SHARP TOTAL VARIATION RESULTS FOR MAXIMAL FUNCTIONS

JOÃO P. G. RAMOS

Abstract. In this article, we prove some total variation inequalities for maximal functions. Our results deal with two possible generalizations of the results contained in [1], one of those considers a variable truncation of the maximal function, and the other one interpolates the centered and the uncentered maximal functions. We also provide counterexamples showing that our methods do not apply outside our parameter ranges, and therefore are sharp.

1. Introduction

An object of major interest in Harmonic Analysis is the Hardy-Littlewood maximal function, which can be defined as

$$Mf(x) = \sup_{t \in \mathbb{R}_+} \frac{1}{2t} \int_{x-t}^{x+t} |f(s)| \mathrm{d}s.$$

Alternatively, one can also define its *uncentered* version as

$$\tilde{M}f(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(s)| \mathrm{d}s.$$

The most classical result about these maximal functions is perhaps the Hardy–Littlewood–Wiener theorem, which states that both M and $\tilde{M}$ map $L^p(\mathbb{R})$ into itself for $1 < p \leq \infty$, and that in the case $p = 1$ they satisfy a weak type inequality:

$$|\{x \in \mathbb{R} \colon Mf(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1,$$

where $C = \frac{11+\sqrt{61}}{12}$ is the best constant possible found by A. Melas [12] for M. The same inequality also holds in the case of $\tilde{M}$ above, but this time with $C = 2$ being the best constant, as shown by F. Riesz [13].

In the remarkable paper [6], J. Kinnunen proves, using functional analytic techniques and the aforementioned theorem, that, in fact, M maps the Sobolev spaces $W^{1,p}(\mathbb{R})$ into themselves, for $1 < p \leq \infty$. Kinnunen also proves that this result holds if we replace the standard maximal function by its uncentered version. This opened a new field of studies, and several other properties of this and other related maximal functions were studied. We mention, for example, [3, 4, 5, 7, 9].

Since the Hardy-Littlewood maximal function fails to be in L^1 for every nontrivial function f and the tools from functional analysis used are not available either in the case $p = 1$, an important question was whether a bound of the form $\|(Mf)'\|_1 \leq C\|f'\|_1$ could hold for every $f \in W^{1,1}$.

In the uncentered case, H. Tanaka [15] provided us with a positive answer to this question. Explicitly, Tanaka proved that, whenever $f \in W^{1,1}(\mathbb{R})$, then $\tilde{M}f$ is weakly differentiable, and it satisfies that $\|(\tilde{M}f)'\|_1 \leq 2\|f'\|_1$. Here, $W^{1,1}(\mathbb{R})$ stands for the Sobolev space $\{f : \mathbb{R} \to \mathbb{R} \colon \|f\|_1 + \|f'\|_1 < +\infty\}$.

Some years later, Aldaz and Pérez Lazaro [1] improved Tanaka's result, showing that, whenever $f \in BV(\mathbb{R})$, then the maximal function $\tilde{M}f$ is in fact *absolutely continuous*, and $\mathcal{V}(\tilde{M}f) = \|(\tilde{M}f)'\|_1 \leq \mathcal{V}(f)$, with $C = 1$ being sharp, where we take the *total variation of a function* to be $\mathcal{V}(f) := \sup_{\{x_1,\ldots,x_N\}=\mathcal{P}} \sum_{i=1}^{N-1} |f(x_{i+1}) - f(x_i)|$, and consequently the space of *bounded variation* *functions* as $BV(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \colon \mathcal{V}(f) < +\infty\}$. In this direction, J. Bobber, E. Carneiro, K. Hughes and L. Pierce [2] studied the discrete version of this problem, obtaining similar results.

In the centered case, many questions remain unsolved. Surprisingly, it turned out to be *harder* than the uncentered one, due to the contrast in smoothnes of Mf and $\tilde{M}f$. In [8], O. Kurka showed the endpoint question to be true, that is, that $\mathcal{V}(Mf) \le C\mathcal{V}(f)$, with $C = 240,004$. Unfortunately, his method does not give the best constant possible, with the standing conjecture being that $C = 1$ is the sharp constant.

In [16], F. Temur studied the discrete version of this problem, proving that for every $f \in BV(\mathbb{Z})$ we have $\mathcal{V}(Mf) \le C'\mathcal{V}(f)$, where $C' > 10^6$ is an absolute constant. The standing conjecture is again that $C' = 1$ in this case, which was in part backed up by J. Madrid's optimal results [11]: If $f \in \ell^1(\mathbb{Z})$, then $Mf \in BV(\mathbb{Z})$, and $\mathcal{V}(Mf) \le 2\|f\|_1$, with 2 being sharp in this inequality.

Our main theorems deal with – as far as the author knows – the first attempt to prove sharp bounded variation results for classical Hardy–Littlewood maximal functions. Indeed, we may see the classical, uncentered Hardy-Littlewood maximal function as

$$\tilde{M}f(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(s)|\mathrm{d}s = \sup_{(y,t)\colon |x-y|\le t} \frac{1}{2t}\int_{y-t}^{y+t} |f(s)|\mathrm{d}s.$$

Notice that this supremum need *not* be attained for every function f and at every point $x \in \mathbb{R}$, but it *is* attained if our points are strict local maxima of $\tilde{M}f$. This way, we may look at this operator as a particular case of the wider class of *nontangential* maximal operators

$$M^\alpha f(x) = \sup_{|x-y|\le \alpha t} \frac{1}{2t}\int_{y-t}^{y+t} f(s)\mathrm{d}s.$$

Indeed, from this new definition, we get directly that

$$\begin{cases} M^\alpha f = Mf, & \text{if } \alpha = 0, \\ M^\alpha f = \tilde{M}f, & \text{if } \alpha = 1. \end{cases}$$

As in the uncentered case, we can still define 'truncated' versions of these operators, by imposing that $t \le R$. These operators are far from being a novelty: several references consider those all around mathematics, among those the classical [14, Chapter 2], and the more recent, yet related to our work, [4]. An easy argument (see Section 5.3 below) proves that, if $\alpha < \beta$, then

$$\mathcal{V}(M^\beta f) \le \mathcal{V}(M^\alpha f).$$

This implies already, by the main Theorem in [8], that there exists a constant $A \ge 0$ such that $\mathcal{V}(M^\alpha f) \le A\mathcal{V}(f)$, for all $\alpha > 0$. In the intention of sharpening this result, our first result reads, then, as follows:

Theorem 1. *Fix any $f \in BV(\mathbb{R})$. For every $\alpha \in [\frac{1}{3}, +\infty)$, we have that*

$$\mathcal{V}(M^\alpha f) \le \mathcal{V}(f). \tag{1}$$

There exist an extremizer f for the inequality (1). If $\alpha > \frac{1}{3}$, then any positive extremizer f to inequality (1) satisfies:

- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x)$.
- *There is x_0 such that f is non-decreasing on $(-\infty, x_0)$ and non-increasing on $(x_0, +\infty)$.*

Conversely, all such functions are extremizers to our problem. Finally, for every $\alpha \ge 0$ and $f \in W^{1,1}(\mathbb{R})$, $M^\alpha f \in W^{1,1}_{loc}(\mathbb{R})$.

Notice that stating that a function $g \in W^{1,1}_{loc}(\mathbb{R})$ is the same as asking it to be absolutely continuous. Our ideas to prove this theorem and theorem 3 are heavily inspired by the ones in

[1]. Our aim will always be to prove that, when $f \in BV(\mathbb{R})$, then the maximal function $M^\alpha f$ is well-behaved on the detachment set

$$E_\alpha = \{x \in \mathbb{R} \colon M^\alpha f(x) > f(x)\}.$$

Namely, we seek to obtain that the maximal function does not have any local maxima in the set where it disconnects from the original function, which should be the main idea behind the *maximal attachment property.* Such property, together with the concept of detachment set E_α, are also far from being new, having already appeared at [1, 3, 4, 15], and recently at [10]. More specific details of this can be found in the next section.

In general, our main ideas are contained in Lemma 2, where we prove that the region in the upper half plane that is taken into account for the supremum that defines

$$M^1_{\equiv R} f = \sup_{x \in I \colon |I| \le 2R} \fint_I |f(s)| \mathrm{d}s,$$

where we define

$$\fint_I g(s)\mathrm{d}s := \frac{1}{|I|}\int_I g(s)\mathrm{d}s,$$

is actually a (rotated) *square*, and not a triangle – as a first glance might impress on someone –, and in the comparison of $M^\alpha f$ and $M^1_{\equiv R}$ over a small interval, in order to establish the maximal attachment property.

We may ask ourselves if, for instance, we could go lower than 1/3 with this method. Our next result, however, shows that this is the optimal bound for this technique:

Theorem 2. *Let $\alpha < \frac{1}{3}$. Then there exists $f \in BV(\mathbb{R})$ and a point $x_\alpha \in \mathbb{R}$ such that x_α is a local maximum of $M^\alpha f$, but $M^\alpha f(x_\alpha) > f(x_\alpha)$.*

We can, however, inquire ourselves whether we can genralize the results from Aldaz and Pérez-Lázaro in yet another direction. With this in mind, we notice that Kurka [8] cites in his paper that his techniques allow one to prove that some Lipschitz truncations of the center maximal function, that is, maximal functions of the form

$$M^0_N f(x) = \sup_{t \le N(x)} \frac{1}{2t}\int_{x-t}^{x+t} |f(s)|\mathrm{d}s,$$

are bounded from $BV(\mathbb{R})$ to $BV(\mathbb{R})$ – with some possibly big constant – if $\mathrm{Lip}(N) \le 1$. Inspired by it, we define the *$N-$truncated uncentered maximal function* as

$$M^1_N f(x) = \sup_{|x-y| \le t \le N(x)} \fint_{y-t}^{y+t} |f(s)|\mathrm{d}s.$$

The next result deals then with an analogous of Kurka's result in the case of the centered maximal functions. In fact, we achieve even more in this case, as we have also the explicit sharp constants for that. In details, the result reads as follows:

Theorem 3. *Let $N : \mathbb{R} \to \mathbb{R}_+$ be a measurable function. If Lip$(N) \le \frac{1}{2}$, we have that, for all $f \in BV(\mathbb{R})$,*

$$\mathcal{V}(M^1_N f) \le \mathcal{V}(f).$$

Moreover, the result is sharp, in the sense that there are non-constant functions f such that $\mathcal{V}(f) = \mathcal{V}(M^1_N f)$.

