}(\mathcal{O}_C) = G(A_E[1/\pi])/\mathcal{G}(A_E).$$

The claim follows from applying the valuative criterion for properness to $\mathrm{Gr}_{\mathcal{G}}$:

$$\mathrm{Gr}_{\mathcal{G}}(C) = \mathrm{Gr}_{\mathcal{G}}(\mathcal{O}_C).$$

□

From Proposition 7.2 we can conclude the following useful criterion.

Corollary 7.3. *Let $\mathcal{P}$ be a $\mathcal{G}$-torsor over U. Then $\mathcal{P}$ extends to* $\mathrm{Spec}(A_E)$ *if and only if the G-torsor*

$$\mathcal{P}_{|V}$$

over $V = \mathrm{Spec}(A_E[1/\pi])$ is trivial. Moreover, every $\mathcal{G}$-torsor on U extends to $\mathrm{Spec}(A_E)$ *if and only if*

$$H^1(V, G) = \{1\}.$$

Proof. By Beauville-Laszlo glueing (cf. [2] resp. Lemma 5.2) there is a bijection of isomorphism classes of $\mathcal{G}$-torsors $\mathcal{P}'$ on U which are trivial on $\mathrm{Spec}(A_E[1/\pi])$ and $\mathrm{Spec}(\mathcal{O}_{\mathcal{E}})$ with the double cosets

$$\mathcal{G}(\mathcal{O}_{\mathcal{E}})\backslash G(\mathcal{E})/G(A_E[1/\pi]).$$

But $\mathcal{O}_{\mathcal{E}}$ is strictly henselian which implies that every $\mathcal{G}$-torsor over $\mathrm{Spec}(\mathcal{O}_{\mathcal{E}})$ is trivial as $\mathcal{G}$ is smooth. With Proposition 7.2 we can conclude the first assertion. Let us prove the second. If

$$H^1(V, G) = \{1\},$$

then by what we have shown so far, every $\mathcal{G}$-torsor on U extends to $\mathrm{Spec}(A_E)$, i.e. is trivial. Conversely, let $\mathcal{P}'$ be a G-torsor on V. As the field $\mathcal{E}$ is of dimension 1 the base change of $\mathcal{P}'$ to $\mathrm{Spec}(\mathcal{E})$ is trivial by Steinberg's theorem (cf. Theorem 5.1). In particular, using Beauville-Laszlo again (cf. Lemma 5.2), we can extend $\mathcal{P}'$ to a $\mathcal{G}$-torsor $\mathcal{P}$ on U. By assumption the $\mathcal{G}$-torsor $\mathcal{P}$ extends to $\mathrm{Spec}(A_E)$ and is thus trivial as we assumed that C (and thus its residue field k') is algebraically closed. In particular, $\mathcal{P}' = \mathcal{P}_{|V}$ is trivial. This finishes the proof. □

We now provide an example showing that Proposition 7.2 fails in the simplest case if $\mathcal{G}$ is not assumed to be parahoric.

Example 7.4. Let $G = \mathbb{G}_{m,E}$ be the multiplicative group and let $\mathcal{G}$ be the smooth model of G over $\mathcal{O}_E$ such that $\mathcal{G}(\mathcal{O}_E) \subseteq \mathcal{O}_E^\times$ is the subgroup of one-units

$$\mathcal{G}(\mathcal{O}_E) = \{a \in \mathcal{O}_E^\times \mid a \equiv 1 \bmod \pi\}$$

(the group scheme $\mathcal{G}$ can be constructed as the dilatation of $\mathbb{G}_{m,\mathcal{O}_E}$ along the unit section of the special fiber). Then

$$\mathcal{G}(\mathcal{O}_\mathcal{E})\backslash G(\mathcal{E})/G(A_E[1/\pi]) \neq \{1\}.$$

In fact,

$$\mathcal{G}(\mathcal{O}_\mathcal{E})\backslash G(\mathcal{E}) \cong \pi^\mathbb{Z} \times C^\times$$

and the image of $G(A_E[1/\pi])$ in $\pi^\mathbb{Z} \times C^\times$ is given by

$$\pi^\mathbb{Z} \times \mathcal{O}_C^\times.$$

Thus we obtain a bijection

$$\mathcal{G}(\mathcal{O}_\mathcal{E})\backslash G(\mathcal{E})/G(A_E[1/\pi]) \cong C^\times/\mathcal{O}_C^\times \neq \{1\}.$$

In particular, we can conclude that there exist $\mathcal{G}$-torsors over U which do not extend to $\mathrm{Spec}(A_E)$ but are trivial on V.

The following corollary will be needed.

Corollary 7.5. *Assume that G is split. Then*

$$H^1(V, G) = \{1\}.$$

Proof. Let $\mathcal{G}$ be the split reductive model of G over $\mathcal{O}_E$. By Proposition 6.5 we now that $\mathcal{G}$-torsors on U extend to $\mathrm{Spec}(A_E)$. Using Corollary 7.3 we can conclude. □

The case $G = \mathrm{PGL}_n$ will be of particular use, when we discuss tori. Note that Corollary 7.5 is wrong for the 2-dimensional regular local noetherian ring $R_E = \mathcal{O}_E[[z]]$ in this case.

Proposition 7.6. *Let T be a torus over E. Then*

$$H^1(V, T) = 0$$

and the torsion subgroup

$$H^2(V, T)_{\mathrm{tor}} = 0$$

of $H^2(V, T)$ is trivial. In particular (cf. Corollary 7.3), every $\mathcal{T}^\circ$-torsor over U under the unique parahoric model $\mathcal{T}^\circ$ of T is trivial.

Proof. Let $\overline{E}$ be a separable closure of E and consider the spectral sequence

$$E_2^{i,j} = H^i(\mathrm{Gal}(\overline{E}/E), H^j(V_{\overline{E}}, T_{\overline{E}}) \Rightarrow H^{i+j}(V, T)$$

with

$$V_{\overline{E}} := V \times_{\mathrm{Spec}(E)} \mathrm{Spec}(\overline{E}).$$

Passing to the limit over finite separable extensions E' of E we can deduce

$$H^1(V_{\overline{E}}, T_{\overline{E}}) = 0$$

from Corollary 4.2 as for some E'/E finite the torus $T_{E'}$ will be split. As E is of cohomologial dimension 1 (thus of strict cohomological dimension ≤ 2, cf. [22, Chapitre I. Proposition 3.2.13]) we get

$$H^i(\mathrm{Gal}(\overline{E}/E), H^j(V_{\overline{E}}, T_{\overline{E}})) = 0$$

for $i \geq 3$ and that the group

$$H^2(\mathrm{Gal}(\overline{E}/E), H^0(V_{\overline{E}}, T_{\overline{E}}))$$

is divisible. To see the second claim, set

$$M := H^0(V_{\overline{E}}, T_{\overline{E}}).$$

Then for $n \in \mathbb{Z} \setminus \{0\}$ the exact sequences

$$0 \to M[n] \to M \to M/M[n] \to 0$$

and

$$0 \to M/M[n] \to M \to M/nM \to 0$$

with $M[n] \subseteq M$ denoting the n-torsion submodule of M furnish that multiplication by n is a surjection on

$$H^2(\mathrm{Gal}(\overline{E}/E), M)$$

as it factors through surjections

$$H^2(\mathrm{Gal}(\overline{E}/E), M) \cong H^2(\mathrm{Gal}(\overline{E}/E), M/M[n]) \twoheadrightarrow H^2(\mathrm{Gal}(\overline{E}/E), M).$$

The above spectral sequence thus yields a (split) short exact sequence

$$0 \to H^2(\mathrm{Gal}(\overline{E}/E), M) \to H^2(V, T) \to H^0(\mathrm{Gal}(\overline{E}/E), H^2(V_{\overline{E}}, T_{\overline{E}})) \to 0.$$

Assuming that $H^2(V_{\overline{E}}, T_{\overline{E}})$ has only trivial torsion, we can derive that there is an isomorphism

$$H^2(\mathrm{Gal}(\overline{E}/E), M) \cong H^2(V, T)_{\mathrm{tor}}$$

on torsion parts, in particular that $H^2(V, T)_{\mathrm{tor}}$ is divisible. Let E'/E be a finite separable extension splitting T. Then the inclusion resp. the norm

$$T \to \mathrm{Res}_{E'/E}(\mathbb{G}_m) \ , \ \mathrm{Res}_{E'/E}(\mathbb{G}_m) \to T$$

compose to multiplication by $[E' : E]$ on T. If $H^2(\mathrm{Spec}(A_{E'}[1/\pi]), \mathbb{G}_m)_{\mathrm{tor}} = 0$, then we can conclude (using Shapiro's isomorphism) that $H^2(V, T)_{\mathrm{tor}}$ is annihilated by $[E' : E]$. Being also divisible this implies

$$H^2(V, T)_{\mathrm{tor}} = 0$$

as desired. Hence, it suffices to prove that the torsion part

$$H^2(V_{E'}, \mathbb{G}_m)_{\mathrm{tor}} = 0$$

vanishes for every separable algebraic extension E'/E. Passing to the limit, we may assume that E'/E is a finite separable extension. As $V_{E'} = \mathrm{Spec}(A_{E'}[1/\pi])$ is affine we can apply Gabbber's theorem (cf. [12] or [17, Corollary 3.1.4.2.][1]) and conclude that each class

