On the strength of mutual stationarity

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Abstract

For $(\kappa_n)_{n<\omega}$ a strictly increasing sequence of regular cardinals $\geqslant \aleph_2$, Foreman and Magidor showed: if every sequence $(S_n)_{n<\omega}$ of sets S_n, which are stationary in κ_n with $\forall \xi \in S_n \ \mathrm{cof}(\xi) = \omega_1$, is mutually stationary then $V \neq L$. We show that the existence of a sequence $(\kappa_n)_{n<\omega}$ with this property is equiconsistent with the existence of a measurable cardinal. In case $(\kappa_n)_{n<\omega}$=$(\aleph_{n+3})_{n<\omega}$ the property implies the existence of inner models with many measurable cardinals .

1 Introduction

The concept of *mutual stationarity* was introduced by M. Foreman and M. Magidor [4] in order to transfer some combinatorial aspects of stationary subsets of regular cardinals to singular cardinals. Together with J. Cummings they further investigated the status of such sequences in [3].

Definition 1 *Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals $\geqslant \aleph_2$ with $\kappa_\omega = sup_{n<\omega}\kappa_n$. A sequence $(S_n)_{n<\omega}$ is called* mutually stationary *in $(\kappa_n)_{n<\omega}$ if every first-order structure $\mathfrak{A}$ of countable type with $\kappa_\omega \subseteq \mathfrak{A}$ has an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ such that $\forall n < \omega \sup |\mathfrak{B}| \cap \kappa_n \in S_n$.*

Note that if $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$ then each $S_n \cap \kappa_n$ is stationary in κ_n. In the following we shall denote the class $\{\xi \in \mathrm{Ord} | \mathrm{cf}(\xi) = \lambda\}$ by Cof_λ. For $X \subseteq \mathrm{Ord}$ a set, we write $\mathrm{ot}(X)$ for its order type.

Definition 2 *Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals and $\lambda < \kappa_0$, λ regular. The mutual stationarity property $\mathrm{MS}((\kappa_n)_{n<\omega}, \lambda)$ is the statement: if $(S_n)_{n<\omega}$ is a sequence of sets $S_n \subseteq \mathrm{Cof}_\lambda$ which are stationary in κ_n then $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$.*

M. Foreman and M. Magidor [4] proved the following two theorems:

Theorem. *For $(\kappa_n)_{n<\omega}$ a strictly increasing sequence of uncountable regular cardinals,* $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega)$ *holds.*

Theorem. $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ *implies $V \neq L$.*

In fact, they proved much more in the latter theorem: assuming $V = L$ they exhibited a double-indexed sequence $S_n^h \subseteq \omega_{n+2} (n < \omega, 1 \leq h < \omega)$, where each $S_n^h = \mathrm{Defcol}(h) \cap \mathrm{Cof}_{\omega_1}$. For α not a cardinal, let $\beta(\alpha) + 1$ be the least level of the L-hierarchy where α is singular, and let $h(\alpha)$ be the least level of definability, so that there is a $\Sigma_h(L_{\beta(\alpha)})$ definable function witnessing the singularity of α. Then $\mathrm{Defcol}(h) = \{\alpha \in \mathrm{Ord} | h(\alpha) = h\}$. Their result then is: For any function $f : \omega \longrightarrow \omega$, $\langle S_n^{f(n)} \rangle$ is mutually stationary if and only if f is eventually constant.

We strengthen this to:

Theorem 1 *The theories* $\mathrm{ZFC} + \exists (\kappa_n)_{n<\omega} \mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ *and* $\mathrm{ZFC} + \exists\kappa(\kappa$ *measurable) are equiconsistent.*

The implication from right to left was proved by J. Cummings, Foreman, and Magidor [3] via Prikry forcing. Again they proved more than this: they showed that a tail of the Prikry generic sequence satisfies $\mathrm{MS}((\kappa_n)_{n<\omega}, \lambda)$ for any $\lambda < \kappa_0$ (or indeed the mutual stationarity of *any* sequence of stationary sets $S_n \subseteq \kappa_n$ irrespective of the cofinalities of the ordinals in the S_n.) This is essentially obtained by utilising the fact that a tail of the Prikry generic sequence remains *coherently Ramsey* in the generic extension. The converse which we prove here uses the core model K of A. J. Dodd and R. B. Jensen (see [2]). We deduce the existence of $O^\sharp$ from $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ in detail. The proof involves the global square principle $\Box$ in L and techniques from the Jensen Covering theorem for L (see [1]). Fine structural details will be presented in the hyperfine structure theory of S. D. Friedman and the first author [5]. Although the hyperfine structure for the Dodd-Jensen Core Model is not yet published we shall nevertheless indicate how to transfer the arguments from L to the Dodd-Jensen K for the proof of the full theorem.

In case $(\kappa_n)_{n<\omega}$ consists of "small" cardinals we can obtain higher consistency strengths:

Theorem 2 *If* $\mathrm{MS}((\aleph_{n+3})_{n<\omega}, \omega_1)$ *holds then there is an inner model with infinitely many measurable cardinals κ of Mitchell order $o(\kappa) = \omega_1$.*

Better results than the above are obtainable, but we leave the precise statement (and a proof of Theorem 2) to a later paper. For these results, the hyperfine structure theory has not been developed, and so there recourse is made to more standard fine structure.