Again, we are also going to use the maximal attachment property in this case. Actually, we are going to prove it both in theorems 1 and 3 for the non-endpoint cases $\alpha > \frac{1}{3}$ and $\mathrm{Lip}(N) < \frac{1}{2}$, while the endpoints are treated with an easy limiting argument.

In the same way, one may ask whether we can ask our Lipschitz constant to be greater than $\frac{1}{2}$ in this result. Regarding this question, we prove in section 4.3 the following negative answer:

Theorem 4. *Let $c > \frac{1}{2}$ and*

$$f(x) = \begin{cases} 1, & \text{if } x \in (-1,0); \\ 0, & \text{otherwise.} \end{cases}$$

Then there is a function $N : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $Lip(N) = c$ and

$$\mathcal{V}(M_N^1 f) = +\infty.$$

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2. Basic definitions and properties

Throughout the paper, I and J will usually denote open intervals, and $l(I), l(J), r(I), r(J)$ their left and right endpoints, respectively. We also denote, for $f \in BV(\mathbb{R})$, the *one-sided limits* $f(a+)$ and $f(a-)$ to be

$$f(a+) = \lim_{x \searrow a} f(x) \text{ and } f(a-) = \lim_{x \nearrow a} f(x).$$

We also define, for a general function $N : \mathbb{R} \to \mathbb{R}$, its *Lipschitz constant* as

$$\text{Lip}(N) := \sup_{x,y \in \mathbb{R}} \frac{|N(x) - N(y)|}{|x-y|}.$$

Throughout the text we will consider a function in $BV(\mathbb{R})$ as *normalized* if $f(x) = \limsup_{y \to x} f(y)$, $\forall x \in \mathbb{R}$. This normalization appears already in [1] in their lemmata, and therefore there is no harm in assuming it here.

We mention also a couple of words about what we called before the *maximal attachment property.* In the paper [1], the authors developed an ingenious way to prove the sharp bounded variation result for the uncentered maximal function. Namely, they proved that, whenever $f \in BV(\mathbb{R})$, then the maximal function $\tilde{M}f$ is actually continuous, and the (open) set

$$E = \{\tilde{M}f > f\} = \cup_j I_j$$

satisfies that, in each of the intervals I_j, $\tilde{M}f$ has no local maxima. More specifically, they observed that every local maximum x_0 of $\tilde{M}f$ satisfies that $\tilde{M}f(x_0) = f(x_0)$. In our case, we are going to need the general version of the maximal attachment property, as the statement with local maxima of $M^\alpha f(x_0)$ may not hold.

We may prove, however, in all cases the following property:

Property 1. *We say that an operator $\mathcal{O}$ defined on the class of bounded variation functions satisfies the* maximal attachment property *if, for every $f \in BV(\mathbb{R})$ and local maximum x_0 of $\mathcal{O}f$ over an interval (a,b), with $Of(x_0) > \max(Of(a), Of(b))$, then there exists an interval $(a,b) \supset I \ni x_0$ such that $\mathcal{O}f$ is constant on I and there is $y \in I$ such that $\mathcal{O}f(y) = f(y)$.*

One may prove, by following the arguments in [1], that such an operator must mandatorily satisfy that $\mathcal{V}(\mathcal{O}f) \leq \mathcal{V}(f)$. Proofs of this fact in our specific cases will eventually appear in our paper.

Finally, we define, for $f \in BV_{loc}(\mathbb{R})$, the *variation of a function over an interval I* as

$$\mathcal{V}_I(f) = \sup_{l(I) \leq x_1 < \cdots < x_N \leq r(I)} \sum_{i=1}^{N-1} |\max(f(x_{i+1}+), f(x_{i+1}-)) - \max(f(x_i+), f(x_i-))|.$$

FIGURE 1. For the function $f = \chi_{(-1,1)}$, we have that the variation $\mathcal{V}_{(-\infty,-1)}(M^0 f) = \mathcal{V}_{(-\infty,-1)}(f)$, whereas, if we defined the variation of an interval as we did for the real line, we would not obtain this equality.

This definition might seem a little artificial in the beginning, but it is going to be the precise one for our sharp results. Moreover, we see directly from the definition that, if $I = \mathbb{R}$, then $\mathcal{V}_{\mathbb{R}} = \mathcal{V}$.

3. Proof of Theorems 1 and 2

In what follows, let $f \in BV(\mathbb{R})$ have our normalization $f(x) = \limsup_{y \to x} f(y)$.

3.1. M^α **satisfies the maximal attachment property.** Here, we prove this cornerstone fact that will facilitate our work. Let then $[a,b]$ be an interval, and suppose that $M^\alpha f$ has a *strict* local maximum at $x_0 \in (a,b)$. Suppose also that $M^\alpha f(x_0) = u(y,t),$ for some $(y,t) \in \{(z,s); |z-x| \leq \alpha t\}$, where we define the function $u : \mathbb{R}^2_+ \to \mathbb{R}_+$ as

$$u(y,t) = \frac{1}{2t} \int_{y-t}^{y+t} |f(s)| \, \mathrm{d}s.$$

This implies, of course, that $M^\alpha f(x_0) = M^\alpha f(y)$. Moreover, we claim that

$$[y - \alpha t, y + \alpha t] \subset (a,b).$$

If this did not hold, then $[y-\alpha t, y+\alpha t] \ni$ either a or b. Let us suppose, without loss of generality, that $a \in [y - \alpha t, y + \alpha t]$. But then

$$a \geq y - \alpha t \Rightarrow |a - y| \leq \alpha t \Rightarrow M^\alpha f(a) \geq M^\alpha f(y) \geq M^\alpha f(x_0),$$

a contradiction to our assumption of strictness of the maximum. This implies that, as for any $z \in [y - \alpha t, y + \alpha t] \Rightarrow |z - y| \leq \alpha t$, the maximal function $M^\alpha f$ is *constant* over the interval $[y - \alpha t, y + \alpha t]$. Moreover, we have that the supremum of

$$u(z,s), \text{ for } (z,s) \in \cup_{z' \in [y-\alpha t, y+\alpha t]} \{(z'', s'') : |z'' - z'| \leq \alpha s''\} =: C(y, \alpha, t),$$

is attained for $(z,s) = (y,t)$.

Our next step is to find a subinterval I of $[y - \alpha t, y + \alpha t]$ and a $R = R(y, \alpha, t)$ such that, over this interval I, it holds that

$$M^1_{\equiv R} f \equiv M^\alpha f.$$

Here, $M^1_{\equiv R}$ stands for the operator $\sup_{x \in I, |I| \leq 2R} \fint_I |f(s)| \mathrm{d}s$. For that, we need to investigate a few properties of the restricted maximal function $M^1_{\equiv R} f$. This is done via the following:

Lemma 1 (Boundary Projection Lemma). *Let $(y,t) \in \mathbb{R}^2_+$. Let us denote*

$$\frac{1}{2t} \int_{y-t}^{y+t} f(s) \mathrm{d}s = u(y,t).$$

If $(y,t) \in \{(z,s); |z-x| \leq s\}$, then

$$u(y,t) \leq \max \left\{ u\left(\frac{x+y-t}{2}, \frac{x-y+t}{2} \right), u\left(\frac{x+y+t}{2}, \frac{y-x+t}{2} \right) \right\}.$$

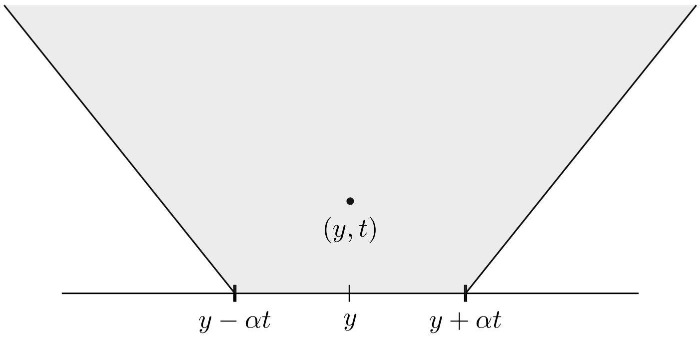

FIGURE 2. The region $C(y,\alpha,t)$.

Proof. The proof is simple: we just have to write

$$\begin{aligned}
u(y,t) &= \frac{1}{2t}\int_{y-t}^{y+t} f(s)\mathrm{d}s = \frac{1}{2t}\int_{y-t}^{x} f(s)\mathrm{d}s + \frac{1}{2t}\int_{x}^{y+t} f(s)\mathrm{d}s \\
&= \frac{x-y+t}{2t}\frac{1}{x-y+t}\int_{y-t}^{x} f(s)\mathrm{d}s \\
&+ \frac{y-x+t}{2t}\frac{1}{y-x+t}\int_{x}^{y+t} f(s)\mathrm{d}s \\
&= \frac{x-y+t}{2t} u\left(\frac{x+y-t}{2}, \frac{x-y+t}{2}\right) \\
&+ \frac{y-x+t}{2t} u\left(\frac{x+y+t}{2}, \frac{y-x+t}{2}\right) \\
&\leq \max\left\{ u\left(\frac{x+y-t}{2}, \frac{x-y+t}{2}\right), u\left(\frac{x+y+t}{2}, \frac{y-x+t}{2}\right)\right\}.
\end{aligned}$$

□

Let $M_{r,A}f(x) = \sup_{0\leq t\leq A} \frac{1}{t}\int_x^{x+t} |f(s)|\,\mathrm{d}s$, and define $M_{l,A}f$ in a same way, there the subindexes "r" and "l" represent, respectively, "right" and "left". There operators are present out of the context of sharp regularity estimates for maximal functions, just like in [13]. In the realm of regularity of maximal function, though, the first to introduce this notion was Tanaka [15]. As a corollary, we may obtain the following:

Corollary 1. *For every $f \in L^1_{loc}(\mathbb{R})$, it holds that*

$$\sup_{|z-x|+|t-R|\leq R} u(z,t) \leq \max\{M_{r,R}f(x), M_{l,R}f(x)\}.$$

From this last corollary, we are able to establish the following important – and, as far as the author knows, new – lemma:

Lemma 2. *For every $f \in L^1_{loc}(\mathbb{R})$, we have also that*

$$M^1_{\equiv R}f(x) = \sup_{|z-x|+|t-R|\leq R} u(z,t).$$

Proof. From Corollary 1, we have that

$$\begin{aligned}
M^1_{\equiv R}f(x) := \sup_{|x-y|\leq t\leq R} u(y,t) &\leq \sup_{|z-x|+|t-R|\leq R} u(z,t) \\
&\leq \max\{M_{r,R}f(x), M_{l,R}f(x)\} \leq M^1_{\equiv R}f(x).
\end{aligned}$$

That is exactly what we wanted to prove. □

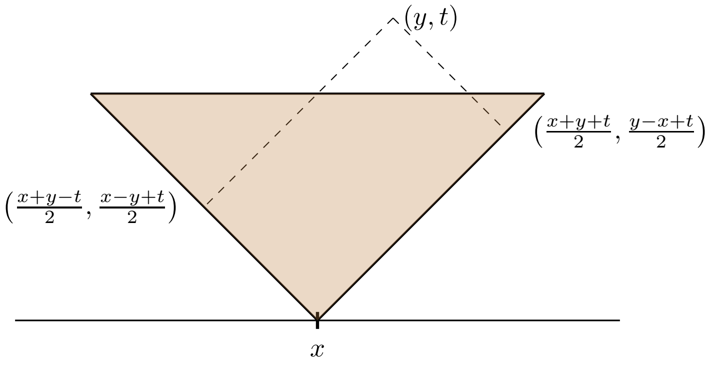

FIGURE 3. Illustration of Lemma 1: the points $\left(\frac{x+y-t}{2}, \frac{x-y+t}{2}\right)$ and $\left(\frac{x+y+t}{2}, \frac{y-x+t}{2}\right)$ are the projections of (y,t) over the lines $t = x - y$ and $t = y + x$, respectively.

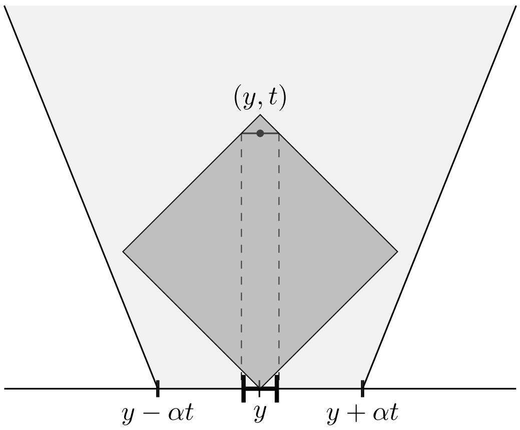

FIGURE 4. In the figure, the **dark gray** area represents the region that our Lemma gives, for some $\frac{1}{2}t < R < \frac{\alpha}{1-\alpha}t$, and the **black** interval is one in which $M^\alpha f = M^1_{\equiv R} f \equiv M^\alpha f(y)$.

Let R be then selected such that $\frac{t}{2} < R < \frac{\alpha}{1-\alpha}t$. For $\alpha > \frac{1}{3}$ this is possible. This condition is *exactly* the condition so that the region

$$\{(z,t) : |z-y| + |t-R| \le R\} \subset C(y,\alpha,t).$$

Now we are able to end the proof: if I is a sufficiently small interval around y, then, by continuity, it must hold true that the regions

$$\{(z,t) : |z-y'| + |t-R| \le R\} \subsetneq C(y,\alpha,t),$$

for all $y' \in I$. This is our desired interval for which $M^\alpha f \equiv M_R f$. But we already know that, from [1, Lemma 3.6], $M^1_{\equiv R} f$ satisfies a *stronger* version of the maximal attachment property. In particular, we know that, if $M^1_{\equiv R} f$ is constant in an interval, then it must be *equal* to the function f at *every* point of that interval. But this is exactly our case, as we have already noticed that $M^\alpha f$ is constant on $[y - \alpha t, y + \alpha t]$, and therefore also on I. This implies, in particular, that

$$M^\alpha f(y) = M^1_{\equiv R} f(y) = f(y),$$

which concludes this proof.

3.2. **Proof of $\mathcal{V}(M^\alpha f) \le \mathcal{V}(f)$, for $\alpha \ge \frac{1}{3}$.** We remark, before beginning, that this strategy, from now on, is essentially the same as the one contained in [1]. First, we say that a function $g : I \to \mathbb{R}$ is *two-part monotone* if there exists a point $c \in I$ such that

$$g\chi_{(l(I),c)} \text{ and } g\chi_{(c,r(I))}$$

are both monotone. We do then our general strategy in two parts:
Part 1: suppose $f \in Lip(\mathbb{R})$. One can easily check then that $M^\alpha f \in C(\mathbb{R})$ in this case. Moreover, we may prove an aditional property about it that will help us later:

Lemma 3 (Reduction to the Lipschtiz case)**.** *Suppose we have that*

$$\mathcal{V}(M^\alpha f) \le \mathcal{V}(f), \ \ \forall f \in BV(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}).$$

Then the same inequality holds for all Bounded Variation functions, that is,

$$\mathcal{V}(M^\alpha f) \le \mathcal{V}(f), \ \ \forall f \in BV(\mathbb{R}).$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a smooth nonnegative function such that $\int_{\mathbb{R}} \varphi(t)\mathrm{d}t = 1$, and call $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$. We define then $f_\varepsilon(x) = f * \varphi_\varepsilon(x)$. Notice that these functions are all Lipschtiz (in fact, smooth) functions. Moreover, by Standard Theorems on Approximate Identities, we have that $f_\varepsilon(x) \to f(x)$ pointwise. Therefore, assuming the Theorem to hold for Lipschitz Functions, we have:

$$\begin{aligned}
\mathcal{V}(M^\alpha f_\varepsilon) &\le \mathcal{V}(f_\varepsilon) \\
&= \sup_{x_1 < \cdots < x_N} \sum_{i=1}^{N-1} |f_\varepsilon(x_{i+1}) - f_\varepsilon(x_i)| \\
&\le \int_{\mathbb{R}} \varphi_\varepsilon(t) \sup_{x_1 < \cdots < x_N} \left(\sum_{i=1}^{N-1} |f(x_{i+1} - t) - f(x_i - t)| \right) \mathrm{d}t \\
&\le \mathcal{V}(f).
\end{aligned}$$

Thus, it suffices to prove that $M^\alpha f_\varepsilon(x) \to M^\alpha f(x)$ for all $x \in \mathbb{R}$. Let us suppose, for the sake of a contradiction, that this does not happen. Therefore, there should exist a real number x_0, a sequence $\varepsilon \to 0$ and a positive real number $\eta > 0$ such that

$$M^\alpha f_\varepsilon(x_0) > (1 + 2\eta) M^\alpha f(x_0).$$

By definition, we may find - depending on ε - then y, r with $|x_0 - y| \le \alpha r$ and

$$⨍_{y-r}^{y+r} f_\varepsilon(s)\mathrm{d}s \ge (1+\eta) M^\alpha f(x).$$

Suppose that $r_\varepsilon \to 0$. In this case, it is clear that $y_\varepsilon \to x_0$. We notice that x_0 must be a discontinuity point of f, as

$$\lim_{\varepsilon \to 0} ⨍_{y-r}^{y+r} [f(s) - f_\varepsilon(s)]\mathrm{d}s \to 0,$$

in case it is not. But then, using Fubini's Theorem,

$$\begin{aligned}
⨍_{y-r}^{y+r} [f(s) - f_\varepsilon(s)]\mathrm{d}s &= ⨍_{y-r}^{y+r} \int_{\mathbb{R}} (f(s) - f(s-t))\varphi_\varepsilon(t)\mathrm{d}t\mathrm{d}s \\
&= \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) \left(⨍_{y-r}^{y+r} [f(s) - f(s-t)]\mathrm{d}s \right) \mathrm{d}t \\
&> 0,
\end{aligned}$$

for $\varepsilon \ll 1$, as s then ranges over a small neighbourhood of x and $f(x) = \limsup_{y \to x} f(y)$. Therefore we also rule this case out, and thus $r \ge c > 0$ for each $\varepsilon > 0$. But, by using Fubini's Theorem again,

$$
\begin{aligned}
\fint_{y-r}^{y+r} f_\varepsilon(s)\mathrm{d}s &= \fint_{y-r}^{y+r}\int_{-\varepsilon} \varphi_\varepsilon(t) f(s-t)\mathrm{d}t\mathrm{d}s \\
&= \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t)\left(\fint_{y-r}^{y+r} f(s-t)\mathrm{d}s\right)\mathrm{d}t \\
&\le \frac{r+\varepsilon}{r}\int_{-\varepsilon}^{\varepsilon}\varphi_\varepsilon(t)\left(\fint_{y-r-\varepsilon}^{y+r+\varepsilon} f(s)\mathrm{d}s\right)\mathrm{d}t \\
&\le \frac{r+\varepsilon}{r} M^\alpha f(x).
\end{aligned}
$$

This already implies that $\frac{r+\varepsilon}{r} > 1+\eta \Rightarrow r < N(\alpha,\eta)$. This boundedness implies that, possibly passing through a subsequence, we may find $(y_{\varepsilon_k}, r_{\varepsilon_k}) \to (y_0, r_0)$, and then

$$
M^\alpha f(x_0) \ge \fint_{y_0-r_0}^{y_0+r_0} f(s)\mathrm{d}s = \lim_{k\to\infty} \fint_{y_{\varepsilon_k}-r_{\varepsilon_k}}^{y_{\varepsilon_k}+r_{\varepsilon_k}} f(s)\mathrm{d}s > (1+\eta) M^\alpha f(x_0),
$$

which is an evident contradiction that ends the proof of this lemma. □

Our main claim is then the following:

Lemma 4. *Let $f \in Lip(\mathbb{R}) \cap BV(\mathbb{R})$. Then, over every interval of the set*

$$
E_\alpha = \{x \in \mathbb{R} \colon M^\alpha f(x) > f(x)\} = \bigcup_{j\in\mathbb{Z}} I_j^\alpha,
$$

it holds that $M^\alpha f$ is either monotone *or* two-fold monotone *in I_j^α.*

Proof. The proof goes roughly as the first paragraph of the proof of Lemma 3.9 in [1]: let $I_j^\alpha = (l(I_j^\alpha), r(I_j^\alpha) =: (l_j, r_j)$, and suppose that $M^\alpha f$ is *not* two-fold monotone there. Therefore, there would be a maximal point $x_0 in I_j^\alpha$ and an interval $J \subset I_j^\alpha$ such that $M^\alpha f$ has a *strict* local maximum at x_0 over J. Then, by the maximal attachment property, we see that we have reached a contradiction from this fact alone, as $J \subset E_\alpha$. We omit further details, as they can be found, as already mentioned, at [1, Lemma 3.9]. □