$$\alpha \in H^2(V_{E'}, \mathbb{G}_m)_{\mathrm{tor}}$$

is represented by some Azumaya algebra, i.e., there exists some n such α lies in the image of

$$H^1(V_{E'}, \mathrm{PGL}_n) \to H^2(V_{E'}, \mathbb{G}_m).$$

Now we can apply Proposition 6.5 to get that

$$H^1(V_{E'}, \mathrm{PGL}_n) = \{1\}$$

is trivial, which implies $\alpha = 0$. The proof of the statement for H^2 is now finished and we turn to show

$$H^1_{et}(V, T) = 0$$

[1] using noetherian approximation for the general case

for every torus T/E. If $T = \mathrm{Res}_{E'/E}(\mathbb{G}_m)$ is an induced torus, then

$$H^1(V,T) = H^1(V_{E'}, \mathbb{G}_m) = 0$$

by Corollary 4.2. In general, let T be an arbitrary torus and chose an exact sequence

$$0 \to T'' \to T' \xrightarrow{\alpha} T \to 0$$

of tori with T' induced. Then there exists a morphism $\beta\colon T \to T'$ such that $\alpha \circ \beta = n$ is multiplication by some non-zero $n \in \mathbb{Z}$. In particular, the group

$$H^1(V,T)$$

is torsion as $H^1(V,T') = 0$. Using that the torsion in

$$H^2(V,T'')$$

vanishes we can thus conclude $H^1(V,T) = 0$ from the exact sequence

$$0 = H^1(V,T') \to H^1(V,T) \to H^2(V,T'').$$

□

We record the following vanishing result for multiplicative coefficients.

Lemma 7.7. *For every finite multiplicative group scheme D/E the second flat cohomology group*

$$H^2_{\mathrm{fl}}(V,M) = 0$$

vanishes.

Proof. We may choose a short exact sequence (for the flat topology)

$$0 \to D \to T \to T' \to 0$$

of multiplicative group schemes with T and T' tori over E. Then the statement follows from Proposition 7.6 by taking the associated long exact sequence in cohomology. □

We record the following corollary of Proposition 7.6.

Lemma 7.8. *Let $1 \to H \to G' \to G \to 1$ be a central extension of two (connected) reductive groups G' resp. G over E. Then*

$$H^1(V,G') = 1 \Leftrightarrow H^1(V,G) = 1.$$

Proof. Let G^{ad} be the adjoint quotient of G. Then G^{ad} is also the adjoint quotient of G'. Argueing for the pairs (G, G^{ad}) resp. (G', G^{ad}) with the respective central extensions reduces to the case that G is adjoint. Let $H^\circ \subseteq H$ be the connected component of the identity. By Proposition 7.6

$$H^1(V,G')$$

vanishes if and only if

$$H^1(V,G'/H^\circ)$$

(noting that the image of the connecting morphism

$$H^1(V,G'/H^\circ) \to H^2(V,H^\circ)$$

lands inside the torsion subgroup as each G'/H°-torsor on A_E is trivial after base change along some finite extension of E). Hence we may assume that $H^\circ = \{1\}$ is

trivial and thus that H is finite. By Lemma 7.7 the group $H^2(\mathrm{Spec}(A_E[1/p]), H)$ vanishes. Hence,

$$H^1(\mathrm{Spec}(A_E[1/p]), G') = \{1\}$$

implies

$$H^1(\mathrm{Spec}(A_E[1/p]), G) = \{1\}.$$

If conversely $H^1(\mathrm{Spec}(A_E[1/p]), G) = \{1\}$, then every G'-torsor $\mathcal{P}$ arises as the pushforward $G' \times^H \mathcal{Q}$ of some H-torsor $\mathcal{Q}$. But the embedding $H \to G'$ factors through some maximal torus $T \subseteq G'$. Hence, the pushforward

$$G' \times^H \mathcal{Q} \cong G' \times^T (T \times^H \mathcal{Q})$$

of every H-torsor to G' is trivial by Proposition 7.6. This finishes the proof. □

We can now turn to our main theorem about extending torsors on the punctured spectrum of A_E. Let $\mathcal{C}$ be the class of pairs (G', E') with E'/E a finite separable field extension and G' a reductive group such that for all parahoric models $\mathcal{G}'$ of G' over $\mathcal{O}_{E'}$ each $\mathcal{G}'$-torsor on $U_{A_{E'}}$ extends to $\mathrm{Spec}(A_{E'}$ (by Corollary 7.3 this is equivalent to $H^1(V_{A_{E'}}, G') = \{1\}$).

By a group of PEL-type we will mean a not necessarily connected group G over E, where E has residue characteristic not 2, whose R-valued points for some E-algebra R are defined as

$$G(R)\colon \{g \in \mathrm{GL}_B(V \otimes_E R) \mid (gv, gw) = c(g)(v, w),\ c(g) \in R^\times\}$$

where B is a finite dimensional central F-algebra for a finite separable E-algebra F which is equipped with an antiinvolution $(-)^*\colon B \to B$, V a finite dimensional B-module and $(-,-)\colon V \times V \to E$ an E-bilinear form satisfying

$$(bv, w) = (v, b^* w)$$

for $b \in B$ and $v, w \in V$.

Theorem 7.9. 1) *Let $\mathcal{P}$ be a $\mathcal{G}$-torsor on U. Then $\mathcal{P}$ extends to* $\mathrm{Spec}(A_E)$ *if the restriction $\mathcal{P}_{|U_{\mathrm{crys}}}$ of $\mathcal{P}$ to the crystalline part $U_{\mathrm{crys}} \subseteq U$ is trivial (cf. Section 2).*

2) *The class $\mathcal{C}$ is closed under Weil restrictions along finite extensions E''/E' over E, direct products of reductive groups, is invariant under central extensions in G' (in particular $\mathcal{C}$ includes all pairs (G', E') with G' a torus) and it contains all pairs (G', E') where G' is either split or of type A or the idendity component of a group of PEL-type.*

Proof. Note that G is quasi-split by our assumption that k is algebraically closed. For part 1) (by 7.3) it suffices to prove that $\mathcal{P}_{|V}$ is trivial. Let

$$T \subseteq B \subseteq G$$

be a maximal torus and a Borel. As it is trivial the torsor

$$\mathcal{P}_{|U_{\mathrm{crys}}}$$

admits a reduction to B over U_{crys}. By Lemma 3.1 and Corollary 3.2 for $s \in V \backslash U_{\mathrm{crys}}$ the local ring

$$\mathcal{O}_{V,s}$$

is a discrete valuation ring. Hence by properness of the quotient $\mathcal{P}_{|V}/B$ over E and the valuative criterion for properness the torsor $\mathcal{P}_{|V}$ admits a reduction

$$\mathcal{P}' \in H^1_(V, B)$$

to B over the whole of V. In other words, there exists a B-torsor $\mathcal{P}'$ over V such that

$$\mathcal{P}' \times^B G \cong \mathcal{P}_{|V}.$$

Let $\mathrm{rad}(B) \subseteq B$ be the unipotent radical of B and consider the natural map

$$H^1_(V, B) \xrightarrow{\Phi} H^1_(V, B/\mathrm{rad}(B)).$$

The fiber $\Phi^{-1}(\Phi(\mathcal{P}'))$ containing $\mathcal{P}' \in H^1_(V, B)$ can naturally be identified with the set

$$H^1_(V, \mathrm{rad}(B)_{\mathcal{P}'})$$

where

$$\mathrm{rad}(B)_{\mathcal{P}'} := (\mathrm{rad}(B) \times_{\mathrm{Spec}(E)} V) \times^B \mathcal{P}'$$

is a twisted form of the constant group scheme $\mathrm{rad}(B) \times_{\mathrm{Spec}(E)} V$ over V where B acts on $\mathrm{rad}(B)$ via conjugation. As $\mathrm{rad}(B)$ admits a canonical, i.e., B-stable, filtration whose graded pieces are vector spaces over E (with B acting linearly) the unipotent group scheme $\mathrm{rad}(B)_{\mathcal{P}'}$ over V admits a filtration with graded pieces vector bundles over V. As V is affine the (étale) cohomology with coefficients in quasi-coherent sheaves, in particular vector bundles, vanishes and therefore

$$H^1_(V, \mathrm{rad}(B)_{\mathcal{P}'}) = \{1\}$$

as well. In particular, the map Φ is injective. By Proposition 7.6 the pointed set

$$H^1(V, B/\mathrm{rad}(B)) \cong H^1(V, T) = \{1\}$$

is trivial and by injectivity of Φ we can conclude that $\mathcal{P}'$, hence $\mathcal{P}_{|V}$, is trivial. Thus we have proven part 1) . Let us proceed with part 2). That $\mathcal{C}$ is stable under Weil restrictions, i.e., that for E''/E' finite separable and $(G'', E'') \in \mathcal{C}$ also $(\mathrm{Res}_{E''/E'}(G''), E') \in \mathcal{C}$, follows from Shapiro's lemma. Stability under direct products is clear. If $G' \to G''$ is a central extension of reductive groups over E', then by 7.8 (G', E') belongs to $\mathcal{C}$ if and only if (G'', E') does. This shows that $\mathcal{C}$ is invariant under central extensions. Assume that G' over E' is split and let $\mathcal{G}'$ be a reductive model of G over $\mathcal{O}_E$. Then the result follows from Corollary 7.5. The case of (G', E') with G' of type A follows from Corollary 9.11 (and Lemma 4.1. Thus assume that G' is the identity component of some group of PEL-type over E' (we note that this in particular implies $p \neq 2$), i.e., defined by some PEL-datum. To simplify notations let $E' = E, G' = G$. Let