2 Order types of Square Sequences

Definition 3 *Let* $\mathrm{Sing} = \{\beta \in \mathrm{Ord} \mid \lim(\beta) \wedge \mathrm{cf}(\beta) < \beta\}$ *be the class of singular limit ordinals. Global square ($\Box$) is the assertion: there is a system* $(C_\beta)_{\beta \in \mathrm{Sing}}$

satisfying:
(a) C_β *is a closed cofinal subset of* β*;*
(b) $\mathrm{ot}(C_\beta) < \beta$;
(c) if $\overline{\beta}$ *is a limit point of* C_β *then* $\overline{\beta} \in \mathrm{Sing}$ *and* $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$.

Jensen [6] introduced the principle $\Box$ and proved it in L. The second author [8] proved $\Box$ in the Dodd-Jensen core model K. From the order types of the square sequences C_ξ we shall define stationary sets S_n to which we shall apply the MS-principle.

Theorem 3 *Let* κ *be a regular cardinal* $\geq \aleph_2$ *and* λ *a regular cardinal* $< \kappa$*. Then for every ordinal* θ *such that* $\theta^+ < \kappa$ *the set*

$$\{\beta \in \mathrm{Cof}_\lambda \cap \kappa \mid \mathrm{ot}(C_\beta) \geq \theta\}$$

is stationary in κ*.*

Proof Let $C \subseteq \kappa$ be closed unbounded in κ. Let $\mu = \max(\lambda, \theta^+)$ which is an uncountable regular cardinal $< \kappa$. Take a singular limit point γ of C of cofinality μ. Then $C \cap C_\gamma$ is closed unbounded in γ of ordertype $\geq \mu$. Take β to be a singular limit point of $C \cap C_\gamma$ such that $\mathrm{cof}(\beta) = \lambda$ and $\mathrm{ot}(C \cap C_\gamma \cap \beta) \geq \theta$. By the coherency property 3 (c), $C_\beta = C_\gamma \cap \beta$. Thus $\beta \in C \cap \{\beta \in \mathrm{Cof}_\lambda \cap \kappa \mid \mathrm{ot}(C_\beta) \geq \theta\} \neq \emptyset$. $\Box$

Note that $(S_n)_{n<\omega}$ with

$$S_n = \{\beta \in \mathrm{Cof}_{\omega_1} \cap \aleph_{n+3} \mid \mathrm{ot}(C_\beta) \geq \aleph_{n+1}\}$$

is a sequence of stationary sets to which we could apply the MS-principle.

3 Hyperfine Singularizations

Let β be a singular ordinal in L. We shall assign to β a level of the fine structural hierarchy and a parameter which canonically witness the singularity of β. We use the hyperfine hierarchy of S. D. Friedman and the first author [5] where the same singularizations were used in the proof of global square.

The hyperfine structural hierarchy refines Gödel's L_α-hierarchy. The levels of the hierarchy are indexed by *locations* $s = (\alpha, \varphi_m, \vec{x})$ where $\alpha \in \mathrm{Ord}$, $\varphi_m(v_0, \ldots, v_{k-1})$ is an $\in$-formula, and $\vec{x} = x_1, \ldots, x_{k-1} \in L_\alpha$. $(\varphi_m)_{m<\omega}$ is an appropriate list of all $\in$-formulas. Then

$$L_s = (L_\alpha, \in, <_L, I, N, S, S^{L_\alpha}_{\varphi_0}, \ldots, S^{L_\alpha}_{\varphi_m} \upharpoonright \vec{x}, \emptyset, \emptyset, \ldots);$$

Here, $<_L$ is the canonical well-ordering of L, I, N, S are an *interpretation function*, a *naming function*, and a *Skolem function* respectively for L; $S^{L_\alpha}_{\varphi_i}$ is a Skolem function for φ_i computed in L_α. Moreover the last function $S^{L_\alpha}_{\varphi_m}$ is restricted to arguments $\vec{y}$ which are lexicographically smaller than $\vec{x}$, where the lexicographical order $<_{\mathrm{lex}}$ is derived from $<_L$.

The locations are well-ordered lexicographically by $\tilde{<}$. For each L_s there is a hulling operator $L_s\{.\}$; $L_s\{X\}$ is the smallest substructure of L_s which contains X. The basic fine structural laws of (L_s) and the associated hulling operations are described in [5].

For a given limit ordinal β which is singular in L we describe its singularization; in view of the intended applications we also assume that $\mathrm{cof}(\beta) \geq \omega_1$. There is a location $s = (\gamma, \varphi, \vec{x})$ and a finite set $p \subseteq L_\gamma$ such that $\gamma \geq \beta$ and

(1) $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}$ *is bounded below* β.

We say that β is *semi-singularized at* (L_s, p). Let $s = s(\beta)$ be the $\tilde{<}$-minimal location such that β is semi-singularized at (L_s, p) for some p. Then let $p = p(\beta)$ be a finite set such that $(L_{s(\beta)}, p)$ semi-singularizes β where p is minimal with respect to the $<_*$-wellordering of finite subsets of L: $p <_* q \leftrightarrow \exists z \in q \setminus p \forall u (u <_L z \rightarrow (u \in p \leftrightarrow u \in q))$.