To finalize the proof in this case for $\alpha > \frac{1}{3}$, we just notice that we can, in fact, bound the variation of $M^\alpha f$ *inside* every interval I_j^α. In fact, we have directly from the last claim that

$$
\begin{aligned}
\mathcal{V}_{I_j^\alpha}(M^\alpha f) &= |\max(M^\alpha f(l(I_j^\alpha)-), M^\alpha f(l(I_j^\alpha)+)) - \max(M^\alpha f(c_j-), M^\alpha f(c_j+))| + \\
&\quad |\max(M^\alpha f(r(I_j^\alpha)-), M^\alpha f(r(I_j^\alpha)+)) - \max(M^\alpha f(c_j-), M^\alpha f(c_j+))| \\
&= |M^\alpha f(l(I_j^\alpha) - M^\alpha f(c_j)| + |M^\alpha f(r(I_j^\alpha)) - M^\alpha f(c_j)| \\
&\le |f(l(I_j^\alpha)) - \max(f(c_j-), f(c_j+))| + |f(r(I_j^\alpha)) - \max(f(c_j-), f(c_j+))| \\
&\le V_{I_j^\alpha}(f).
\end{aligned}
$$

The way to formally end the proof is the following: Let $x_1 < \cdots < x_N$ be an arbitrary sequence of real numbers. As adding points to the sequence is a non-decreasing operation in terms of the variation of the function, then we may assume that the endpoints of all intervals I_j^α are contained in such a sequence. From this point on, we split the variation into two parts: the points inside E_α and the ones outside. It was seen above that

$$
\mathcal{V}_{E_\alpha}(M^\alpha f) \le \mathcal{V}_{E_\alpha}(f).
$$

As $M^\alpha f \equiv f$ on $\mathbb{R}\backslash E_\alpha$, it is then obvious that

$$
\mathcal{V}_{x_1<\cdots<x_N}(M^\alpha f) = \sum_{i=1}^{N-1} |M^\alpha f(x_{i+1}) - M^\alpha f(x_i)| \le \mathcal{V}(f).
$$

Inequality (1) is proved for $\alpha > \frac{1}{3}$ by just taking the supremum over any such a sequence of real numbers. For $\alpha = \frac{1}{3}$, we just notice that, for any such a sequence,

$$\sum_{i=1}^{N-1} |M^{\frac{1}{3}} f(x_{i+1}) - M^{\frac{1}{3}} f(x_i)| \overset{\text{Fatou}}{\leq} \lim_{\alpha \searrow \frac{1}{3}} \sum_{i=1}^{N-1} |M^{\alpha} f(x_{i+1}) - M^{\alpha} f(x_i)| \leq \mathcal{V}(f).$$

The theorem follows again in this case by taking a supremum.
Part 2: let $f \in BV(\mathbb{R})$ be a general function, normalized as $f(x) = \limsup_{y\to x} f(y)$. The argument here is morally the same, with just a couple of minor modifications – and with the use of the facts we proved above, namely, that the result *already* holds. Therefore, this section might seem a little bit superfluous now, even though its reason of being is going to be shown while we prove the characterization of extremizers.

Claim 1. *Let $E_\alpha = \{x \in \mathbb{R} : M^\alpha f(x) > f(x)\}$. This set is open for any $f \in BV(\mathbb{R})$ normalized as $f(x) = \limsup_{y\to x} f(y)$ and therefore can be decomposed as*

$$E_\alpha = \cup_{j\in\mathbb{Z}} I_j^\alpha,$$

where each I_j^α is an interval. Furthermore, the restriction of $M^\alpha f$ to each of those intervals is either a monotone function or a two-part monotone function with a minimum at $c_j \in I_j^\alpha$. Moreover, $M^\alpha f(c_j) < \min\{M^\alpha f(l(I_j^\alpha)), M^\alpha f(r(I_j^\alpha))\}$.

Proof of the claim. The claim seems quite sophisticated, but its proof is simple, once one has the maximal attachment property. The fact that E_α is open is easy to see. In fact, let $x_0 \in E_\alpha$. By definition, there must be a $t > 0$ and y such that $|x - y| \leq \alpha t$ and $M^\alpha f(x) = u(y,t)$. This itself already implies that

$$\liminf_{z\to x_0} M^\alpha f(z) \geq M^\alpha f(x_0) > f(x_0) = \limsup_{z\to x_0} f(z).$$

This shows that, for z close to x_0, the strict inequality should still hold, as desired.
The second part follows in the same fashion as the proof of Lemma 4, and we therefore omit it. ☐

To finish the proof of the fact that $\mathcal{V}_{I_j^\alpha}(M^\alpha f) \leq \mathcal{V}_{I_j^\alpha}(f)$ also in this case we just need one more lemma:

Lemma 5. *For every (maximal) open interval $I_j^\alpha \subset E_\alpha$ we have that*

$$\max(M^\alpha f(l(I_j^\alpha)-), M^\alpha f(l(I_j^\alpha)+)) = f(l(I_j^\alpha)),$$

and an analogous identity holds for $r(I_j^\alpha)$.

Proof. It is easy to prove that $M^\alpha f(x) \geq \min(M^\alpha f(x-), M^\alpha f(x+))$ for all $x \in \mathbb{R}$. We separate therefore into two cases:

- If $M^\alpha f$ is continuous at $l(I_j^\alpha)$, then we have automatically that $M^\alpha f(l(I_j^\alpha)) \geq f(l(I_j^\alpha))$. If the inequality were *strict,* we would have that $l(I_j^\alpha) \in E_\alpha$. But then there would be an open interval $E_\alpha \supset J \ni l(I_j^\alpha)$, which contradicts the maximality of I_j^α.
- If $M^\alpha f(l(I_j^\alpha)+) > M^\alpha f(l(I_j^\alpha)-)$, then, as $f \in BV(\mathbb{R})$, we can select two sequences $\{x_n\}_{n\geq1}, \{y_n\}_{n\geq1}$ such that $y_n \nearrow l(I_j^\alpha), x_n \searrow l(I_j^\alpha)$ and f is continuous at each point of $\{y_n\}_{n\geq1}, \{x_n\}_{n\geq1}$. By those properties, we have that $M^\alpha f(x_n) \geq f(x_n), M^\alpha f(y_n) \geq f(y_n)$. This shows that

$$\max(M^\alpha f(l(I_j^\alpha)+), M^\alpha f(l(I_j^\alpha)-)) \geq \max(f(l(I_j^\alpha)-), f(l(I_j^\alpha+)).$$

  In the case where the inequality above is *strict,* then suppose, without loss of generality, that the sequence $\{y_n\}_{n\geq1}$ satisfies, for n sufficiently large, there is $\varepsilon_0 > 0$ with the property that

$$M^\alpha f(y_n) - \max(f(l(I_j^\alpha)-), f(l(I_j^\alpha+)) > \varepsilon_0.$$

  This implies that there must be a sequence of points (z_n, t_n) such that

$$u(z_n,t_n) - \max(f(l(I_j^\alpha)-), f(l(I_j^\alpha+)) \geq \frac{\varepsilon_0}{2},\ |y_n - z_n| \leq \alpha t_n,\ \frac{1}{\delta_0} \geq t_n \geq \delta_0$$

for $n \gg 1$, – ortherwise either $\lim_{n\to\infty} u(z_n, t_n) = \lim_{n\to\infty} f(y_n)$ or we get $M^\alpha f(l(I_j^\alpha)+) = M^\alpha f(l(I_j^\alpha)-)$, – and therefore we may extract a subsequence $\{(z_{n_k}, t_{n_k})\}_{k\geq 1}$ such that $(z_{n_k}, t_{n_k}) \to (z_0, t_0)$ that satisfies $|z_0 - l(I_j^\alpha)| \leq \alpha t_0$. This implies already that

$$M^\alpha f(l(I_j^\alpha)) - \max(f(l(I_j^\alpha)-), f(l(I_j^\alpha+)) \geq \frac{\varepsilon_0}{2}.$$

This is again a contradiction, by the same argument as in the first case. This finishes the proof of the Lemma.

The finish in this case uses Lemma 5 in a direct fashion, combined with the strategy for Part 1: namely, the estimate

$$\begin{aligned}\mathcal{V}_{I_j^\alpha}(M^\alpha f) &= |\max(M^\alpha f(l(I_j^\alpha)-), M^\alpha f(l(I_j^\alpha)+)) - \max(M^\alpha f(c_j-), M^\alpha f(c_j+))|+ \\ &\quad |\max(M^\alpha f(r(I_j^\alpha)-), M^\alpha f(r(I_j^\alpha)+)) - \max(M^\alpha f(c_j-), M^\alpha f(c_j+))| \\ &\leq |M^\alpha f(l(I_j^\alpha) - M^\alpha f(c_j)| + |M^\alpha f(r(I_j^\alpha)) - M^\alpha f(c_j)| \\ &\leq |f(l(I_j^\alpha)) - \max(f(c_j-), f(c_j+))| + |f(r(I_j^\alpha)) - \max(f(c_j-), f(c_j+))| \\ &\leq V_{I_j^\alpha}(f)\end{aligned}$$

still holds, by Lemma 5 and by the fact that $c_j \in I_j^\alpha$. This finishes finally the most general version of Theorem [**?**]. ☐

3.3. **Absolute continuity on the detachment set.** We prove briefly the fact that, for $f \in W^{1,1}(\mathbb{R})$, then we have that $M^\alpha f \in W^{1,1}_{loc}(\mathbb{R})$ for *any* $1 > \alpha > 0$. Actually, we may prove something a little bit stronger: for $f \in BV$, with our standard normalization – that is, $\limsup_{y\to x} f(y) = f(x)$–, then the restriction of $M^\alpha f$ to $E_\alpha = \{M^\alpha f > f\}$ is absolutely continuous.