$$G^\circ \subseteq G$$

be the connected component of the identity of G and let $\mathcal{G}$ be a smooth affine model of G defined by some integral PEL-datum defined as in [18, Appendix to Chapter 3] (cf. [21, Theorem 21.5.4], note that the arguments in both references work equally well if E is an arbitrary complete discrete valuation field with residue characteristic not 2). By [21, Corollary 21.5.6.] (building on [18, Theorem 3.11, Theorem 3.16]) $\mathcal{G}$-torsors over some $\mathcal{O}_E$-scheme Y are equivalent to the groupoid of polarized lattice chains of type $\mathcal{L}$ over Y. Let $(M_\Lambda)_{\Lambda \in \mathcal{L}}$ be a polarized lattice chain over $Y = U$. By Lemma 4.1 it is clear that the chain (M_Λ) extends to a polarized chain on $\mathrm{Spec}(A_E)$ except that we have to check that condition (2) in

[21, Definition 21.5.1] is still satisfied. Using Morita invariance in order to replace the maximal order in the matrix algebra B (defining G) by the maximal order in some field extension this follows from Lemma 4.3. In particular we obtain that $\mathcal{G}$-torsors on U are trivial. Now we want to show that

$$H^1(V, G^\circ) = \{1\}.$$

Thus let $\mathcal{P}$ be a G°-torsor on V and let $\mathcal{P}' := \mathcal{P} \times^{G^\circ} G$ be the push forward of $\mathcal{P}$. Then the pull back

$$\mathcal{P}' \times_V \mathrm{Spec}(\mathcal{E})$$

is trivial where $\mathcal{E}$ denotes the fraction field of the local ring of U at the prime ideal $(\pi) \subseteq A_E$ (by Steinberg's theorem Theorem 5.1). In particular, $\mathcal{P}'$ extends to a $\mathcal{G}$-torsor on U and is thus trivial as we saw. In other words, $\mathcal{P}$ lies in the image of the connecting morphism

$$\pi_0(G)(V) \to H^1(V, G^\circ)$$

associated with the short exact sequence

$$1 \to G^0 \to G \to \pi_0(G) \to 1$$

of affine algebraic groups over E. Hence, it suffices to show that

$$G(V) \to \pi_0(G)(V)$$

is surjective. But this follows because

$$\pi_0(G)(V) = \pi_0(G)(E)$$

and

$$G(E) \hookrightarrow \pi_0(G)(E)$$

(by Steinberg's theorem applied to E). This finishes the proof. □

Thus the pairs (G', E') missing in $\mathcal{C}$ are (essentially) of ramified outer forms of type E_6, ramified triality groups (both in arbitrary residue characteristic) or (non-trialitarian) ramified outer forms of type D in residue characteristic 2.

8. Extending torsors to $\mathrm{Spec}(R_E)$

We now turn to the question of extending torsors on the punctured spectrum of (some) regular 2-dimensional local noetherian rings. We continue to use the notation from Section 2, thus E denotes a complete discretely valued field, $\mathcal{O}_E$ its ring of integers, etc. Furthermore we let R_E be given by

$$R_E := \mathcal{O}_E[[z]].$$

We again denote by

$$s := s_{R_E} \in \mathrm{Spec}(R_E)$$

the unique closed point and by

$$U := U_{R_E} := \mathrm{Spec}(R_E) \setminus \{s\}$$

its complement. Moreover, set

$$V := V_{R_E} := \mathrm{Spec}(R_E[1/\pi]).$$

If confusion with our previous notation for A_E from Section 2 may be possible we will add subscripts. First of all let us recall that vector bundles on U extend

uniquely to $\mathrm{Spec}(R_E)$. Note that R_E is a regular 2-dimensional local noetherian ring.

Lemma 8.1. *For vector bundles the restriction functor*

$$\mathrm{Bun}(\mathrm{Spec}(R_E)) \to \mathrm{Bun}(U)$$

is an equivalence.

Proof. Fully faithfullness follows from $H^0(U, \mathcal{O}_U) = R_E$ which is implied by normality of R_E. Conversely let $\mathcal{V}$ be a vector bundle on U and let $M := H^0(U, \mathcal{V})$. By the Auslander-Buchsbaum formula

$$\mathrm{pd}(M) + \mathrm{depth}(M) = 2$$

where $\mathrm{pd}(M)$ and $\mathrm{depth}(M)$ are the projective dimension and depth of M. Hence, it suffices to proof $\mathrm{depth}(M) = 2$. For this it suffices to proof that M/π is torsionfree over R_E/π. But applying cohomology to the exact sequence

$$0 \to \mathcal{V} \xrightarrow{\pi} \mathcal{V} \to \mathcal{V}/\pi \to 0$$

we see that M/π embeds into $H^0(U, \mathcal{V}/\pi)$ which is torsionfree over R_E/π as $\mathcal{V}/\pi$ is a vector bundle on $\mathrm{Spec}(R_E/\pi) \cap U = \mathrm{Spec}(\mathrm{Frac}(R_E/\pi))$. □

Hence, we can apply our general results from Section 6, in particular Lemma 6.3. For this let us define a morphism of $\mathcal{O}_E$-algebras

$$f \colon R_E \to A_E.$$

Namely let $\varpi \in \mathfrak{m}_C \setminus \{0\}$ be an arbitrary element and define f by

$$f(z) := [\varpi].$$

Lemma 8.2. *The morphism $f \colon R_E \to A_E$ is faithfully flat.*

Proof. The proof in [3, Lemma 4.30] works in this situation as well. □

Thus we are able to apply the descent lemma Lemma 6.3. First of all let us draw the following proposition, which in the tamely ramified case appears (using Beauville-Laszlo) as Step 3 in [14, Proposition 1.4.3].

Proposition 8.3. *For every parahoric group scheme $\mathcal{G}$ over $\mathcal{O}_E$ a $\mathcal{G}$-torsor $\mathcal{P}$ on U extends to $\mathrm{Spec}(R_E)$ if it is trivial when base changed to $V_{R_{E'}} = \mathrm{Spec}(R_{E'}[1/\pi]$ for some unramified finite extension E'/E (i.e., $\mathcal{O}_{E'}$ is étale over $\mathcal{O}_E$).*

Proof. We recall that the base change of a parahoric group scheme under an étale extension $\mathcal{O}_E \to \mathcal{O}_{E'}$ is again parahoric (cf. [5, 4.6.20.Remarques]), hence we may assume that $E = E'$ and $\mathcal{P}_{|V}$ is trivial. Then the statement follows from descent (cf. Lemma 8.2 and Lemma 6.3) from Corollary 7.3. □

We remark that contrary to the case of A_E handled in Section 7 it may happen that $H^1(V, G) \neq \{1\}$ but every $\mathcal{G}$-torsor on U extends to $\mathrm{Spec}(R_E)$ (i.e., not every G-torsor on V extends to U). For example this happens if $G = \mathrm{PGL}_n$.

The following theorem slightly extends [14, Proposition 1.4.3] to some wildly ramified cases (or groups containing a direct factor of type E_8).

Let $\mathcal{D}$ be the class of pairs $(\mathcal{G}', E')$ where E'/E is a finite extension and $\mathcal{G}'$ is a parahoric group scheme over $\mathcal{O}_{E'}$ such that every $\mathcal{G}'$-torsor on

$$U_{E'} = \mathrm{Spec}(R_{E'}) \setminus \{s\}$$

extends to $\mathrm{Spec}(R_{E'})$. For $\mathcal{G}'$ let

$$G' := \mathcal{G}' \otimes_{\mathcal{O}_{E'}} E'$$

be its reductive generic fiber over E'.