(2) $(L_{s(\beta)}, p(\beta))$ *exists and semi-singularizes* β*; it is called the* L*-singularization of* β*.*

We give some more information about the L-singularization. Note that by $\mathrm{cof}(\beta) \geq \omega_1$ we are in the "generic case" of [5].

(3) $s(\beta) = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$.

(4) *There is* $\alpha_0 = \alpha_0(\beta) < \beta$ *minimal such that* $L_s\{\alpha_0 \cup p\}$ *is unbounded below* s*, i.e., for all* $\vec{y} <_{lex} \vec{x}$*,* $|\vec{y}| = |\vec{x}|$ *there is* $\vec{z} \in L_s\{\alpha_0 \cup p\}, |\vec{z}| = |\vec{x}|$ *such that* $\vec{y} <_{lex} \vec{z}$*; in case that* $\vec{x} = \vec{0}$ *we have to require instead that* $L_s\{\alpha_0 \cup p\}$ *is cofinal in* $(L_s, <_L)$*.*

In the construction of the canonical $\square$-sequence C_β some ordinal $\alpha \leq \alpha_0$ will be used as a "steering ordinal". As a brief sketch, we want to define the C_β sequence with reference to a cofinalising sequence in the location s. If α_0 is a limit ordinal, then we shall take α_0 itself as α. Otherwise $\alpha_0 = \alpha_0' + 1$, and we have some $\alpha_1 < \alpha_0'$ so that $L_s\{\alpha_1 \cup \{p, \alpha_0'\}\}$ is unbounded below s; if $\alpha_1 > 0$ but is $\alpha_1' + 1$, we repeat, and see that $L_s\{\alpha_2 \cup \{p, \alpha_0' \, \alpha_1'\}\}$ is unbounded below s for some $\alpha_2 < \alpha_1$. After a finite number k of steps we find that α_k is zero (in which case we deduce that the cofinality of $\beta = \omega$) or a limit. In the latter case, by recursion on $\iota \leq \alpha_k$ we define an increasing sequence of hulls in the location whose suprema below β will be the elements of what will ultimately contain the C_β We thus have bounded the order type of C_β by this "steering ordinal" $\alpha_k(\beta) \leq \alpha_0$. Hence

(5) $\mathrm{otp}(C_\beta) \leq \alpha_0 < \beta$.

This restriction on order types will later conflict with the choice of $(S_n)_{n<\omega}$ described in section 2 and conclude a proof by contradiction.

4 Lifting up Singularizations

The following argument is an upward extensions of embeddings construction as known from the proof of Jensen's Covering Theorem:

Theorem 4 *Let $\pi : (L_\beta, \in) \to (L_{\beta^*}, \in)$ be an elementary cofinal map between* ZF^-*-models. Let β be singular in L and $\mathrm{cof}(\beta) \geq \omega_1$, let (L_s, p) be the L-singularization of β as described in the previous paragraph. Then there are a uniquely defined structure preserving map $\pi^* : L_s \to L_{s^*}$ and a parameter p^* satisfying:*

a) $\pi^* \upharpoonright L_\beta = \pi$, $\pi^* “p = p^*$;

b) (L_{s^*}, p^*) *is the L-singularization of β^*.*

Proof. The proof of $\square$ in L shows that L_s can be represented as $L_s = \bigcup_{i<\tau} L_{s_i}\{\beta_i \cup p\}$ for strictly increasing sequences $(\beta_i)_{i<\tau}$ and $(s_i)_{i<\tau}$ converging to β and s resp., such that each transitive collapse $\sigma_i : M_i \cong (L_{s_i}\{\beta_i \cup p\}, p)$ is the singularization of β_i.
For $i \leq j < \tau$ let $\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i : M_i \to M_j$. The minimality of s implies that each $M_i \in L_\beta$ and $\sigma_{ij} \in L_\beta$. $(M_i)_{i<\tau}, (\sigma_{ij})_{i \leq j < \tau}$ is a directed system of L-singularizations all of whose components are elements of L_β.

We can now map the directed system pointwise to L_{β^*}: for $i < \tau$ let $M_i^* = \pi(M_i)$ and $\sigma_{ij}^* = \pi(\sigma_{ij})$. $(M_i^*)_{i<\tau}, (\sigma_{ij}^*)_{i \leq j < \tau}$ is a commutative system of L-singularizations for the ordinals $\beta_i^* = \pi(\beta_i)$.

(1) The direct limit of $(M_i^*)_{i<\tau}, (\sigma_{ij}^*)_{i \leq j < \tau}$ is well founded.