The proof of the second fact claimed above is just a simple adaption of [8, Claim 7.2] and [1, Lemma 3.8]. Indeed, let

$$E_{\alpha,k} = \{x \in E_\alpha \colon \exists (y,t), M^\alpha f(x) = \frac{1}{2t}\int_{y-t}^{y+t} |f(s)|\mathrm{d}s \text{ and } 2t \geq \frac{1}{k}\}.$$

Then we see that $E_\alpha = \cup_{k\geq 1} E_{\alpha,k}$. Moreover, for $x, y \in E_{\alpha,k}$, let then (y_1, t_1) have this property for x. Suppose also, without loss of generality, that $y \geq x$ and $M^\alpha f(x) \geq M^\alpha f(y)$. We have that

$$\begin{aligned}M^\alpha f(x) - M^\alpha f(y) &\leq \frac{1}{2t_1}\int_{y_1-t_1}^{y_1+t_1} |f(s)|\mathrm{d}s - \frac{1}{\frac{2}{1+\alpha}(y - y_1 + t_1)}\int_{y_1-t_1}^{y_1+t_1} |f(s)|\mathrm{d}s \\ &\leq \frac{\frac{2}{1+\alpha}(y-y_1) - \frac{2\alpha}{1+\alpha}t_1}{2t_1 \cdot \frac{2}{1+\alpha}(y - y_1 + t_1)}\int_{y_1-t_1}^{y_1+t_1} |f(s)|\mathrm{d}s \\ &\leq \frac{\frac{2}{1+\alpha}|y-x|}{\frac{2}{1+\alpha}(y - y_1 + t_1)}\|f\|_\infty \leq \frac{|x-y|}{2t_1}\|f\|_\infty \leq k|x-y|\|f\|_\infty.\end{aligned}$$

This shows that $M^\alpha f$ is Lipschitz continuous with constant $\leq n\|f\|_\infty$ on each $E_{\alpha,n}$. This proves already the second assertion. The proof of the first one, however, follows from the second one, using the well-known Banach-Zarecki lemma:

Lemma 6 (Banach-Zarecki)**.** *A function $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if and only if the following conditions hold simultaneously:*

(A) *g is continuous;*
(B) *g is locally of bounded variation;*
(C) *$g(S)$ has measure zero for every set S with $|S| = 0$.*

Let S be then a null-measure set on the real line and $f \in W^{1,1}(\mathbb{R})$ – which implies that $M^\alpha f \in C(\mathbb{R})$ –, and let us estimate:

$$\begin{aligned}|M^\alpha f(S)| &\le |M^\alpha f(S\cap E_\alpha^c)| + \sum_{k\ge 1} |M^\alpha f(S\cap E_{\alpha,k})| \\ &\le |f(S\cap E_\alpha^c)| \\ &= 0,\end{aligned}$$

where we used that $f \in W^{1,1}(\mathbb{R})$ (and particularly f is absolutely continuous) on the last inequality. This completes the proof.

3.4. **Sharpness of the inequality and extremizers.** In this part, we prove that the best constant in such inequalities is indeed 1, and characterize the extremizers for such. Namely, we mention promptly that the inequality *must* be sharp, as $f = \chi_{(-1,0)}$ reaches equality.
It is easy to see that, to do so, we may assume that $f(x) = \limsup_{y\to x} f(y)$. We have to be especially careful now, as we can have points on $\mathbb{R}$ for which $f > M^\alpha f$. We define then two more sets

$$E_\alpha^* = \{x \in \mathbb{R}; f(x) > M^\alpha f(x)\} \text{ and } E_\alpha^0 = \{x \in \mathbb{R}; f(x) = M^\alpha f(x)\}.$$

As we already now, the set E_α^* is at most countable. We divide the proof in this section into several claims:

Claim 2. *$E_\alpha^* \subset \partial E_\alpha = \{x \in \mathbb{R} \colon x \in \overline{E_\alpha} \setminus E_\alpha\}$. Moreover, the points where that happens are discontinuity points of f.*

Proof. Let $a \in E_\alpha^*$, and $f(a) = f(a+) > f(a-)$ – as, if they were equal, a would be a Lebesgue point of f. Therefore, as also

$$M^\alpha f(a) \ge \frac{(1+\alpha)f(a+) + (1-\alpha)f(a-)}{2} > \frac{f(a_+) + f(a-)}{2},$$

and

$$\liminf_{x\to_-} M^\alpha f(x) \ge M^\alpha f(a),$$

$$M^\alpha f(x) > \frac{f(a_+) + f(a_-)}{2} > f(x), \text{ for all } x \text{ close to } a \text{ at its left.}$$

This finishes the proof of the claim. □

Claim 3. *Let $f \in BV(\mathbb{R})$ normalized as before satisfy $\mathcal{V}(f) = \mathcal{V}(M^\alpha f)$. If we decompose then $E_\alpha = \cup_j I_j^\alpha$, where each of the I_j^α is open, then*

$$\mathcal{V}_{I_j^\alpha}(f) = \mathcal{V}_{I_j^\alpha}(M^\alpha f).$$

Proof. Let $\mathcal{P}, \mathcal{Q}$ be two finite partitions of $\mathbb{R}$ such that

$$\begin{cases}\mathcal{V}(M^\alpha f) & \le \mathcal{V}_{\mathcal{P}}(M^\alpha f) + \varepsilon. \\ \mathcal{V}(f) & \le \mathcal{V}_{\mathcal{Q}}(f) + \varepsilon.\end{cases}$$

Now let the mutual refinement of those be $\mathcal{S} = \mathcal{P} \cup \mathcal{Q}$. As our partition is finite, there is a finite number of intervals from those in E_α that contain at least a point in $\mathcal{S}$. We will focus on those intervals: with each of those, we add the following points:

(A) If $f = M^\alpha f$ on the boundary of an interval I_j^α, we add to the collection both endpoints $r(I_j^\alpha), l(I_j^\alpha)$.
(B) If $f > M^\alpha f$ at at least one of the points on the boundary of one of those intervals, select then a point $\tilde{r}_j$ next to $r_j := r(I_j^\alpha)$ such that
- $f(\tilde{r}_j) \le M^\alpha f(\tilde{r}_j)$;
- $|M^\alpha f(\tilde{r}_j) - f(r_j)| \le \varepsilon 2^{-20j}$;
- $|f(\tilde{r}_j) - f(r_j)| \le \varepsilon 2^{-20j}$;
- $|\tilde{r}_j - r_j| \le \varepsilon 2^{-20j}$.

Add then such a point to our collection.

(C) If $M^\alpha f$ is *not* monotone on the interval I_j^α, then, as we showed it has at most one local minimum, add this point c_j to the collection.
(D) Finally, add a finite quantity of points on each I_j^α such that the variation of f according to the partition between the points $\tilde{l_j}$ and $\tilde{r_j}$ selected as above is at least greater than $\mathcal{V}_{I_j^\alpha}(f) - \varepsilon 2^{-20j}$.

It is easy to see that, from the prescriptions above, if we denote by $\mathcal{S}'$ this new partition, then, as

$$|\mathcal{V}_{\mathcal{S}'}(f) - \mathcal{V}_{\mathcal{S}'}(M^\alpha f)| \le 2\varepsilon,$$

We will get that

$$\sum_{j\in\mathbb{Z}} \mathcal{V}_{I_j^\alpha}(f) - 2\varepsilon \le \sum_{j\in\mathbb{Z}} \mathcal{V}_{(l_j,r_j)\cap\mathcal{S}'}(f) \le \sum_{j\in\mathbb{Z}} \mathcal{V}_{I_j^\alpha}(M^\alpha f) + 2\varepsilon.$$

As ε was arbitrary, comparing the first and last terms above and looking back to our proof that in each of the I_j^α the variation of f controls that of the maximal function, we conclude that, for each $j \in \mathbb{Z}$,

$$\mathcal{V}_{I_j^\alpha}(f) = \mathcal{V}_{I_j^\alpha}(M^\alpha f). \tag{2}$$

This finishes the proof of this claim. ☐

Claim 4. *Let f, I_j^α as above. Then f and $M^\alpha f$ have to be monotone in I_j^α.*

Proof. Suppose this is not the case. This would imply the existence of a point inside the interval $c_j \in I_j^\alpha$ such that

$$\min\{f(c_j+), f(c_j-)\} < \min\{f(l(I_j^\alpha)), f(r(I_j^\alpha))\}.$$

Suppose the minimum on the left is attained for $f(c_j+)$, without loss of generality. Then

$$\begin{aligned}
\mathcal{V}_{I_j^\alpha}(f) &\ge |f(l(I_j^\alpha)) - f(c_j+)| + |f(r(I_j^\alpha)) - f(c_j+)| \\
&\ge f(l(I_j^\alpha)) - 2f(c_j+) + f(r(I_j^\alpha)) \\
&\ge \max(M^\alpha f(r(I_j^\alpha)-), M^\alpha f(r(I_j^\alpha)+)) - 2M^\alpha f(y) + \max(M^\alpha f(l(I_j^\alpha)-), M^\alpha f(l(I_j^\alpha)+)), \\
&\forall y \in I_j^\alpha.
\end{aligned}$$

But this last expression is always *strictly* greater than $\mathcal{V}_{I_j^\alpha}(M^\alpha f)$ in the case when $f(c_j) < M^\alpha f(c_j)$, which is our case here, by definition. We get immediately to a contradiction, which shows our claim to be true. ☐

Notice that this last claim proves also that, if I_j^α is *bounded*, f is *non-decreasing* over it and l_j is its left endpoint, then $f(l_j-) \le f(l_j+)$, as otherwise we would arrive at a contradiction with the fact that $\mathcal{V}_{I_j^\alpha}(f) = \mathcal{V}_{I_j^\alpha}(M^\alpha f)$. An analogous statement holds for the right endpoint, and analogous conclusions if f is *non-increasing* instead of non-decreasing over the interval.

Next, we suppose without loss of generality that the function f is non-decreasing on I_j^α, as the other case is completely analogous.

Claim 5. *Such an f is, in fact, non-decreasing on $(-\infty, r(I_j^\alpha)]$.*

Proof. Our proof of this fact will go by contradiction:

First, let $a_j = \inf\{t \in \mathbb{R}; f \text{ is non-decreasing in } [t, r(I_j^\alpha)]\}$, and define $b_j < a_j$ such that the minimum of f in $[b_j, r_j]$ happens *inside* (b_j, r_j). Of course, such a minimum need not happen at a point, but it surely does happen at a *lateral limit* of a point.

Subclaim 1. *$M^\alpha f(a_j) = f(a_j)$ and $f(a_j-) = f(a_j+)$.*

Proof. We have to consider some different cases:

- *If $M^\alpha f(a_j) > f(a_j)$*, then there exists an open interval $E_\alpha \supset J_j' \ni a_j$, and, as we proved before, f must be monotone in such an interval. By the definition of a_j, f must be non-decreasing, which is a contradiction to the definition of a_j.