Theorem 8.4 (cf. Theorem 1.2)**.** 1) *Let $\mathcal{P}$ be a $\mathcal{G}$-torsor on U. Then $\mathcal{P}$ extends (necessarily uniquely) to $\mathrm{Spec}(R_E)$ if the restriction $\mathcal{P}_{|\mathrm{Spec}(\mathrm{Frac}(R_E))}$ of $\mathcal{P}$ is trivial.*

2) *The class $\mathcal{D}$ is stable under Weil restrictions, direct products and contains all pairs $(\mathcal{G}', E')$ with either G' split by some tamely ramified extension of E', G' of type A, G' a torus or G' simply-connected.*

Proof. Let us prove the first part. We may replace E by some finite unramified extension E'/E. Hence, we may assume that $G = \mathcal{G} \otimes_{\mathcal{O}_E} E$ is quasi-split. By Corollary 7.3 and Lemma 6.3 it suffices to show that $\mathcal{P}_{|V_{A_E}}$ is trivial where $V_{A_E} = \mathrm{Spec}(A_E[1/\pi])$. As in the proof of Theorem 7.9 we see that $\mathcal{P}_{|V_{R_E}}$ has a reduction to a maximal torus $T \subseteq G$. By Proposition 7.6 we can conclude that the base change $\mathcal{P}_{|V_{A_E}}$ is trivial, thus finishing the proof of part 1). For the second part, the stability under Weil restrictions and direct products is clear. If G' is split by some tamely ramified extension then $(\mathcal{G}', E') \in \mathcal{D}$ by in [14, Proposition 1.4.3.]. The case G' a torus follows from Theorem 7.9 by descent. If G' is simply connected without factors of type E_8, then the result follows from 1) and the proven Conjecture II by Serre (cf. [9] and [14, Lemma 1.4.6.]). Finally for G' of type E_8, then G' is split by some unramified extension of E' as E_8 has no non-trivial outer automorphism (and is quasi-split by some unramified extension). Thus the statement follows from Theorem 7.9 by descent. □

9. Hermitian quadratic forms

In this section we will work through the case of unitary groups with respect to some ramified extension L/K of degree 2. We want to construct a concrete affine smooth model and describe the category of torsors under it (cf. Theorem 9.10). By work in progress of D. Kirch these models are special parahorics. From our description we can conclude extension of torsors on the punctured spectrum of A_E or R_E for these ramified unitary groups (cf. Corollary 9.11). To construct these models we will study certain quadratic forms, which we call hermitian quadratic forms.

Let K be a complete discretely valued field and let L/K be a separable, ramified extension of degree 2. We denote by

$$(-)^* \colon \mathcal{O}_L \to \mathcal{O}_L$$

the non-trivial Galois involution on L. More generally, for every $\mathcal{O}_K$-algebra R we again denote by

$$(-)^* \colon R \otimes_{\mathcal{O}_K} \mathcal{O}_L \to R \otimes_{\mathcal{O}_K} \mathcal{O}_L,\ r \otimes l \mapsto r \otimes l^*$$

the base change of $(-)^*$. In particular, we obtain the multiplicative norm

$$\mathrm{N}_{L/K} \colon R \otimes_{\mathcal{O}_K} \mathcal{O}_L \to R,\ r \mapsto rr^*$$

and the R-linear trace

$$\mathrm{Tr}_{L/K} \colon R \otimes_{\mathcal{O}_K} \mathcal{O}_L \to R,\ r \mapsto r + r^*.$$

Definition 9.1. Let R be an $\mathcal{O}_K$-algebra, let M be a finite locally free $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$-module and let $\mathcal{L}$ be an invertible R-module. We call a quadratic form

$$q\colon M \to \mathcal{L}$$

with associated symmetric R-bilinear form

$$f\colon M \times M \to \mathcal{L}$$

a $\mathcal{L}$-valued hermitian quadratic form on M if they satisfy the equations

i) $q(xm) = N_{L/K}(x)q(m)$
ii) $f(m,n) = q(m+n) - q(m) - q(n)$
iii) $f(xm,n) = f(m,x^*n)$
iv) $f(xm,m) = \mathrm{Tr}_{L/K}(x)q(m)$

for $x \in R \otimes_{\mathcal{O}_K} \mathcal{O}_L$ and $m,n \in M$.

The forth equation is actually a consequence of the first and second. In the following we want to derive a normal form for certain hermitian quadratic forms. Let us fix a uniformizer $\Pi \in \mathcal{O}_L$ and write

$$\Pi^2 = t\Pi - \pi$$

with $t := \mathrm{Tr}_{L/K}(\Pi)$ and $pi := N_{L/K}(\Pi)$. As L/K is ramified π is a uniformizer of $\mathcal{O}_K$.

Lemma 9.2. *For an $\mathcal{O}_K$-algebra R and $n \geq 1$ there is a bijection between hermitian quadratic forms*

$$q\colon M \to R$$

on the trivial $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$-module $M := (R \otimes_{\mathcal{O}_K} \mathcal{O}_L)$ with standard basis $e_1, \dots, e_n$ over $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$ and pairs (A,B) of $n \times n$-matrices A, B with entries in R such that

- *A is symmetric*
- *$B + B^{\mathrm{tr}} = t\tilde{A}$*
- *$B_{ii} = tA_{ii}$ for $i = 1, \dots, n$*

where $\tilde{A}$ is the $n \times n$-matrix

$$\tilde{A}_{i,j} = \begin{cases} A_{i,j} & \text{if } i \neq j \\ 2A_{i,i} & \text{if } i = j \end{cases}$$

by sending q to

$$\tilde{A}_{i,j} = f(e_i, e_j), A_{ii} := q(e_i), B_{i,j} := f(e_i, \Pi e_j).$$

In particular, the scheme representing hermitian quadratic forms on M is represented by the affine space $\mathbb{A}^N_R \cong \mathrm{Spec}(\mathcal{O}_K[A_{i,j}, B_{k,l} | k < l, i \leq j])$ of relative dimension $n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n^2$ over R.

Proof. Let $m = \sum_{i=1}^n x_i e_i + \sum_{i=1}^n y_i \Pi e_i \in M$. If q is a hermitian quadratic form on M, then

$$\begin{aligned} q(m) &= q(\sum_{i=1}^n x_i e_i) + \pi q(\sum_{i=1}^n y_i e_i) + f(\sum_{i=1}^n x_i e_i, \sum_{i=1}^n \Pi y_i e_i) \\ &= \sum_{1 \leq i \leq j \leq n} A_{i,j} x_i x_j + \pi \sum_{1 \leq i \leq j \leq n} A_{i,j} y_i y_j + \sum_{i,j=1}^n x_i y_j B_{i,j} \end{aligned}$$

with

$$A_{i,i} := q(e_i), A_{i,j} := f(e_i, e_j)$$

for $i < j$ and

$$B_{i,j} = f(e_i, \Pi e_j)$$

where f is the symmetric bilinear form associated to q. In particular, q is uniquely determined by (A, B). We check that the displayed relations hold. Namely, A is symmetric as f is symmetric. Moreover,

$$B_{i,j} + B_{j,i} = f(e_i, \Pi e_j) + f(e_j, \Pi e_i) = f(e_i, (\Pi + \Pi^*)e_j) = t\tilde{A}_{i,j}$$

and

$$f(e_i, \Pi e_i) = tq(e_i) = tA_{i,i}$$

for all $1 \leq i, j \leq n$.

Conversely, for given matrices A, B with A symmetric we can define a quadratic form $q_{A,B}$ by the above formula and check that if A,B satisfy the displayed relation that $q_{A,B}$ is a hermitian quadratic form. □

In order to define non-degenerate hermitian quadratic forms we introduce the discriminant.

Definition 9.3. Let $(M, q, \mathcal{L})$ be a $\mathcal{L}$-valued hermitian quadratic form over some $\mathcal{O}_K$-algebra R with $\mathrm{rk}_R M = 2n$. Then we define the discriminant as the morphism

$$\mathrm{disc} := \Lambda_R^{2n}(f^\sharp) \colon \Lambda_R^{2n} M \to \Lambda_R^{2n}(M^\vee \otimes_R \mathcal{L}) \cong \Lambda_R^{2n} M^\vee \otimes_R \mathcal{L}^{2n}$$

induced by

$$f^\sharp \colon M \to M^\vee \otimes_R \mathcal{L},\ m \mapsto f(m, -).$$

where

$$M^\vee := \mathrm{Hom}_R(M, R)$$

is the R-dual of M. If n is even we call q non-degenerate if $\mathrm{disc}(q)$ is an isomorphism.

However, as will follow from Lemma 9.6 non-degenerate hermitian quadratic forms should only be expected if $\mathrm{rk}_{R \otimes_{\mathcal{O}_K} \mathcal{O}_L} M$ is even. Thus if n is odd we need a replacement for the discriminant, which we will call the divided discriminant.

Let $\theta = (4\pi - t^2) \subseteq \mathcal{O}_K$ be the discriminant of L/K.