Proof. The indexing ordinal τ has cofinality $\geq \omega_1$. So any descending ω-sequence in the direct limit is already represented in some M_j^* with $j < \tau$. But M_j^* is transitive. $\square$(1)

Let $M^*, (\sigma_i^*)_{i<\tau}$ be the direct limit of the system $(M_i^*), (\sigma_{ij}^*)$. An argument similar to the proof of the condensation theorem in [5] shows that M^* is a level of the hyperfine hierarchy, say $M^* = L_{s^*}$. Define the map $\pi^* : L_s \to L_{s^*}$ by $\sigma_i(z) \mapsto \sigma_i^*(\pi(z))$. π^* is a homomorphism by general facts about direct limits. If $z \in L_\beta$, then $\sigma_i(z) = z$ for sufficiently high $i < \tau$, and so
(2) $\pi^* \supseteq \pi$.
Let $p^* = \pi^* “p$.
(3) $\pi^* : L_s \to L_{s^*}$ is cofinal with respect to the well-ordering $\tilde{<}$ of locations.
Proof. The location s^* is determined as the $\tilde{<}$-minimal location such that $\sigma_i^* : M_i^* \to M^*$ is a well-defined homomorphism. This property is equivalent to: for all $i < \tau$ and $M_i^* = L_{s_i^*}$ and for all $t \tilde{<} s^*$ holds $\sigma_i^*(t) \tilde{<} s^*$.
Consider $r^* \tilde{<} s^*$. Then take $i < \tau, M_i^* = L_{s_i^*}$ and some $t_0 \tilde{<} s_i^*$ such that $r^* \tilde{\leq} \sigma_i^*(t_0)$.
Take j, $i < j < \tau$ such that $s_i \in L_{s_j}\{\beta_j \cup p\}$. Let $s_i = \sigma_j(s_j'), M_i = L_{\tilde{s}_i}$.
Then

$$\forall t \tilde{<} \tilde{s}_i : \sigma_{ij}(t) \tilde{<} s_i', \quad \forall t \tilde{<} s_i^* : \sigma_{ij}^*(t) \tilde{<} \pi(s_i'); \quad \forall t \tilde{<} s_i^* : \sigma_i^*(t) \tilde{<} \sigma_j^*(\pi(s_i')) = \pi^*(s_i).$$

In particular: $r^* \tilde{\leq} \sigma_i^*(t_0) \tilde{<} \pi^*(s_i)$, as required. $\square(3)$

(4) (L_{s^*}, p^*) is the L-singularization of β^*.
Proof. Take $\delta < \beta$ such that $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\} \subseteq \delta$. Set $\delta^* = \pi(\delta)$. We claim that $\{\eta < \beta^* \mid \eta = \beta^* \cap L_{s^*}\{\eta \cup p^*\}\} \subseteq \delta^*$. Let $\eta \geq \delta^*$. Take $\overline{\beta} < \beta$ minimal such that $\pi(\overline{\beta}) \geq \eta$. Then $\overline{\beta} \geq \delta$ and $\overline{\beta} \not\subseteq \beta \cap L_s\{\overline{\beta} \cup p\}$. Take an L_s-term t and $\vec{x} \subseteq \overline{\beta}$ such that $\overline{\beta} \leq t^{L_s}(\vec{x}, p) < \beta$. Since π^* is a homomorphism,

$$\eta \leq \pi(\overline{\beta}) \leq t^{L_{s^*}}(\pi(\vec{x}), p^*) < \beta^*, \text{and } \pi(\vec{x}) \subseteq \pi''\overline{\beta} \subseteq \eta.$$

Hence $\eta \neq \beta^* \cap L_{s^*}\{\eta \cup p^*\}$, and s^* satisfies the semi-singularity property for β^*.
To show that s^* is minimal semi-singularizing β^* consider $r^* \tilde{<} s^*$. By the cofinality property (3) take $r \tilde{<} s$ such that $r^* \tilde{<} \pi^*(r)$. By the minimality of s, $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p\}\}$ is unbounded in β. Let $\overline{\beta} < \beta$, $\quad \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p\}\}$. Take $i < \tau$ such that $r \tilde{<} s_i$, $\quad \overline{\beta} < \beta$, $\quad r \in L_{s_i}\{\beta_i \cup p\}$. Then:
$L_{s_i} \models \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p\}$,
$M_i \models \overline{\beta} = \beta_i \cap L_{\sigma_i^{-1}(r)}\{\overline{\beta} \cup \sigma_i^{-1}\text{“}p\}$,
$M_i^* \models \pi(\overline{\beta}) = \pi(\beta_i) \cap L_{\pi(\sigma_j^{-1}(r))}\{\pi(\overline{\beta}) \cup \pi(\sigma_i^{-1}\text{“}p)\}$.
Apply σ_i^*:
$\pi(\overline{\beta}) = \beta^* \cap L_{\pi^*(r)}\{\pi(\overline{\beta}) \cup p^*\}$, and
$\pi(\overline{\beta}) = \beta^* \cap L_{r^*}\{\pi(\overline{\beta}) \cup p^*\}$. Since the set of such $\pi(\overline{\beta})$ is cofinal in β^*, r^* does not semi-singularize β^*, as required.
Now we examine the properties of p^*. By construction:
$(*) L_{s^*} = L_{s^*}\{\beta^* \cup p^*\}$.
Suppose that some $q^* <_* p^*$ also satisfies $(*)$. Then $p^* = t(\vec{x}, q^*)$ for some term t and $\vec{x} < \beta^*$.
$L_{s^*} \models \exists \vec{x} < \beta^* \exists q^* <_* p^* \quad p^* = t(\vec{x}, q^*)$.
This existential property can be pulled back to L_s via the directed systems:
$L_s \models \exists \vec{x} < \beta \exists q <_* p \quad p = t(\vec{x}, q)$,
which contradicts the minimal choice of p. $\square$

The previous proof shows that π^* is cofinal in the locations. This affects the "steering ordinal" α_0 as follows:

Lemma 1 *In the situation of the previous theorem,* $\alpha_0(\beta^*) \leq \pi(\alpha_0(\beta))$ *and* $\mathrm{otp}(C_{\beta^*}) \leq \pi(\alpha_0(\beta))$.