- *If $M^\alpha f(a_j) < f(a_j)$,* then, as we have seen before, there exists an interval $E_\alpha \supset J_j''$ such that a_j is the right end point of J_j'', as we have seen that $E_\alpha^* \subset \partial E_\alpha$. But then f must be non-decreasing on J_j''. In fact, it is easy to see that, for this present case to be possible, we are forced to have $f(a_j+) > f(a_j-)$. But then, if f were non-increasing on J_j'', we would have that $\mathcal{V}_{J_j''}(f) > \mathcal{V}_{J_j''}(M^\alpha f)$. Therefore, f is indeed non-decreasing on this interval, and this is a contradiction to the definition of a_j.

Now for the second equality: if it were not true, then a_j would be, again, one of the endpoints of a maximal interval $J_j \subset E_\alpha$. If a_j is the left-endpoint, then it means that $f(a_j-) > f(a_j+)$. But this is a contradiction, as f then must be non-decreasing on J_j, and therefore we would again have that $\mathcal{V}_{J_j}(f) > \mathcal{V}_{J_j}(M^\alpha f)$. Therefore, a_j is the right endpoint, and also $f(a_j-) < f(a_j+)$. At the present moment an analysis as in the proof of the case $M^\alpha f(a_j) = f(a_j)$ is already available, and thus we conclude that f shall be non-decreasing on J_j, which is again a contradiction to the definition of J_j. □

We must prove yet another fact that will help us:

Subclaim 2. *Let*

$$\mathcal{D} = \{x \in (b_j, r_j) \colon \min(f(x-), f(x+)) \text{ attains the minimum in } (b_j, r_j)\}.$$

Then there exists $d \in \mathcal{D}$ such that $f(d-) = f(d+)$ and $M^\alpha f(d) = f(d)$.

Proof. If $a_j \in \mathcal{D}$, then our assertion is proved by Subclaim 1. If not, then $\mathcal{D} \subset (b_j, a_j)$. In this case, pick any point d_0 in this intersection.

Case 1:$f(d_0+) = f(d_0-)$. In this case, we claim that we have already that $f(d_0) = M^\alpha f(d_0)$. Otherwise, we would have that $M^\alpha f(d_0) > f(d_0)$, and then there would be an interval $E_\alpha \supset J_0 \ni d_0$. By the fact that *all* the points in $\mathcal{D}$ must lie in (b_j, a_j), and that f is monotone on J_0, we see automatically that either $J_0 \subset (b_j, r_j)$ or $M^\alpha f(b_j) \le M^\alpha f(d_0)$, a contradiction. Suppose also that f is non-decreasing on J_0, and the other case can be handled analogously. We must have then that f is *continuous* on l_0 – the left endpoint of J_0, and as othwersie we would have some point $y' \in (b_j, a_j)$ with $f(y') < f(d_0)$. Then naturally $M^\alpha f(l_0) = f(l_0)$ there. This $l_0 \in \mathcal{D}$ is our desired point.

Case 2:$f(d_0+) > f(d_0-)$. It is easy to see that, in this case, there is an open interval $J \subset E_\alpha$ such that either $J \ni d_0$ or d_0 is its right endpoint. In either case, we see that f must be non-decreasing over this interval J, and let again l_0 be its left endpoint. As we know, $l_0 \in \mathcal{D}$ again, $l_0 \in (b_j, r_j)$ and, by the comments following Claim 4, we must have that $f(l_0-) = f(l_0+)$. Of course, by being the endpoint we have automatically again that $M^\alpha f(l_0) = f(l_0)$. This concludes again this case, and therefore the proof of the subclaim. □

The concluding argument for the proof of the Claim 5 goes as follows: let d be the point that Subclaim 2 gives. Then we must have that

$$f(d) = M^\alpha f(d) \ge Mf(d) \ge \fint_{d-\delta}^{d+\delta} f.$$

varying δ, it is easy to see that we get a contradiction from that. This contradiction came from the fact that we supposed that $a_j > -\infty$, and our claim is established. □

Now we finish the proof: If $M^\alpha f \le f$ always, we get to the case of a *superharmonic function*, i.e., a function which satisfies $\fint_{x-r}^{x+r} f(s)\mathrm{d}s \le f(x)$ for all $r > 0$. That is going to be handled in a while. If not, then we analyze the detachment set:

(A) If all intervals in the detachment set are either *increasing* or *decreasing*, our function must then admit a point x_0 such that f is either non-decreasing on $(-\infty, x_0]$ (resp. non-decreasing on $[x_0, +\infty)$,) and $f = M^\alpha f$ on $(x_0, +\infty)$ (resp. on $(-\infty, x_0)$).
(B) If there is at least one interval of each type, then we must have an interval $[R, S]$ such that
   - f is non-decreasing on $(-\infty, R]$;
   - f is non-increasing on $[R, +\infty)$;

- $f = M^\alpha f$ on (R, S).

The analysis is then easily completed for every one of the cases above: If $f = M^\alpha f$ over an interval, then, as $M^\alpha f \geq Mf$, we conclude that f must be superharmonic there, where by "locally subharmonic" we mean a function that satisfies $f(x) \geq \fint_{x-r}^{x+r} f(s)\mathrm{d}s$ for all $0 \leq s \ll_x 1$. As superharmonic in one dimension coincides with concave, and concave functions have *at most* one global maximum, then the first case above gives that f is either monotone or has exaclty one point x_1 such that it is exactly two-fold monotone on $\mathbb{R}$. The case of monotone functions is easily ruled out, as if $\lim_{x\to\infty} f = L$, $\lim_{x\to-\infty} f = M \Rightarrow \mathcal{V}(f) = |M-L|, \mathcal{V}(M^\alpha f) \leq \frac{|M-L|}{2}$. The second case is treated in the exact same fashion, and the result is the same: in the end, the only possible extremizers for this problem are functions f such that there is a point x_1 such that f is non-decreasing on $(-\infty, x_1)$, and f is non-increasing on $(x_1, +\infty)$. The theorem is then complete.

3.5. **Proof of Theorem 2.** We start our discussion by pointing out that the measure $\mathrm{d}\mu = \delta_0 + \delta_1$ satisfies our Theorem.

Proposition 1. *Let $0 \leq \alpha < \frac{1}{3}$. Then*

$$+\infty = M^\alpha\mu(0) > M^\alpha\mu\left(\frac{1}{3}\right) < M^\alpha\mu\left(\frac{1}{2}\right) > M^\alpha\mu\left(\frac{2}{3}\right).$$

That is, $M^\alpha\mu$ has a nontrivial local maximum.

Before proving our Proposition, we mention that our choice of $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ was not random: $\frac{1}{2}$ is actually a *maximum point* of $M^\alpha\mu$, while $\frac{1}{3}, \frac{2}{3}$ are *local minima.*

Proof. By the symmetries of our measure, $M^\alpha\mu\left(\frac{1}{3}\right) = M^\alpha\mu\left(\frac{2}{3}\right)$. Therefore, let's calculate $M^\alpha\mu\left(\frac{1}{3}\right)$ and $M^\alpha\mu\left(\frac{1}{2}\right)$ explicitly.

As we have concentrated our mass at $\{0, 1\}$, we will have at most three averages where the supremum could be achieved in either case. Explicitly, for $\frac{1}{3}$, we have to choose averages that pick up either $\{0\}, \{1\}$, or both. If it picks only $\{0\}$, then we have the existance of (y, t) with $y - t < 0$ and $|\frac{1}{3} - y| \leq \alpha t$. If $|\frac{1}{3} - y| < \alpha t$, then, by changing y, we may get an interval (y', t) that *still* contains zero and $|\frac{1}{3} - y'| = \alpha t$. If $y - t < 0$, we move both y and t so that $y' - t' = 0$ and $|y' - \frac{1}{3}| = \alpha t'$. Therefore, we are restricted to minimizing for those intervals. But then we have only two possibilities, which are

$$t = \frac{1}{3(\alpha+1)} \text{ or } t = \frac{1}{3|\alpha - 1|}.$$

However, the second option, when $\alpha \geq \frac{1}{3}$ makes us *also* include $\{1\}$ in our average. Doing a similar analysis to the one we did above, we see that, for us to include only $\{1\}$ in our average, we will have to take an interval of length at least

$$\frac{2}{3\alpha+3}.$$

Finally, to get both of them, we can take, in the case $\alpha \geq \frac{1}{3}$, $y = \frac{1}{2}$ and $t = \frac{1}{2}$. For $\alpha < \frac{1}{3}$, the interval which maximizes matches exactly with the above mentioned interval that optimizes for taking "only" $\{1\}$. Thus we see that

$$M^\alpha\mu\left(\frac{1}{3}\right) = \frac{3\alpha+3}{2},\ \alpha < \frac{1}{3}.$$

As $M^\alpha\mu\left(\frac{1}{2}\right) \geq M\mu\left(\frac{1}{2}\right) = 2 > \frac{3\alpha+3}{2} \iff \alpha < \frac{1}{3}$, we are done with the proof of this proposition. □

Proof of Theorem 2. Let $f_n(x) = n(\chi_{[0,\frac{1}{n}]} + \chi_{[1-\frac{1}{n},1]})$. It is easy to see that $\int g f_n \mathrm{d}x \to \int g \mathrm{d}\mu(x)$, for each $g \in L^\infty(\mathbb{R})$ that is continuous on $[0, t_0) \cup (t_1, 1]$, for some $t_0 < t_1$.

We prove that $M^\alpha f_n(x) \to M^\alpha \mu(x),\ \forall x \in [0,1]$. This is clearly enough to conclude our Theorem, as then, if we fix $\alpha < \frac{1}{3}$, there will be $n(\alpha) > 0$ such that, for $N \ge n(\alpha)$,

$$M^\alpha f_N\left(\frac{1}{3}\right) < M^\alpha f_N\left(\frac{1}{2}\right) > M^\alpha f_N\left(\frac{2}{3}\right).$$

To prove convergence, we argue in two steps.