Lemma 9.4. *Let $n = 2r + 1 \geq 1$ be an odd integer and $(M, q, \mathcal{L})$ a hermitian quadratic form over some $\mathcal{O}_K$-algebra R such that $\mathrm{rk}_{R \otimes_{\mathcal{O}_K} \mathcal{O}_K} M = n$. Then there exists a functorial factorisation*

$$\begin{tikzcd}
\Lambda_R^{2n} M \arrow[r, "\mathrm{disc}"] \arrow[d, "\mathrm{disc}'"] & \Lambda_R^{2n} M^\vee \otimes_R \mathcal{L}^{2n} \\
\Lambda_R^{2n} M^\vee \otimes_R \mathcal{L}^{2n} \otimes_{\mathcal{O}_K} \theta \arrow[ur, "\mathrm{can}"] &
\end{tikzcd}$$

Proof. It suffices to check the statement in the universal case, i.e., over the ring

$$R := \mathcal{O}_K[A_{i,j}, B_{i,j} | 1 \leq i, j \leq n]/I$$

with

$$I := (A_{i,j} - A_{j,i}, B_{k,l} + B_{k,l} - tA_{k,l}, B_{i,i} = tA_{i,i} | 1 \leq i, j \leq n, k < l, h)$$

from Lemma 9.2 with its quadratic form

$$q(\sum_{i=1}^n x_i e_i + \sum_{i=1}^n \Pi y_i e_i) := \sum_{1 \leq i \leq j \leq n} A_{i,j} x_i x_j + \pi \sum_{1 \leq i \leq j \leq n} A_{i,j} y_i y_j + \sum_{i,j=1}^n x_i y_j B_{i,j}.$$

The discriminant $\mathrm{disc}(q)$ defines a Cartier divisor D on Spec(R) and we must check that D contains the vanishing locus of $4\pi - t^2$. As $\mathrm{Spec}(R)$ is smooth over $\mathcal{O}_K$ this may be checked at the local ring R' of the generic point of the special fiber of R. But there the elements $A_{i,j}, B_{i,j}$ are units. In the basis $e_1, \dots, e_n, \Pi e_1, \dots, \Pi e_n$ the bilinear form f is represented by

$$\begin{pmatrix} \tilde{A} & B \\ t\tilde{A} - B & \pi\tilde{A} \end{pmatrix}$$

where

$$\tilde{A}_{i,j} := f(e_i, e_j).$$

As $\tilde{A}_{i,j} = A_{i,j}$ for $i \neq j$ is a unit in R' and $\tilde{A}_{i,i} = 2A_{i,i}$ is divisible by 2 we may do a coordinate change $e_i \mapsto e_i'$ do achieve first that $\tilde{A}$ is a block diagonal matrix with r blocks given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and one block of size 1. Using further matrix manipulations we may then change the $\Pi e_i \mapsto e_i''$ such that

$$f(e_i', e_j'') = 0$$

for $1 \leq i \leq n-1, 1 \leq j \leq n$. Expressing f in the R'-basis $e_1', \dots, e_n', e_1'', \dots, e_n''$ shows that the determinant of

$$C := \begin{pmatrix} 2A_{n,n} & B_{n,n} \\ B_{n,n} & \pi 2A_{n,n} \end{pmatrix}$$

divides disc(q). But $B_{n,n} = tA_{n,n}$ and thus

$$\det(C) = A_{n,n}(4\pi - t^2)$$

which implies $(4\pi - t^2) | \mathrm{disc}(q)$ as claimed. □

If n is odd, the morphism

$$\mathrm{disc}' \colon \Lambda_R^{2n} M \to \Lambda_R^{2n}(M^\vee) \otimes_R \mathcal{L}^{2n} \otimes_{\mathcal{O}_K} \theta$$

from 9.4 will be called the divided discriminant. Moreover, in this case we call q non-degenerate if

$$\mathrm{disc}'(q)$$

is an isomorphism.

Example 9.5. Let us compute the divided discriminant for a hermitian quadratic space of rank 1. That is, let $(M, q, \mathcal{L})$ be a hermitian quadratic space with M free of rank 1 over $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Let $x \in M$ be a generator over $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$. In the R-basis $x, \Pi x$ the matrix (with entries in $\mathcal{L}$) for the associated bilinear form f is given by

$$\begin{pmatrix} 2q(x) & tq(x) \\ tq(x) & 2\pi q(x) \end{pmatrix}$$

(recall that $\Pi^2 = t\Pi - \pi$) as $f(x, \Pi x) = tq(x)$ and $f(\Pi x, \Pi x) = \pi f(x, x)$. Thus

$$\mathrm{disc}'(q) = q(x)^2.$$

while

$$\mathrm{disc}(q) = (4\pi - t^2)q(x)^2.$$

The following crucial lemma is taken from the unpublished [13] and we heartily thank Daniel Kirch for sharing his notes.

Lemma 9.6. *Let R be an $\mathcal{O}_K$-algebra such that π is nilpotent in R and let $(M, q, \mathcal{L})$ be an $\mathcal{L}$-valued hermitian quadratic form over $\mathrm{Spec}(R)$. Assume that $x, y \in M$ are elements such that*

$$f(x, \Pi y) = 1.$$

Then there exists $x', y' \in \langle x, y\rangle_{R\otimes_{\mathcal{O}_K}\mathcal{O}_L}$ such that $q(x') = q(y') = 0$ and in the elements $x', y', \Pi x', \Pi y'$ the bilinear form f is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, the elements x', y' are part of an $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$-basis of M and if $f^\sharp$ is an isomorphism, then $2|\mathrm{rk}_{R\otimes_{\mathcal{O}_K}\mathcal{O}_L}$.

Proof. First we want to argue that we can assume that $q(x) = q(y) = 0$. Let $r \in R$. Then

$$f(x + r\Pi y, \Pi y) = 1 + rN_{L/K}(\Pi)f(y, y)$$

is a unit in R as $\pi = N_{L/K}(\Pi)$ is nilpotent. Moreover,

$$q(x + r\Pi y) = r + q(x) + r^2\pi q(y).$$

As $\pi \in R$ is nilpotent we can apply the (converging) Newton iteration

$$r_1 := -q(x)$$
$$r_{i+1} := r_i - \tfrac{r_i + q(x) + r_i^2\pi q(y)}{1 + 2r_i\pi q(y)}$$

to find some $r \in R$ such that

$$q(x + r\Pi y) = 0.$$

Replacing x by $\frac{x+r\Pi y}{1+r\pi f(y,y)}$ yields x, y satisfying $f(x, \Pi y) = 1$ and $q(x) = 0$. Then replacing y by $\frac{y-q(y)\Pi^* x}{1-q(y)\pi f(x,y)}$ yields x, y satisfying x, y satisfying $f(x, \Pi y) = 1$ and $q(x) = q(y) = 0$. We now want to obtain moreover that $f(x, y) = 0$. For this set

$$a := \frac{1}{1 - f(x, y)f(x, \Pi^2 y)}$$

(note that $f(x, \Pi^2 y) = -\pi f(x, y) + t$ is nilpotent in R) and

$$b := -af(x, y).$$

Let

$$x' := (a + b\Pi^*)x.$$

Then

$$f(x', y) = af(x, y) + b = 0$$

(using $f(\Pi^* x, y) = f(x, \Pi y) = 1$) while

$$f(x', \Pi y) = a - bf(x, \Pi^2 y) = 1$$

and

$$q(x') = N_{L/K}(a + b\Pi^*)q(x) = 0.$$

Thus we can replace x, y by x', y such that $f(x, \Pi y) = 1$, $q(x) = q(y) = 0$ and $f(x, y) = 0$. Then $f(\Pi x, x) = tq(x) = 0$ (recall $t = \mathrm{Tr}_{L/K}(\Pi)$) and $f(y, \Pi x) = f(x, \Pi^* y) = tf(x, y) - f(x, \Pi y) = -1$ and the lemma follows. □

In particular, we can define for any $n \geq 1$ a standard example of a hermitian quadratic space of rank n.

Definition 9.7. We define

$$M_{\mathrm{std},2} := \mathcal{O}_L e_1 \oplus \mathcal{O}_L e_2$$

with hermitian quadratic form q_{std} defined (using 24) by

$$q_{\mathrm{std}}(e_1) = q_{\mathrm{std}}(e_2) = 0, f_{\mathrm{std}}(e_1, e_2) = 0$$

and $f_{\mathrm{std}}(e_1, \Pi e_2) = 1$. Moreover, for $n = 2r$ even we set

$$M_{\mathrm{std},n} := M_{\mathrm{std}_2}^{\oplus r}$$

as the r-fold orthogonal sum of $M_{\mathrm{std},2}$. and for $n = 2r+1$ odd we set

$$M_{\mathrm{std},n} := M_{\mathrm{std},n-1} + \mathcal{O}_L e_n$$

as the orthogonal sum where $q_{\mathrm{std}}(e_n) := 1$.

Thus if n is odd the hermitian quadratic form is given on $\mathcal{O}_L e_n$ by the norm $N_{L/K}$.

If $(M, q, \mathcal{L})$ and $(M', q', \mathcal{L}')$ are two hermitian quadratic spaces over some $\mathcal{O}_K$-algebra R, then we call a similtude $\gamma\colon (M, q, \mathcal{L}) \to (M', q', \mathcal{L}')$ a pair of isomorphisms $\gamma_1\colon M \to M'$ and $\gamma_2\colon \mathcal{L} \to \mathcal{L}'$ such that

$$\gamma_2(q(m)) = q'(\gamma_1(m))$$

for all $m \in M$. We denote by

$$\mathrm{Sim}((M, q, \mathcal{L}), (M', q', \mathcal{L}'))$$

or simply $\mathrm{Sim}(M, M')$ if no confusion is possible the group of such similtudes and by

$$\underline{\mathrm{Sim}}(M, M')\colon R' \to \mathrm{Sim}(M \otimes_R R', M' \otimes_R R')$$

the functor of similtudes. If $\mathcal{L} = \mathcal{L}'$, then it makes sense to look at the subgroup of isomorphisms

$$\mathrm{Isom}(M, M'),$$

that is, at similtudes $\gamma = (\gamma_1, \gamma_2)$ such that $\gamma_2 = \mathrm{Id}_{\mathcal{L}}$. If $(M, q, \mathcal{L}) = (M', q', \mathcal{L}')$ we further abbreviate

$$\underline{\mathrm{Sim}}(M) := \underline{\mathrm{Sim}}(M, M').$$

We need the following lemma.