5 Getting $O^\sharp$

Theorem 5 *If* $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ *holds then* $O^\sharp$ *exists.*

Proof. Assume $\neg O^\sharp$. Without loss of generality we may assume that $\kappa_0 \geq \aleph_3$. Set $\kappa_\omega = \sup_{n<\omega} \kappa_n$. Define a sequence $(S_n)_{n<\omega}$ of stationary sets as in section

2: $S_0 = \mathrm{Cof}_{\omega_1} \cap \kappa_0$, $S_1 = \mathrm{Cof}_{\omega_1} \cap \kappa_1$, and for $n \geq 2$:

$$S_n = \{\beta \in \mathrm{Cof}_{\omega_1} \cap \kappa_n \mid \mathrm{otp}(C_\beta) \geq \kappa_{n-2}\}.$$

Take a first-order structure $\mathfrak{A} = (L_{\kappa_\omega^+}, \cdots)$ with countable language which has a family of Skolem functions f_i for $L_{\kappa_\omega^+}$, constants $\kappa_0, \kappa_1, \cdots, \kappa_\omega$ and functions g_i, n :

$$g_{i,n}(\vec{x}) = \sup\{f_i(\vec{x}, \vec{y}) \mid \vec{y} < \aleph_2\} \cap \kappa_n.$$

Applying $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ to $(S_n)_{n<\omega}$ and the structure $\mathfrak{A}$ yields some $X \prec L_{\kappa_\omega^+}$ such that $\{\kappa_n \mid n \leq \omega\} \subseteq X$, $\quad \forall n < \omega \sup(X \cap \kappa_n) \in S_n$, and $\omega_2 \subseteq X$. Let $\pi : (L_\delta, \in) \cong (X, \in)$, and $\beta_n = \pi^{-1}(\kappa_n)$ for $n \leq \omega$. For each $n < \omega : \beta_n \geq \aleph_2$ and $\mathrm{cf}(\beta_n) = \omega_1$. The Jensen Covering Theorem for L implies that every β_n is a singular ordinal in L. For $n < \omega$ let (L_{s_n}, p_n) be the singularization of β_n.

(1) If $s_n = (\gamma, -, -)$ then $\gamma \geq \beta_\omega$, since inside L_{β_ω}, β_n is a regular cardinal.

(2) $s_{n+1} \tilde{\leq} s_n$.
Proof. We show that s_n singularizes β_{n+1} as well as β_n :
$L_{s_n} = L_{s_n}\{\beta_n \cup p_n\} \supseteq \beta_\omega \supseteq \beta_{n+1}$, and so
$\{\overline{\beta} < \beta_{n+1} \mid \overline{\beta} = \beta_{n+1} \cap L_{s_n}\{\overline{\beta} \cup p_n\}\} \subseteq \beta_n$. $\square$(2)

Since $\tilde{<}$ is a well-order there is $n_0 < \omega$ such that $s_{n_0} = s_{n_0+1} = s_{n_0+2} = \ldots$. Set $s = s_{n_0}$.

(3) For $n_0 \leq n < \omega : p_{n+1} \leq_* p_n$.
Proof. We show that p_n satisfies the property in the definition of p_{n+1}.
$L_s = L_s\{\beta_n \cup p_n\}$ and so $L_s = L_s\{\beta_{n+1} \cup p_n\}$. $\square$(3)

Since $<_*$ is a well-order there is some $n_1 < \omega$, $\;n_1 \geq n_0$ such that $p_{n_1} = p_{n_1+1} = p_{n_1+2} = \ldots$. Set $p = p_{n_1}$. Then (L_s, p) is the L-singularization of $\beta_{n_1}, \beta_{n_1+1}, \ldots$. Let $\alpha = \alpha_0(\beta_{n_1}) < \beta_{n_1}$ as defined in section 3. As the location s for singularization of the β_m is the same for $m \geq n_1$, the definition of $\alpha_0(\beta_m)$ is independent of $m \geq n_1$. Thus $\alpha = \alpha_0(\beta_m)$ for $n_1 \leq m < \omega$. For $\beta = \beta_{n_1+2}, \beta^* = \sup(X \cap \kappa_{n_1+2})$, we have

$$\pi \restriction L_\beta : L_\beta \to L_{\beta^*}$$

cofinally as required in Theorem 4. Then Lemma 1 yields

$$\mathrm{otp}(C_{\beta^*}) \leq \pi(\alpha_0(\beta)) = \pi(\alpha) < \pi(\beta_{n_1}) = \kappa_{n_1}.$$