The first step is to prove that $\liminf_{n\to+\infty} M^\alpha f_n(x) \ge M^\alpha \mu(x)$. It clearly holds for $x \in \{0,1\}$. For $x \in (0,1)$, we see that

$$M^\alpha f_n(x) = \sup_{|x-y|\le \alpha t \le 3\alpha} \frac{1}{2t}\int_{y-t}^{y+t} f_n(s)\mathrm{d}s.$$

But then

$$\begin{aligned} M^\alpha \mu(x) &= \sup_{|x-y|\le \alpha t\le 3\alpha} \frac{1}{2t}\int_{y-t}^{y+t} \mathrm{d}\mu(s) \\ &= \sup_{|x-y|\le \alpha t \le 3\alpha; t \ge \delta(x) > 0} \lim_{n\to\infty} \frac{1}{2t}\int_{y-t}^{y+t} f_n(s)\mathrm{d}s \\ &\le \liminf_{n\to\infty} M^\alpha f_n(x), \end{aligned}$$

where $\delta(x) > 0$ is a multiple of the minimum of the distances of x to either 1 or 0. This completes this part.

The second step is to establish that, for every $\varepsilon > 0$, $(1+\varepsilon)M^\alpha\mu(x) \ge \limsup_{N\to\infty} M^\alpha f_N(x)$. This readily implies the result.

To do so, notice that, as $1 > x > 0$, then for N sufficiently large, the average that realizes the supremum on the definition of M^α has a positive radius bounded bellow in N. Specifically, we have that

$$M^\alpha f_N(x) = \fint_{y_N - t_N}^{y_N + t_N} f_N(s)\mathrm{d}s, \quad t_N \ge \delta(x) > 0.$$

This shows also that $\{y_N\}$ must be a bounded sequence. Therefore, passing to subsequences if necessary, we may suppose that $y_N \to y, t_N \to t$. Hence

$$\begin{aligned} \limsup_{N\to\infty} M^\alpha f_N(x) &= \limsup_{N\to\infty} \fint_{y_N-t_N}^{y_N+t_N} f(s)\mathrm{d}s \\ &\le (1+\eta)\frac{1}{2t}\limsup_{N\to\infty}\int_{y-(1+\varepsilon/2)t}^{y+(1+\varepsilon/2)t} f_N(s)\mathrm{d}s \\ &= (1+\eta)(1+\varepsilon/2)\fint_{y-(1+\varepsilon/2)t}^{y+(1+\varepsilon/2)t} \mathrm{d}\mu(s) \\ &\le (1+\varepsilon)M^\alpha\mu(x), \end{aligned}$$

if we manage to make N sufficiently large, and take η depending on ε such that $(1+\eta)(1+\varepsilon/2) < 1+\varepsilon$. □

Remark. Actually, our proof holds only for a subsequence, as we had to assume that $(y_N, t_N) \to (y,t)$. As our method did not take the limit (y,t) directly into account, we can change it so that for every accumulation point (y', t') of $\{(y_N, t_N)\}$ and every subsequence converging to it, we are able to prove our final result.

4. Proof of Theorems 3 and 4

The idea for this proof is basically the same as before: we prove that the maximal attachment property still holds in this Lipschitz case, *if* the Lipschitz constant into consideration is less than $\frac{1}{2}$. By the end, we sketch on how to build the mentioned counterexamples.

4.1. **Maximal attachment property for Lip$(N) < \frac{1}{2}$.** Let first (a, b) be an interval on the real line, such that there exists a point x_0 inside it with the property that

$$M^1_N f(x_0) > \max\{M^1_N f(a), M^1_N f(b)\}.$$

Therefore, we wish to prove that, for either that point or another one in (a, b), $M^1_N f = f$. We begin with the general strategy: let us suppose that this is not the case. Then there must be an average $u(y, t) = \frac{1}{2t}\int_{y-t}^{y+t} |f(s)|\mathrm{d}s$ with $N(x) \geq t > 0$, $|x - y| \leq t$ and $M^1_N f(x) = u(y, t)$. Now we look for a neighbourhood of $x_0 \in I$ such that there is $R > 0$ such that, for all $x \in I$, $M^1_{\equiv R} f(x) = M^1_N f(x_0)$.

By Lemma 1, we can suppose that either $y = x_0 - t$ or $y = x_0 + t$, as we can show that $y \in (a, b)$. Without loss of generality, let us assume the first equality.

Case (a): $t < N(x_0)$. This is the easiest case, and we rule out with a simple observation: let I be an interval for which x_0 is an endpoint and such that, for all $x \in I$, $N(x) > t$. We claim then that, for $x \in I$, $M^1_{\equiv t+\varepsilon} f(x) = M^1_N f(x_0)$, if ε is sufficiently small. Indeed, if ε is sufficiently small, then $M^1_{\equiv t+\varepsilon} f(x) \leq M^1_N f(x) (\leq M^1_N f(x_0))$ for every $x \in I$. But then we see also that $(x_0 - t, t)$ belongs to the region $\{y : |x - y| \leq s \leq N(x)\}$, as then $|(x_0 - t) - x| = x + t - x_0 \leq t < t + \varepsilon < N(x)$. This shows that

$$M^1_N f(x_0) \leq \inf_{x \in I} M^1_{\equiv t+\varepsilon} f(x) \leq \sup_{x \in I} M^1_{\equiv t+\varepsilon} f(x) \leq M^1_N f(x_0).$$

As before, we finish this case with [1, Lemma 3.6], as then it guarantees us that $M^1_{\equiv t+\varepsilon} f(x) = f(x)$ for every point in this interval I.

Case (b): $t = N(x_0)$. In this case, we have to use Lemma 2. Namely, we wish to include the point $(x_0 - N(x_0), N(x_0))$ in the region

$$\{(z, s) : |z - x| + |s - N(x)| \leq N(x)\},$$

for $x < x_0$ but sufficiently close to it.

Let then $\varepsilon > 0$ and x close to x_0 be such that $N(x) \geq N(x_0) - \varepsilon$. We have already a comparison of the form

$$M^1_N f(x) \geq M^1_{\equiv N(x_0)-\varepsilon} f(x).$$

We want to conclude that there is an interval I such that $M^1_{\equiv N(x_0)-\varepsilon} f$ is constant on I. We want then the point $(x_0 - N(x_0), N(x_0))$ to lie on the set

$$\{(z, s) : |z - x| + |s - N(x_0) + \varepsilon| \leq N(x_0) - \varepsilon\}.$$

But this is equivalent to

$$x - x_0 + N(x_0) + \varepsilon \leq N(x_0) - \varepsilon \iff |x - x_0| \geq 2\varepsilon.$$

So, we can only afford to to this if x is somewhat not too close to x_0. But, as $\mathrm{Lip}(N) < \frac{1}{2}$ in this case, we see that

$$|N(x) - N(x_0)| \leq \mathrm{Lip}(N)|x - x_0| \Rightarrow N(x) \geq N(x_0) - \mathrm{Lip}(N)|x - x_0| > N(x_0) - \varepsilon \iff$$
$$|x - x_0| \leq \frac{1}{\mathrm{Lip}(N)}\varepsilon.$$

Therefore, we conclude that, on the non-trivial set

$$\{x \in \mathbb{R} : \frac{1}{\mathrm{Lip}(N)}\varepsilon \geq |x - x_0| \geq 2\varepsilon\},$$

it holds that $M^1_N f(x_0) \geq M^1_N f(x) \geq M^1_{\equiv N(x_0)-\varepsilon} f(x) \geq M^1_N f(x_0) \geq M^1_N f(x)$. By [1, Lemma 3.6], $M^1_{\equiv N(x_0)-\varepsilon} f(x) = M^1_N f(x) = f(x)$. This concludes the proof of this fact, and also finishes this part

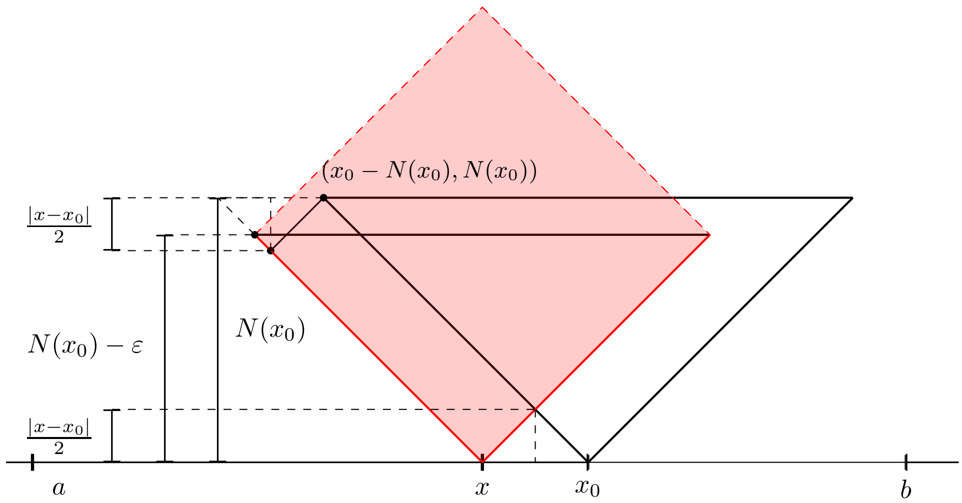

FIGURE 5. Illustration of proof of case (b).

of the section, as the finish here is then the same as the one used in Theorem 1, and we therfore omit it.

4.2. **The critical case Lip**$(N) = \frac{1}{2}$**.** The argument is pretty simple: we build explicitly a suitable sequence of approximations of N such that they all have Lipschitz constants less than $\frac{1}{2}$. By our already proved results, we this will give us the result also in this case.