Lemma 9.8. *Let A be a noetherian ring and let $X \to \mathrm{Spec}(A)$ be a morphism locally of finite type. Let $g \in A$ be some element. If the base changes*

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/g^m) \to \mathrm{Spec}(A/g^m)$$

for all $m \geq 1$ and

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[1/g]) \to \mathrm{Spec}(A[1/g])$$

are smooth, then $X \to \mathrm{Spec}(A)$ is smooth.

Proof. This is proven in [23, Tag 0A43]. Namely, as A is noetherian it suffices to test formal smoothness on local artinian rings. But a morphism from the spectrum of some local artinian ring to $\mathrm{Spec}(A)$ will factor through $\mathrm{Spec}(A[1/g])$ or $\mathrm{Spec}(A/g^m)$ for some $m \geq 1$. □

Next we can prove smoothness of the functor of similitudes.

Proposition 9.9. *For any $n \geq 1$ the sheaf of groups over $\mathcal{O}_K$*

$$\mathcal{G}_{\mathrm{std},n} := \underline{\mathrm{Sim}}(M_{\mathrm{std},n})$$

is represented by an affine smooth group scheme with generic fiber a unitary group of similitudes for L/K.

Proof. It is clear that $\mathcal{G}_{\mathrm{std},n}$ is represented by an affine group scheme of finite type over $\mathcal{O}_K$. We apply Lemma 9.8. We base change from K to L (note that the definitions of a hermitian quadratic form makes sense for L not necessarily a field) and calculate

$$G := \mathrm{G}_{\mathrm{std},n} \otimes_{\mathcal{O}_K} L.$$

Set $K' := L$ and $L' := L \otimes_K L$ and

$$M := M_{\mathrm{std},n} \otimes_K K'.$$

Then $L' = Le_1 \oplus Le_2$ with two non-trivial idempotents e_1 and e_2. Accordingly,

$$M = M_1 + M_2$$

with $M_i = e_i M$. Moreover, for $x \in M_i$ we get

$$q(x) = q(e_i x) = N_{L'/K'}(e_i) q(x) = 0$$

as $N_{L/K}(e_i) = 0$. As $f_{\mathrm{std}} \otimes_K K'$ is non-degenerate it must induce an isomorphism

$$f_{\mathrm{std}}^{\sharp} \colon M_1^{\vee} \cong M_2$$

As every $\gamma \in G$ acts L' linearly it must preserve the decomposition $M = M_1 + M_2$ and we obtain that

$$G \cong \mathrm{GL}(M_1) \times \mathbb{G}_m$$

by mapping $\gamma = (\gamma_1, \gamma_2)$ to its restriction $\gamma_{|M_1}$ and its similtude factor γ_2 and conversely, mapping $(g_1, g_2) \in \mathrm{GL}(M_1) \times \mathbb{G}_m$ to the automorphism

$$(g_1, g_2 f_{\mathrm{std}}^{\sharp} \circ {g_1^{-1}}^{\vee} \circ {f_{\mathrm{std}}^{\sharp}}^{-1}).$$

Thus we obtain that the generic fiber of $\mathcal{G}_{\mathrm{std},n}$ is smooth and in fact a unitary group of similtudes associated with L/K. In order to finish we prove that $\mathcal{G}_{\mathrm{std},n}$ satisfies the lifting criterion for formal smoothness on $\mathcal{O}_K$-algebras R such that π is nilpotent in R. Let $R \to \overline{R}$ be a surjection of such $\mathcal{O}_K$-algebras with kernel I nilpotent. We claim more generally that for every hermitian quadratic space $(M, q, \mathcal{L})$ over R each similtude

$$\varphi \colon M_{\mathrm{std}} \otimes_{\mathcal{O}_K} \overline{R} \cong M \otimes_R \overline{R}$$

can be lifted. For $x \in M$ we denote by $\overline{x} \in M \otimes_R \overline{R}$ its reduction. Let

$$e_1, \ldots, e_n \in M_{\mathrm{std},n}$$

be the standard basis and let $x_1, \ldots, x_n \in M$ such that $\overline{x_i} = \varphi(e_i)$. It suffices to show that after possibly changing the lifts x_i) the basis $x_1, \ldots, x_n$ of M can be brought into the standard form (at least up to some similtude) without changing the reductions $\overline{x_1}, \ldots, \overline{x_n}$. First let us assume that $n \geq 2$. Then

$$f(x_1, \Pi x_2)$$

is a unit because this is true mod I and by rescaling x_1 we may assume that

$$f(x_1, \Pi x_2) = 1.$$

Then we can apply the same reasoning as in Lemma 9.6 to conclude that we can arrange

$$f(x_1, \Pi x_2) = 1, q(x_1) = q(x_2) = 0, f(x_1, x_2) = 0$$

(note that as $q(x_1), q(x_2), f(x_1, x_2), f(x_2, x_2) \in I$ the procedure in Lemma 9.6 does not change $\overline{x_1}, \overline{x_2}$). Moreover, as f is non-degenerate on

$$N := \langle x_1, x_2 \rangle_{R \otimes_{\mathcal{O}_K} \mathcal{O}_L}$$

we may change the lifts $x_3, \ldots, x_n$ to lie in the orthogonal complement of N. Then we may argue by induction to reduce to the case $n = 0$ (if n is even) or $n = 1$ (if n is odd). Note that for n even we do not need to pass to similitudes. However, assume $n = 1$ and let

$$\lambda := q(x_1).$$

Then $\lambda \in \mathcal{L}^\times$ is a generator as this is true mod I. But then

$$\gamma = (\gamma_1, \gamma_2) \colon (M_{\mathrm{std}_1}, q_{\mathrm{std},1}, R) \to (R \otimes_{\mathcal{O}_K} \mathcal{O}_L x_1, q, \mathcal{L})$$

with $\gamma_1(e_1) = x_1$ and $\gamma_2 \colon R \to \mathcal{L},\ 1 \mapsto \lambda$ defines a similtude lifting φ. □

As the proof shows if n is even, the sheaf of isomorphisms

$$\underline{\mathrm{Isom}}(M_{\mathrm{std},n})$$

is affine and smooth, but this does not happen in general in the case when n is odd. Namely, if $n = 1$, then

$$M_{\mathrm{std},1} = \mathcal{O}_L$$

equipped with the norm $N_{L/K}$ and if K has residue characteristic 2, then the torus

$$\mathcal{T} := (\mathrm{Res}_{\mathcal{O}_L/\mathcal{K}} \mathrm{G}_m)^{N_{L/K}=1}$$

of norm 1 elements is not smooth. However, this phenomen does not happen if L/K is tamely ramified, i.e., the residue characteristic of K is odd.

We will denote by

$$\mathcal{G}_{\mathrm{std},n} = \underline{\mathrm{Sim}}(M_{\mathrm{std},n})$$

the smooth affine group scheme from 9.9 (cf. Proposition 9.9).

Theorem 9.10. *Let n be an integer and let R be an $\mathcal{O}_K$-algebra. Then there is equivalence of categories between*

$$\{\mathcal{G}_{\mathrm{std},n} - \textit{torsors for the étale topology over } \mathrm{Spec}(R)\}$$

and non-degenerate hermitian quadratic spaces of rank n over $\mathrm{Spec}(R)$, i.e., triples

$$(M, q, \mathcal{L})$$

with M a finite projective $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$-module of rank n, $\mathcal{L}$ an invertible R-module and $q \colon M \to \mathcal{L}$ a $\mathcal{L}$-valued hermitian quadratic form on M such that the discriminant

$$\mathrm{disc}(q) \colon \Lambda_R^{2n}(M) \to \Lambda_R^{2n}(M^\vee) \otimes_R \mathcal{L}^{2n}$$

if n is even resp. the divided discriminant

$$\mathrm{disc}'(q) \colon \Lambda_R^{2n}(M) \to \Lambda_R^{2n}(M^\vee) \otimes_R \mathcal{L}^{2n} \otimes_{\mathcal{O}_K} \theta$$

if n is odd, is an isomorphism.