But $\beta^* \in S_{n_1+2}$ and $\mathrm{otp}(C_{\beta^*}) \geq \kappa_{n_1}$. Contradiction! $\square$

6 Singularizations in Core Models

For stronger results, we have to apply *core models* instead of the inner model L. We use models of the form $K = L[E]$ where E is a sequence of measures on ordinals. For Theorem 1 we use the Dodd-Jensen core model below one measurable cardinal [2], (and for Theorem 2 we should have to use core model for sequences of measures [7], where E is a sequence of total and partial measures on K, together with the more usual fine structure - rather than hyperfine structure). Since our proofs are dependent on a range of results and techniques from core model theory the further presentation has to omit many details and tries to convey basic ideas. We are forced to make several simplifying assumptions and have to argue by analogy with the L-case. The general reference to core model theory is the book [9] by Martin Zeman.

We use the fine structure as developed by Jensen, For small core models, where the only measures that appear are of Mitchell order 0, an extender E with critical point κ is a filter indexed by some ν which will be the successor cardinal of κ in the ultrapower of the model by E. For higher core models containing sequences of measures, or extenders proper, then larger indices are used (see [9], Chapter 8 for details).

Subsequently, the letter K stands for the Dodd-Jensen core model. Global Square is proved in K by carefully assigning singularizing sequences to singular ordinals in K. We describe the singularization of an ordinal β in terms of the Jensen fine structure for measure sequences ("mice"). It will consist of a level of the fine structural hierarchy and a parameter(-sequence) which canonically witness the singularity of β.

Definition 4 *Let $M = J_\alpha[E]$ be a mouse and let $p \in M$ be some finite parameter. Then (M, p)* semi-singularizes *β, if $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap M\{\overline{\beta} \cup p\}\}$ is bounded below β. Here $M\{X\}$ denotes the fine structural hull of X in M. For simplicity, we shall say "*singularize*" instead of "semi-singularize". M as above is called a canonical singularization of β if*

a) $M \models$ "β is regular" or $\beta = \omega\alpha$;

b) $M = M\{\beta \cup p_M\}$;

c) (M, p_M) singularizes β where p_M is the standard parameter of M.

Again, we only say "K-singularization*" instead of "canonical singularization".*

From a K-singularization M of β one can readily define a subset C_β of β as in the proof of $\Box$ which is cofinal in β of ordertype $< \beta$. Let us indicate some elements of that definition. In view of the intended applications we also assume that $\mathrm{cof}(\beta) \geq \omega_1$. For simplicity we may assume that the first projectum $\rho^1_M < \beta$ so that we can use the relatively simple Σ_1-finestructure. There is $\alpha_0 = \alpha_0(\beta) < \beta$ minimal such that $M\{\alpha_0 \cup p_M\}$ is unbounded in M. In the construction of the canonical $\Box$-sequence C_β some ordinal $\alpha \leq \alpha_0$ will be used as a "steering ordinal" which will imply that $\mathrm{otp}(C) \leq \alpha_0 < \beta$. This restriction on order types will later conflict with the choice of $(S_n)_{n<\omega}$ described above and conclude a proof by contradiction. The coherency property of $\Box$ is due to the coherency between various K-singularizations.

Lemma 2 *If M and N are K-singularizations of β and $M \upharpoonright \beta = N \upharpoonright \beta$ then $M = N$. Also p_M is the least parameter p such that $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap M\{\overline{\beta} \cup p\}\}$ is bounded below β.*

Proof Coiterate M and N up to $\tilde{M}$and $\tilde{N}$. As $M \upharpoonright \beta = N \upharpoonright \beta$, and we are here dealing with measure filters, no critical point of any measure used in this coiteration is below β. Thus if $\tilde{M} \in \tilde{N}$ then $\tilde{N}$ contains a code for M and hence $N \models$"β is singular" which contradicts the definition of a K-singularization. Hence $\tilde{M} = \tilde{N}$. By the preservation of standard parameters, p_M and p_M are both mapped to the standard parameter of $\tilde{M}$. Therefore M and N are both the β-core of $\tilde{M}$ and thus equal. Assume there is some $p < p_M$ such that $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap M\{\overline{\beta} \cup p\}\}$ is bounded below β. Let $(Q, \bar{p}) \cong (M\{\beta \cup p\}, p)$ be the transitivization. Then $(Q, \bar{p})$ singularizes β. The uniqueness argument above shows that $Q = M$ and $M = M\{\beta \cup \bar{p}\ \}$ with $\bar{p} < p_M$which contradicts the minimality of p_M. $\square$

7 Lifting up K-Singularizations

We transfer the upward extensions of embeddings technique to the core model situation:

Theorem 6 *Let $\pi : (J_\beta[\bar{E}], \in) \to (J_{\beta^*}[E], \in)$ be an elementary cofinal map between ZF^--models with $\mathrm{cof}(\beta) \geq \omega_1$. Let $M = J_\alpha[\widetilde{E}]$ be a K-singularization of β which end extends $J_\beta[\bar{E}]$, i.e., $\alpha > \beta$ and $\widetilde{E} \upharpoonright \beta = \bar{E} \upharpoonright \beta$. Then there is a uniquely defined structure preserving map $\pi^* : M \to M^*$, $M^* = J_{\alpha^*}[\widetilde{E}^*]$ satisfying:*

a) $\pi^* \upharpoonright J_\beta[\bar{E}] = \pi$, π^* "$p_M = p_{M^*}$;

b) M^* is the unique K-singularization of β^* satisfying $\widetilde{E}^* \upharpoonright \beta = E \upharpoonright \beta$.