Explicitly, let N be such that $\mathrm{Lip}(N) = \frac{1}{2}$ and $f \in BV(\mathbb{R})$. Let then $\mathcal{P} = \{x_1 < \cdots < x_M\}$ be any partition of the real line. Let $J \gg 1$ be a large integer, and divide the interval $[x_1, x_M]$ into J equal parts, that we call (a_j, b_j). Define also the numbers

$$\Delta_j = \frac{N(b_j) - N(a_j)}{b_j - a_j}.$$

We know, by hypothesis, that $\Delta_j \in [-1/2, 1/2]$. Let then $\tilde{\Delta}_j = \frac{1}{2} - \frac{1}{J^3}$, and define the function

$$\tilde{N}(x) = \begin{cases} N(x_1), & \text{if } x \leq x_1, \\ N(x_1) + \tilde{\Delta}_1(x - x_1), & \text{if } x \in (a_1, b_1), \\ \tilde{N}(b_{j-1}) + \tilde{\Delta}_j(x - b_{j-1}), & \text{if } x \in (a_j, b_j), \\ \tilde{N}(b_J), & \text{if } x \geq x_N. \end{cases}$$

It is obvious that this function is continuous and Lipschitz with constant $\frac{1}{2} - \frac{1}{J^3}$. If $x \in [x_1, x_N]$, then

$$|\tilde{N}(x) - N(x)| = |\tilde{N}(x) - \tilde{N}(x_1) + N(x_1) - N(x)| \leq \int_{x_1}^{x} |\frac{1}{2} - \frac{1}{J^3} - \frac{1}{2}| \mathrm{d}t \leq \frac{|x_M - x_1|}{J^3}.$$

We now choose J such that the right hand side above is less than $\delta > 0$, which is going to be chosen as follows: for the same partition $\mathcal{P}$, we let $\delta > 0$ be such that

$$|\tilde{N}(x_i) - N(x_i)| < \delta \Rightarrow |M^1_{N(x_j)} f(x_j) - M^1_{\tilde{N}(x_j)} f(x_j)| < \frac{\varepsilon}{2M}.$$

This can obviously, by continuity, always be accomplished. This implies that, using the previous case,

$$\mathcal{V}_{\mathcal{P}}(M^1_N f) \leq \mathcal{V}_{\mathcal{P}}(M^1_{\tilde{N}} f) + \varepsilon \leq \mathcal{V}(M^1_{\tilde{N}} f) + \varepsilon \leq \mathcal{V}(f) + \varepsilon.$$

Taking the supremum over all possible partitions and then taking $\varepsilon \to 0$ finishes also this case, and thus the proof of Theorem 3.

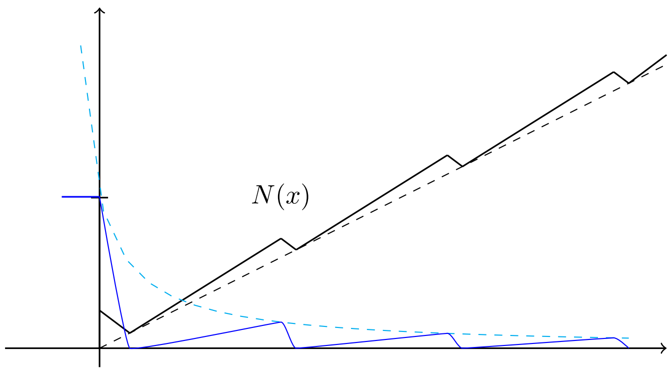

FIGURE 6. Such a counterexample in the case of $\alpha = \frac{3}{4}$. The dashed lines are the graphs of $\frac{x}{2}$ and $\frac{1}{1+x}$, and the non-dashes ones the graphs of $M_N^1 f$ and N in this case.

4.3. **Counterexample for Lip**$(N) > \frac{1}{2}$**.** Finally, we build examples of functions with $\mathrm{Lip}(N) > \frac{1}{2}$ and $f \in BV(\mathbb{R})$ such that

$$\mathcal{V}(M_N f) = +\infty.$$

Fix then $\alpha > \frac{1}{2}$ and let a function N with $\mathrm{Lip}(N) = \alpha$ be defined as follows:

(A) First, let $x_0 = \frac{2}{2\alpha+1}$. Let then $N(0) = 1$, $N(x_0) = \frac{x_0}{2}$ and extend it linearly in $(0, x_0)$.
(B) Let x_K' be the real number that satisfies the equation $\alpha x - \alpha x_{K-1} + \frac{x_{K-1}}{2} = \frac{x+1}{2} \iff x_K' = x_{K-1} + \frac{1}{\alpha - \frac{1}{2}}$.
(C) At last, take $x_K = x_K' + \frac{1}{2\alpha+1}$, and define for all $K \geq 1$ $N(x_K) = \frac{x_K}{2}$, $N(x_K') = \frac{x_K'+1}{2}$, extending it linearly on (x_{K-1}, x_K') and (x_K', x_K).

As $\{x_K'\}_{K\geq 1}$ is an arithmetic progression, we see that

$$\sum_{K\geq 0} \frac{1}{x_K'} = +\infty.$$

Moreover, define $f(x) = \chi_{(-1,0)}(x)$. We will show that, for this N, we have that

$$\mathcal{V}(M_N f) = +\infty.$$

In fact, it is not difficult to see that:

(A) $M_N^1 f(x_K) = 0, \forall K \geq 0$. This is due to the fact that the maximal intervals $(y-t, y+t)$ that satsify $|x_K - y| \leq t \leq N(x_K)$ are still contained in $[0, +\infty)$, which is of course disjoint from $(-1, 0)$.
(B) $M_N^1 f(x_K') \geq \frac{1}{x_K'+1}$. This follows from

$$M_N^1 f(x_K') \geq \frac{1}{2Nf(x_K')} \int_{-1}^{x_K'} f(t)\mathrm{d}t = \frac{1}{x_K'+1}.$$

This shows that

$$\mathcal{V}(M_N^1 f) \geq \sum_{K=0}^{\infty} |M_N^1 f(x_K') - M_N^1 f(x_K)| = \sum_{K=0}^{\infty} \frac{1}{x_K'+1} = +\infty.$$

This construction therefore works, and Theorem 4 is proved.

5. Comments and remarks

5.1. **Nontangential maximal functions and classical results.** Here, we investigated mostly the regularity aspect of our family M^α of nontangential maximal functions, and looked for the sharp constants in such bounded variation inequalities. One can, however, still ask about the most classical aspect studied by Melas [12]: Let C_α be the least constant such that we have the following inequality:

$$|\{x \in \mathbb{R} \colon M^\alpha f(x) > \lambda\}| \leq \frac{C_\alpha}{\lambda}\|f\|_1.$$

By [12], we have that, for when $\alpha = 0$, then $C_0 = \frac{11+\sqrt{61}}{12}$, and the classical argument of Riesz [13] that $C_1 = 2$. Therefore, $\frac{11+\sqrt{61}}{12} \leq C_\alpha \leq 2,\ \forall \alpha \in (0,1)$. Nevertheless, the exact values of those constants is, as long as the author knows, still unknown.

5.2. **Bounded variation results for mixed Lipschitz and nontangential maximal functions.** In Theorems 3 and 4, we proved that, for the *uncentered* Lipschitz maximal function M_N, we have sharp bounded variation results for $\text{Lip}(N) \leq \frac{1}{2}$, and, if $\text{Lip}(N) > \frac{1}{2}$, we cannot even assure *any* sort of bounded variation result.

We can ask yet another question: if we define the *nontangential Lipschitz maximal function*

$$M_N^\alpha f(x) = \sup_{|x-y|\leq \alpha t \leq \alpha N(x)} \frac{1}{2t}\int_{y-t}^{y+t} |f(s)|\mathrm{d}s,$$

then what should be the best constant $L(\alpha)$ such that, for $\text{Lip}(N) \leq L(\alpha)$, then we have some sort of bounded variation result like $\mathcal{V}(M_N^\alpha f) \leq A\mathcal{V}(f)$, and, for each $\beta > L(\alpha)$, there exists a function N_β and a function $f_\beta \in BV(\mathbb{R})$ such that $\text{Lip}(N_\beta) = \beta$ and $\mathcal{V}(M_{N_\beta} f_\beta) = +\infty$? Regarding this question, we cannot state any kind of sharp constant bounded variation result, but the following is still attainable: it is possible to show that the first two lemmas of O. Kurka [8] are adaptable in this context *if* we suppose that

$$\text{Lip}(N) \leq \frac{1}{\alpha+1},$$

and then we obtain our results, with a constant that is even independent of $\alpha \in (0,1)$. On the other hand, our example used above in the proof of Theorem 4 is easily adaptable as well, and therefore one might prove the following Theorem:

Theorem 5. *Let $\alpha \in [0,1]$ and N be a Lipschitz function with $Lip(N) \leq \frac{1}{\alpha+1}$. Then, for every $f \in BV(\mathbb{R})$, we have that*

$$\mathcal{V}(M_N^\alpha f) \leq C\mathcal{V}(f),$$

where C is independent of N, f, α. Moreover, for all $\beta > \frac{1}{\alpha+1}$, there is a function N_β and

$$f(x) = \begin{cases} 1, & \text{if } x \in (-1,0); \\ 0, & \text{otherwise}, \end{cases}$$

with $Lip(N_\beta) = \beta$ and $\mathcal{V}(M_{N_\beta}^\alpha f) = +\infty$.

5.3. **Increasing property of maximal $BV-$norms.** Theorem 1 proves that, if we define

$$B(\alpha) := \sup_{f \in BV(\mathbb{R})} \frac{\mathcal{V}(M^\alpha f)}{\mathcal{V}(f)},$$

then $B(\alpha) = 1$ for all $\alpha \in [\frac{1}{3}, 1]$. We can, however, with the same technique, show that $B(\alpha)$ is increasing in $\alpha > 0$, and also that $B(\alpha) \equiv 1\ \forall \alpha \in [\frac{1}{3}, +\infty)$. Indeed, we show that, for $f \in BV(\mathbb{R})$ and $\beta > \alpha$, then $\mathcal{V}(M^\alpha f) \geq \mathcal{V}(M^\beta f)$. The argument uses the maximal attachment property in the following way: let, as usual, (a,b) be an interval where $M^\beta f$ has a local maximum *inside* it, at, say, x_0, and

$$M^\beta f(x_0) > \max(M^\beta f(a), M^\beta f(b)).$$

Then, as we have that $M^\beta f \geq M^\alpha f$ *everywhere*, we have two options:

- *If* $M^\beta f(x_0) = f(x_0)$, we do not have absolutely anything to do, as then also $M^\alpha f(x_0) = M^\beta f(x_0)$.

- *If* $M^{\beta}f(x_0) = u(y,t)$, *for* $t > 0$, we have – as in the proof of Theorem 1 – that $(y - \beta t, y + \beta t) \subset (a,b)$. But it is then obvious that

$$M^{\alpha}f(y) \geq u(y,t) = M^{\beta}f(x_0) \geq M^{\beta}f(y) \geq M^{\alpha}f(y).$$

Therefore, we have obtained a form of the maximal attachment property, and therefore we can apply the standard techniques that have been used through the paper to this case, and it is going to yield our result.

This shows directly that $B(\alpha) \leq 1, \forall \alpha \geq 1$, but taking $f(x) = \chi_{(0,1)}$ as we did several times shows that actually $B(\alpha) = 1$ in this range.

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