Proof. By Proposition 9.9 the group $\mathcal{G}_{\mathrm{std},n}$ is represented by an affine smooth group scheme. Let $(M, q, \mathcal{L})$ be a non-degenerate hermitian quadratic space over $\mathrm{Spec}(R)$. It suffices to prove that the sheaf

$$\underline{\mathrm{Sim}}((M_{\mathrm{std},n}, q_{\mathrm{std},n}, R), (M, q, \mathcal{L}))$$

of similtudes is represented by an affine smooth, surjective scheme over $\mathrm{Spec}(R)$. Clearly, it is represented by an affine scheme. Smoothness follows from the proof of Proposition 9.9 and we are left with surjectivity. Thus we may assume that R is the spectrum of an algebraically closed field. First assume that $\pi \in R^\times$. As $R \otimes_K L \cong R \times R$ we see as in Proposition 9.9 that $M = M_1 \oplus M_2$ decomposes into to isotropic subspaces and that f induces a perfect pairing $f\colon M_1 \times M_2 \to K$. The same does happen for $M_{\mathrm{std},n} \otimes_K R$ and we obtain our desired isomorphism. Thus assume that $\pi R = 0$. To lighten notation we may even assume that $R = k$ is the residue field of $\mathcal{O}_K$, the general case is handled similarly or deduced by a suitable unramified base change $\mathcal{O}_K \to \mathcal{O}_{K'}$ with $\mathcal{O}_{K'}$ having residue field R. Then $R \otimes_{\mathcal{O}_L} \mathcal{O}_L \cong k[\Pi]$ with $\Pi^2 = 0$. If $n = 1$, let $x \in M$ be a generator of M over $\mathcal{O}_L$. The divided discriminant in this case is given by $q(x)^2$ (cf. Example 9.5). By assumption it is a generator of $\mathcal{L}^2$. In particular, $q(x)$ generates $\mathcal{L}$. The pair

$$\gamma = (\gamma_1, \gamma_2)\colon (M_{\mathrm{std},1}, q_{\mathrm{std},1}, R) \to (M, q, \mathcal{L})$$

with

$$\gamma_1\colon M_{\mathrm{std},1} \to M,\ e_1 \mapsto x$$

and

$$\gamma_2\colon R \to \mathcal{L},\ 1 \mapsto q(x)$$

defines a similtude as we searched for. Now assume that $n \geq 2$. We want to construct $x, y \in M$ such that

$$f(x, \Pi y) = 1.$$

If n is even, then this follows from non-degeneracy of f. Namely, take any $y \in M \setminus \Pi M$. Then there exists some $x \in M$ such that

$$f(x, \Pi y) = 1$$

because f is non-degenerate and $\Pi y \neq 0$. Hence, we may assume n odd, and thus $n \geq 3$. We may assume $\mathcal{L} = k$ is trivial. Let us assume that there do no exist $x, y \in M$ such that $f(x, \Pi y) \neq 0$, i.e., that

$$f(x, \Pi y) = 0$$

for all $x, y \in M$. Let $e_1, \ldots, e_n \in M$ be a basis of M over $k[\Pi]$ and set

$$A_{ii} := q(e_i)$$

and

$$\tilde{A}_{i,j} := f(e_i, e_j)$$

for $1 \leq i, j \leq n$ (as in Lemma 9.2). If

$$f(e_i, e_j) \neq 0$$

for some i, j with $i \neq j$, manipulating the basis $e_1, \ldots, e_n$ we can achieve that $f(e_1, e_2) = 1$ and that the spaces

$$N_1 := \langle e_1, e_2 \rangle_{k[\Pi]}$$

and

$$N_2 := \langle e_3, \ldots, e_n \langle_{k[\Pi]}$$

are orthogonal. Let $q_1 \colon N_1 \to k$ and $q_2 \colon N_2 \to k$ be the restriction of q to N_1 and N_2. Lifting N_1 and N_2 to orthogonal subspaces in a lift of M to $\mathcal{O}_K$ we can see that

$$\mathrm{disc}'(q) = \mathrm{disc}(q_1)\mathrm{disc}'(q_2).$$

But $\mathrm{disc}(q_1) = 0$ as f on N_1 in the k-basis $e_1, e_2, \Pi e_1, \Pi e_2$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus we obtain $f(e_i, e_j) = 0$ for $i \neq j$. Then we can lift M to a hermitian quadratic space $\tilde{M}$ over $\mathcal{O}_K$ which is the orthogonal sum of $k[\Pi]$-submodules of rank 1. Calculating the discriminant of $\tilde{M}$ we see that it is at least divisible by $(4\pi - t^2)^n$. As $n \geq 3$ this is a contradiction and we see that we do find $x, y \in M$ such that $f(x, \Pi y) = 1$. Hence, now in any case (n even or odd) we may assume that there are $x, y \in M$ satisfying

$$f(x, \Pi y) = 1.$$

By Lemma 9.6 the form f is non-degenerate on the space $N := \langle x, y \rangle_{k[\Pi]}$ and thus decomposes into an orthogonal direct sum

$$M = N \oplus N'$$

with N' the orthogonal complement of N and N isomorphic to the standard space $\cong M_{\mathrm{std},2}$. Now, we may apply induction to N' and conclude. □

Recall the situation of Section 6, thus let A be a ring, $U \subseteq \mathrm{Spec}(A)$ quasi-compact open and assume that the restriction functor

$$\mathrm{Bun}(\mathrm{Spec}(A)) \cong \mathrm{Bun}(U)$$

for vector bundles is an equivalence. Furthermore assume that A is an $\mathcal{O}_K$-algebra and that similarly the restriction defines an equivalence

$$\mathrm{Bun}(A \otimes_{\mathcal{O}_K} \mathcal{O}_L) \cong \mathrm{Bun}(U \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathcal{O}_L).$$

We can derive the following corollary.

Corollary 9.11. *Let $n \geq 1$ and let $\mathcal{P}$ be a $\mathcal{G}_{\mathrm{std},n}$-torsor on U. Then $\mathcal{P}$ extends to* $\mathrm{Spec}(A)$.

Proof. This follows from Theorem 9.10 as the data therein extends from U to $\mathrm{Spec}(A)$ still satisfying a non-degenerate hermitian quadratic form. □

Let us end this section with some comments on normalforms of lattices for even orthogonal groups. Thus, let G/K be the orthogonal group $G = O(q)$ associated to some non-degenerate quadratic form $q \colon V \to K$ with $\dim(V)$ even. It would be desirable to find (in arbitrary residue characteristic) some smooth affine model $\mathcal{G}$ of G over $\mathcal{O}_K$ (which probably turns out to be parahoric) and a concrete description of torsors under it, similar to Theorem 9.10. Namely, this would (probably) imply extension results in wildly ramified cases for non-trialitarian groups of type D, a case missing in 7.9 and 8.4. But we are doubtful that such a description is possible if K has residue characteristic 2, because of the following problem. It is natural to expect that such a linear algebra description would involve quadratic forms or

symmetric bilinear forms.[2] But in this linear algebra description each of these forms has its discriminant which is a square *flat* locally, but not necessarily *étale* locally. But as the searched for model $\mathcal{G}$ is required to be smooth, its categories of flat and étale torsors are equivalent. Thus in this linear algebra description of torsors under $\mathcal{G}$ the discriminants must by some reason forced to be a square étale locally. We note that this problem does not occur for defining an affine smooth model $\mathcal{G}$ of G, only for a description of torsors under it. In fact in [6] smooth models for orthogonal groups are constructed in every residue characteristic (if at least K has characteristic not 2) by concrete lattice chains with symmetric bilinear forms and quadratic forms. In the unitary case of this section this type of a problem does not appear, due to the hermitian property of the quadratic forms considered. For example, in the non-degenerate odd case the divided discriminant is always a square as follows from Example 9.5.

10. A specialization map between mixed-characteristic affine Grassmannians

Let k be an algebraically closed field of char $p > 0$ and let $C/W(k)[1/p]$ be an algebraically closed, non-archimedean field with residue field k'. After possibly enlarging k we may without loosing generality assume $k = k'$. In this section we want to use Theorem 7.9 to concoct for a parahoric group scheme $\mathcal{G}$ over $W(k)$ a canonical specialization map

$$\mathrm{sp}\colon \mathrm{Gr}_{\mathcal{G}}^{B_{\mathrm{dR}}^+}(C) \to \mathrm{Gr}_{\mathcal{G}}^{W}(k)$$

between the mixed characteristic affine Grassmannians $\mathrm{Gr}_{\mathcal{G}}^{B_{\mathrm{dR}}^+}$ and $\mathrm{Gr}_{\mathcal{G}}^{W}$. The existence of the specialization map is motivated by results of Richarz [19, Theorem 1.19] and the definition of a mixed-characteristic Beilinson-Drinfeld Grassmannian [20, Definition 20.4.1.] resp. [21, Definition 20.3.1.]. In fact, using Theorem 7.9 it is in fact possibly to prove that the mixed-characteristic Beilinson-Drinfeld Grassmannian is ind-proper (cf. [21, Section 21.2]).

Let us recall the definition of both affine Grassmannians (we content ourselves with their k resp. C-valued points.[3] We will denote by $C^\flat$ the tilt of C. By definition (cf. [24] or [4]) the k-valued points of the Witt vector affine Grassmannian for $\mathcal{G}$ (or better Witt vector affine flag variety) are pairs

$$(\mathcal{P}, \alpha)$$

with $\mathcal{P}$ a $\mathcal{G}$-torsor on $\mathrm{Spec}(W(k))$ and α a trivialization of $\mathcal{P}_{|\mathrm{Spec}(W(k)[1/p])}$. On the other hand, the C-valued points of the B_{dR}^+-affine Grassmannian (cf. [20, Definition 20.4.1.] resp. [21, Definition 19.1.1.]) are pairs

$$(\mathcal{P}', \alpha')$$

with $\mathcal{P}'$ a $\mathcal{G}$-torsor on $\mathrm{Spec}(B_{\mathrm{dR}}^+(C))$ and α' a trivialization of $\mathcal{P}_{|\mathrm{Spec}(B_{\mathrm{dR}}(C)}$. We note that, as B_{dR}^+ contains an algebraic closure $\overline{W(k)[1/p]}$ of $W(k)[1/p]$, a $\mathcal{G}$-torsor over $\mathrm{Spec}(B_{\mathrm{dR}}^+(C))$ is just a torsor under the split reductive geometric generic fiber $\mathcal{G}_{\overline{W(k)[1/p]}}$ of $\mathcal{G}$.