Proof The proof of $\square$ in K shows that M can be represented as $M = \bigcup_{i<\tau} J_{\alpha_i}[\widetilde{E}]\{\beta_i \cup p_M\}$ for strictly increasing sequences $(\beta_i)_{i<\tau}$ and $(\alpha_i)_{i<\tau}$ converging to β and α respectively, such that each transitive collapse $\sigma_i : M_i \cong J_{\alpha_i}[\widetilde{E}]\{\beta_i \cup p_M\}$ is the K-singularization of β_i. For $i \leq j < \tau$ let $\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i : M_i \to M_j$. Since β is a cardinal in M and by acceptability, each $M_i \in J_\beta[\bar{E}]$ and each $\sigma_{ij} \in J_\beta[\bar{E}]$. $(M_i)_{i<\tau}, (\sigma_{ij})_{i \leq j < \tau}$ is a directed system of K-singularizations all of whose components are elements of M.

We can now map the directed system pointwise to $J_{\beta^*}[E]$: for $i < \tau$ let $M_i^* = \pi(M_i)$ and $\sigma_{ij}^* = \pi(\sigma_{ij})$. $(M_i^*)_{i<\tau}, (\sigma_{ij}^*)_{i \leq j < \tau}$ is a commutative system of singularizations for the ordinals $\beta_i^* = \pi(\beta_i)$.

(1) The direct limit of $(M_i^*)_{i<\tau}, (\sigma_{ij}^*)_{i \leq j < \tau}$ is well founded.
Proof. The indexing ordinal τ has cofinality $\geq \omega_1$. So any descending ω-sequence in the direct limit is already represented in some M_j^* with $j < \tau$. But M_j^* is transitive. $\square$(1)

Let $M^*, (\sigma_i^*)_{i<\tau}$ be the direct limit of the system $(M_i^*), (\sigma_{ij}^*)$. M^* is a level of a J-hierarchy, say $M^* = J_{\alpha^*}[\widetilde{E}^*]$.

(2) M^* is a mouse.

Proof. This runs similar to the proof of (1): if M^* were not iterable fine structurally then this would be testified in some M_j^* with $j < \tau$. But M_j^* is iterable since M_j is iterable and π is elementary. $\square(2)$

Define the map $\pi^* : M \to M^*$ by $\sigma_i(z) \mapsto \sigma_i^*(\pi(z))$. π^* is a homomorphism by general facts about direct limits. If $z \in J_\beta$, then $\sigma_i(z) = z$ for sufficiently high $i < \tau$, and so

(3) $\pi^* \supseteq \pi$.

(4) $\pi^* : M \to M^*$ is $\in$-cofinal.

Let $p^* = \pi^*$"p_M. By the direct limit construction:

(5) $M^* = M^*\{\beta^* \cup p^*\}$.

(6) $p^* = p_{M^*}$.

Proof. $p^* \geq p_{M^*}$ If $p^* > p_{M^*}$ then this would be reflected in some M_j^* with $j < \tau$ but then the elementarity of π would yield the contrary. $\square(6)$

(7) (M^*, p_{M^*}) singularizes β^*.

Proof. Take $\delta < \beta$ such that $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap M\{\overline{\beta} \cup p_M\}\} \subseteq \delta$. Set $\delta^* = \pi(\delta)$. We claim that $\{\eta < \beta^* \mid \eta = \beta^* \cap M^*\{\eta \cup p_{M^*}\}\} \subseteq \delta^*$. Let $\eta \geq \delta^*$. Take $\overline{\beta} < \beta$ minimal such that $\pi(\overline{\beta}) \geq \eta$: then $\overline{\beta} \geq \delta$ and $\overline{\beta} \not\subseteq \beta \cap M\{\overline{\beta} \cup p_M\}$. Take a term t and $\vec{x} \subseteq \overline{\beta}$ such that $\overline{\beta} \leq t^M(\vec{x}, p_M) < \beta$. Since π^* is a homomorphism,

$$\eta \leq \pi(\overline{\beta}) \leq t^{M^*}(\pi(\vec{x}), p_{M^*}) < \beta^*, \text{and } \pi(\vec{x}) \subseteq \pi''\overline{\beta} \subseteq \eta.$$

Hence $\eta \neq \beta^* \cap M^*\{\eta \cup p^*\}$, and M^* satisfies the semi-singularity property for β^*. $\square(7)$

The uniqueness of the K-singularization M^* follows from Lemma 2. $\square$

We saw in the previous proof that $\pi^* : M \to M^*$ is cofinal. This again affects the "steering ordinal" α_0 as follows.

Lemma 3 *In the situation of the previous theorem,* $\alpha_0(\beta^*) \leq \pi(\alpha_0(\beta))$ *and thus* $\mathrm{ot}(C_{\beta^*}) \leq \pi(\alpha_0(\beta))$.