[2]More seriously, looking at the local Dynkin diagram the reductive quotients of the special fibers of parahoric models of G are again orthogonal groups, thus defined by quadratic forms.

[3]For the precise geometric structure as a v-sheaf we confer to [21, Section 20.3].

Now, let us construct the specialization map

$$\mathrm{sp}\colon \mathrm{Gr}_{\mathcal{G}}^{B_{\mathrm{dR}}^+}(C) \to \mathrm{Gr}_{\mathcal{G}}^{W}(k).$$

Let $(\mathcal{P}', \alpha') \in \mathrm{Gr}_{\mathcal{G}}^{B_{\mathrm{dR}}^+}(C)$ be given. The kernel of Fontaine's map

$$\theta\colon A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$$

is generated by a non-zero divisor ξ. In fact, we may simply take ξ of the form

$$\xi = p - [\varpi]$$

for a suitable $\varpi \in \mathfrak{m}_{C^\flat}$. The ξ-adic completion of $A_{\mathrm{inf}}[1/p]$ is by definition Fontaine's ring

$$B_{\mathrm{dR}}^+.$$

Using the Beauville-Laszlo gluing lemma (cf. [2]) and the given data $(\mathcal{P}', \alpha')$ we can modify the trivial $\mathcal{G}$-torsor $\mathcal{P}_0$ on

$$\mathrm{Spec}(A_{\mathrm{inf}}) \setminus \{s\}$$

at the point $\infty \in \mathrm{Spec}(A_{\mathrm{inf}}) \setminus \{s\}$ defined by ξ (cf. Lemma 3.1). Thus we obtain canonically a $\mathcal{G}$-torsor $\mathcal{P}_1$ over $\mathrm{Spec}(A_{\mathrm{inf}}) \setminus \{s\}$ with an isomorphism

$$\mathcal{P}_{1|\mathrm{Spec}(A_{\mathrm{inf}})\setminus\{s,\infty\}} \cong \mathcal{P}_{0|\mathrm{Spec}(A_{\mathrm{inf}})\setminus\{s,\infty\}}.$$

In particular, the torsor $\mathcal{P}_1$ is trivial when restricted to the crystalline part $U_{\mathrm{crys}} \subseteq \mathrm{Spec}(A_{\mathrm{inf}})$ (cf. Lemma 3.1). By Theorem 7.9 the $\mathcal{G}$-torsor $\mathcal{P}_1$ extends uniquely to a $\mathcal{G}$-torsor $\mathcal{P}_2$ on $\mathrm{Spec}(A_{\mathrm{inf}})$. In particular, we still have a canonical trivialization

$$\mathcal{P}_{2|\mathrm{Spec}(A_{\mathrm{inf}})\setminus\{s,\infty\}} \cong \mathcal{P}_{1|\mathrm{Spec}(A_{\mathrm{inf}})\setminus\{s,\infty\}} \cong \mathcal{P}_{0|\mathrm{Spec}(A_{\mathrm{inf}})\setminus\{s,\infty\}}$$

of $\mathcal{P}_2$ on $\mathrm{Spec}(A_{\mathrm{inf}}) \setminus \{s, \infty\}$. Now set

$$\mathcal{P} := \mathcal{P}_{2|\mathrm{Spec}(W(k)}$$

as the restriction of $\mathcal{P}_2$ along the canonical morphism $A_{\mathrm{inf}} \to W(k)$ and α as the canonical trivialization

$$\mathcal{P}_{|\mathrm{Spec}(W(k)[1/p]} \cong \mathcal{P}_{2|\mathrm{Spec}(W(k)[1/p]} \cong \mathcal{P}_{0|\mathrm{Spec}(W(k)[1/p]}.$$

The data $(\mathcal{P}, \alpha)$ defines a k-valued point in the Witt vector affine Grassmannian and we set

$$\mathrm{sp}(\mathcal{P}', \alpha') := (\mathcal{P}, \alpha).$$

This finishes the construction of sp. In a more compact form, the specialization map is given as the chain of equivalences and maps

$$\begin{gathered} G(B_{\mathrm{dR}}(C))/G(B_{\mathrm{dR}}^+(C)) \\ \cong \{(\mathcal{G}-\text{torsor } \mathcal{P} \text{ on } \mathrm{Spec}(A_{\mathrm{inf}}) \setminus \{s\},\ \alpha \text{ a trivialization of } \mathcal{P}_{|\mathrm{Spec}(A_{\mathrm{inf}}[1/\xi])})\} \\ \cong \{(\mathcal{G}-\text{torsor } \mathcal{P} \text{ on } \mathrm{Spec}(A_{\mathrm{inf}}),\ \alpha \text{ a trivialization of } \mathcal{P}_{|\mathrm{Spec}(A_{\mathrm{inf}}[1/\xi])})\} \\ \to \{(\mathcal{G}-\text{torsor } \mathcal{P} \text{ on } \mathrm{Spec}(W(k)),\ \alpha' \text{ a trivialization of } \mathcal{P}_{|\mathrm{Spec}(W(k)[1/p])})\} \\ \cong G(W(k)[1/p])/\mathcal{G}(W(k)). \end{gathered}$$

Here the first and last $\cong$'s are the description of the affine Grassmannian via torsors (using Beauville-Laszlo, Lemma 5.2, for the first), the second equivalence is deduced from Theorem 7.9 (and Proposition 6.2) and the arrow $\to$ is simply base change along $A_{\mathrm{inf}} \to W(k)$ (which maps the ideal (ξ) to the ideal (p)).

Using this description it follows that the specialization map

$$\mathrm{sp}\colon G(B_{\mathrm{dR}}(C))/G(B_{\mathrm{dR}}^+(C)) \to G(W(k)[1/p])/\mathcal{G}(W(k))$$

is equivariant for the action of the subgroup $G(A_{\mathrm{inf}}[1/\xi]) \subseteq G(B_{\mathrm{dR}}(C))$ on the trivialization α of $\mathcal{P}_{|\mathrm{Spec}(A_{\mathrm{inf}}[1/\xi])}$. For tori we can provide a different description of sp. Let $\mathcal{T}$ be a parahoric group scheme over $W(k)$ such that

$$T := \mathcal{T}_{\mathrm{Spec}(W(k)[1/p]}$$

is a torus. Then there are canonical bijections

$$\mathrm{Gr}_{\mathcal{T}}^{B_{\mathrm{dR}}^+}(C) \cong X_*(T)$$

(by observing that $B_{\mathrm{dR}}^+(C)$ is abstractly isomorphic to $C[[\xi]]$) and

$$\mathrm{Gr}_{\mathcal{T}}^{W}(k) \cong X_*(T)_\Gamma$$

where Γ is the absolute Galois group of $W(k)[1/p]$ (cf. [24, Proposition 1.21.]).

Lemma 10.1. *For $\mathcal{T}$ as above the diagram*

$$\begin{array}{ccc} \mathrm{Gr}_{\mathcal{T}}^{B_{\mathrm{dR}}^+}(C) & \xrightarrow{\mathrm{sp}} & \mathrm{Gr}_{\mathcal{T}}^{W}(k) \\ \downarrow \cong & & \downarrow \cong \\ X_*(T) & \xrightarrow{\mathrm{can}} & X_*(T)_\Gamma \end{array}$$

with $\mathrm{can}\colon X_*(T) \to X_*(T)_\Gamma$ *the canonical projection commutes.*

Proof. We first handle the case $T = \mathbb{G}_m$ (which implies $\mathcal{T} = \mathbb{G}_m$). Then $X_*(T) \cong \mathbb{Z}$ and $j \in \mathbb{Z}$ is mapped to the class of $\xi^j \in \mathrm{Gr}_{\mathcal{T}}^{B_{\mathrm{dR}}^+}(C)$. This class corresponds to the trivial line bundle $\mathcal{L}$ on $\mathrm{Spec}(A_{\mathrm{inf}}) \setminus \{s\}$ with trivialization ξ^j on $\mathrm{Spec}(A_{\mathrm{inf}}[1/\xi])$. The line bundle $\mathcal{L}$ extends canonically to the trivial line bundle, again denoted $\mathcal{L}$, on $\mathrm{Spec}(A_{\mathrm{inf}})$. Hence, the specialization map sends ξ^j to the class in $\mathrm{Gr}_{\mathcal{T}}^{W}(k)$ corresponding to the pair

$$(W(k) = \mathcal{L} \otimes_{A_{\mathrm{inf}}} W(k), \bar{\xi}^j \colon W(k)[1/p] \cong W(k)[1/p]).$$

But $\bar{\xi}^j = p^j$, which shows the claim for $T = \mathbb{G}_m$. As in [19, Lemma 1.21] we can use this to deal with the case that T is induced by using that $X_*(T)_\Gamma$ is torsionfree in such cases. In the general case, choose a surjection

$$T' \to T$$

with T' induced and connected kernel T''. As

$$\mathrm{Gr}_{T'}^{B_{\mathrm{dR}}^+}(C) \twoheadrightarrow \mathrm{Gr}_{T}^{B_{\mathrm{dR}}^+}(C)$$

is surjective (by Steinberg's theorem as T'' is connected, cf. Theorem 5.1) the general case follows then from naturality of the specialization map. □

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