8 Getting an Inner Model with a Measurable Cardinal

We modify the proof of Theorem 5 to yield the existence of an inner model with a measurable cardinal. We assume $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ and work with the Dodd-Jensen core model K under the assumption that there is no inner model with a measurable cardinal. By the Dodd-Jensen covering theorem for K every ordinal $\beta \geq \omega_2$ with $\mathrm{cof}(\beta) \leq \omega_1$ is singular in K. In particular $\kappa_\omega = \sup_{n<\omega} \kappa_n$ is singular in K. Take the sequence $(S_n)_{n<\omega}$ of stationary sets $S_n \subseteq \kappa_n$ as in the proof of Theorem 5. Define the first-order structure $\mathfrak{A} = (H^K_{\kappa_\omega^+}, \cdots)$ in analogy to that proof. The mutual stationarity property yields some $X \prec H^K_{\kappa_\omega^+}$

such that

$$\{\kappa_n \mid n \leq \omega\} \subseteq X, \quad \forall n < \omega \ \ (\sup X \cap \kappa_n) \in S_n, \text{ and } \omega_2 \subseteq X.$$

Let $\pi : (\bar{K}, \in) \cong (X, \in)$ where $\bar{K}$ is transitive, and $\beta_n = \pi^{-1}(\kappa_n)$ for $n \leq \omega$. $\bar{K}$ is a mouse without a total measure. For $n < \omega$ take $M_n = (J_{s_n}[E], p_n)$ to be the K-singularization of β_n (by which we mean the least location s_n in the K-hierarchy where we take $E = E^K$).

Coiterate the mice $\bar{K}$ and M_n. $\bar{K}$ comes out below M_n because M_n has information for singularizing β_n whereas β_n is regular in $\bar{K}$. So in the coiteration there is no truncation on the $\bar{K}$-side and M_n either is an end-extension of $\bar{K}$, or will coiterate up to one.

(1) If M_n is not an end-extension of $\bar{K}$, let $(\lambda_i \mid i \leq \theta)$ be the sequence of critical points of the M_n-side of the coiteration. Then $\lambda_\theta \geq \beta_\omega$ and $\beta_\omega \notin \{\lambda_i \mid i \leq \theta\}$.

Proof. Suppose M_n is not an end-extension of $\bar{K}$. If $\lambda_\theta < \beta_\omega$ then either the M_n-side had a total measure on λ_θ which $\bar{K}$ does not have, or M_n were a proper initial segment of $\bar{K}$. Both possibilities lead to a contradiction.
If $\beta_\omega = \lambda_i$, then the i-th iterate M_n^i of M_n would contain $\mathcal{P}(\beta_\omega) \cap \bar{K}$. Since β_ω is singular in $\bar{K}$, M_n^i would contain a cofinal subset of β_ω of small ordertype. But λ_i is regular in M_n^i. $\square(1)$

So the coiterate M_n^θ is the minimal iterate of M_n whose critical point is $> \beta_\omega$, or is M_n itself. In the former case, by (1) there is some maximal $i < \theta$ such that $\lambda_i < \beta_\omega$. Then the iterate M_n^{i+1} is generated from $\lambda_i + 1$ together with some finite parameter, and the critical point of M_n^{i+1} is $> \beta_\omega$. So in this former case, M_n^{i+1} semi-singularizes all β_m such that $\lambda_{i+1} < \beta_m < \beta_n$. However in the latter case, since $M_n = M_n\{\beta_n \cup p_{M_n}\}$ and $On^{M_n} \geq \beta_\omega$ it is clear that M_n itself semi-singularizes all β_m for $m \geq n$. This implies:

(2) For all $n < \omega$ there exists $n' < \omega, n' \geq n$ such that for all $m, n' \leq m < \omega$: $M_m \leq^* M_n$.

Since $\leq^*$ is a pre-wellorder of mice, one can choose a $\leq^*$-minimal element of $\{M_n \mid n < \omega\}$. Choose $n_0 < \omega$ such that

(3) for all $m, n_0 \leq m < \omega : M_m \leq^* M_{n_0}$ and $M_{n_0} \leq^* M_m$.

By the properties of the $\leq^*$-relation:

(4) M_{m+1} is an iterate of M_m, for $m \geq n_0$.

Then $(M_m)_{m \geq n_0}$ is a subsequence of the M_{n_0}-side of the coiteration of $\bar{K}$ and M_{n_0}.

By (1), β_ω is not a critical point in that iteration of M_{n_0}. So there must be some $n_1 < \omega, \ \ n_1 \geq n_0$, so that

(5) $M_{m+1} = M_m$, for $m \geq n_1$.

Set $M = M_{n_1}$. As in the L-case:

(6) $p_{m+1} \leq_* p_m$, for $m \geq n_1$.

By the wellfoundedness of $\leq_*$ take $n_2 < \omega$, $n_2 \geq n_1$ such that $p = p_{n_2} = p_{n_2+1} = \ldots$. So (M, p) is a common K-singularization of $\beta_{n_2}, \beta_{n_2+1}, \ldots$. We can then conclude the proof by contradiction as in the proof of Theorem 5.

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