New operadic actions on the homology of moduli spaces

BONN INTERNATIONAL GRADUATE SCHOOL OF MATHEMATICS

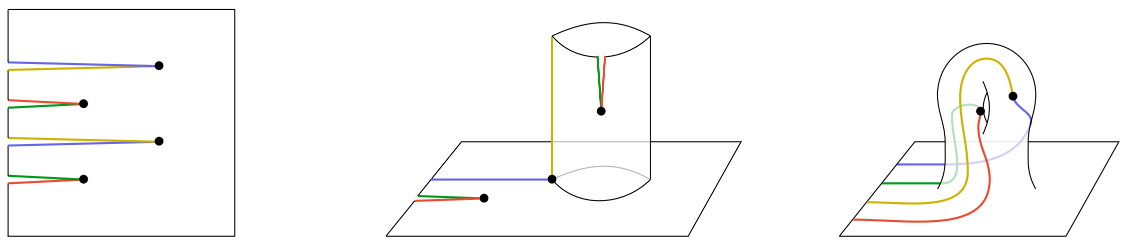
UNIVERSITÄT BONN

Kranhold, Florian

Advisor: Prof. Dr. Carl-Friedrich Bödigheimer

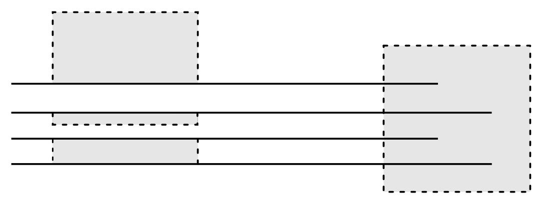
Background and aim

Let $\mathfrak{M}_{g,n}^m$ be the moduli space of Riemann surfaces of genus g with $n \geq 1$ parametrised boundary curves and m punctures, i. e. its points are equivalence classes of conformal structures up to complex isomorphism. As a classifying space for the mapping class group $\Gamma_{g,n}^m$, $\mathfrak{M}_{g,n}^m$ classifies orientable surface bundles. Our approach uses a simplicial model $p : \mathfrak{P}_{g,1}^m \xrightarrow{\simeq} \mathfrak{M}_{g,1}^m$ from [Böd90a] and [ABE08], whose points are given by configurations of slits on the complex plane and additional combinatorial gluing data: Starting with such a *slit configuration*, the corresponding surface (and its conformal structure) is given by gluing the plane along paired slits.



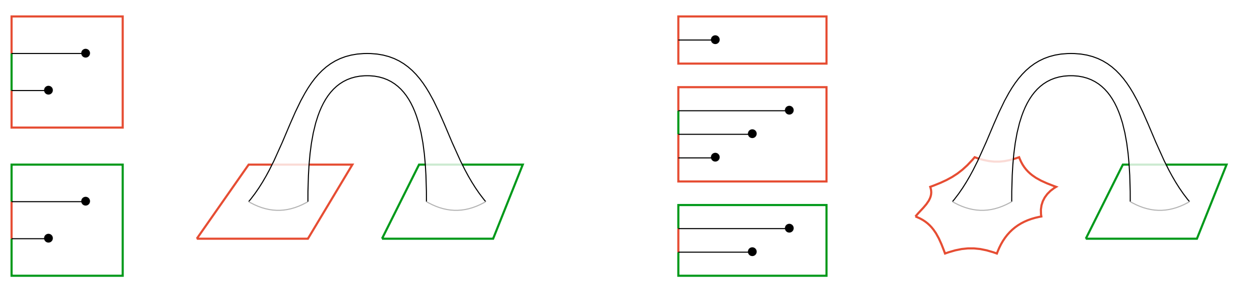
We know from [Böd90b] that the little 2-cube operad $\mathcal{C}$ acts on $\mathfrak{P}_1 := \coprod_{g,m} \mathfrak{P}_{g,1}^m$ by implanting slit configurations into each other: We start from right to left and pose in each step the next box inside the partially glued surface. Thus, we get graded structure maps

$$\mathcal{C}(r) \times \prod_{i=1}^r \mathfrak{P}_{g_i,1}^{m_i} \longrightarrow \mathfrak{P}_{g_1+\dots+g_r,1}^{m_1+\dots+m_r}.$$



From the general theory of iterated loop spaces, [CLM76], $H_*(\mathfrak{M}_{g,1}^m)_{g,m}$ becomes a Gerstenhaber algebra (i. e. we have a Pontryagin product and a Browder bracket $[-,-]$ of degree 1 satisfying the Jacobi identity, such that $[a,-]$ is a derivation). This has turned out to be useful to describe some generators of $H_*(\mathfrak{M}_{g,1}^m)$ for small g and m, see for example [Meh11], [BH14], [BB19]. There are two generalisations depicted below:

- We can consider $n \geq 2$ boundary curves: The model $\mathfrak{P}_{g,n}^m$ is again a cell complex whose points are now given by configurations of slits on n copies of the complex plane (left picture).
- Given a tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 1}^n$, there is a model $\mathfrak{P}_g^m[\lambda_1, \dots, \lambda_n]$ for $\mathfrak{M}_{g,n}^m$ consisting of slit configurations on $\lambda_1 + \dots + \lambda_n$ planes where λ_i layers "belong" to the i^{th} boundary curve (right picture).



Our aim is to find operadic actions on these generalised models and to understand the induced operations on the homology of the moduli spaces $\mathfrak{M}_{g,n}^m$.

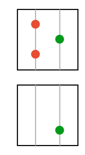
The vertical operad $\mathcal{V}$

In [BT13], Bödigheimer and Tillmann proposed an operad $\mathcal{V}$ reflecting the structure of slit configurations on multiple layers: For $k, n \geq 1$, consider the k-fold ordered configuration space of $\mathbb{R}^{\sqcup n} \longrightarrow \mathbb{C}^{\sqcup n} \xrightarrow{\text{Re}} \mathbb{R}$

$$\text{Conf}^{\perp}\binom{k}{n} := \left\{(z_1, \dots, z_k) \in (\mathbb{C}^{\sqcup n})^k ; \, \text{Re}(z_1) = \dots = \text{Re}(z_k) \text{ and } z_i \neq z_j \text{ for } i \neq j\right\}.$$

Given $k_1, \dots, k_r \geq 1$, we define the *vertical configuration space* as the multi-fibrewise configuration space

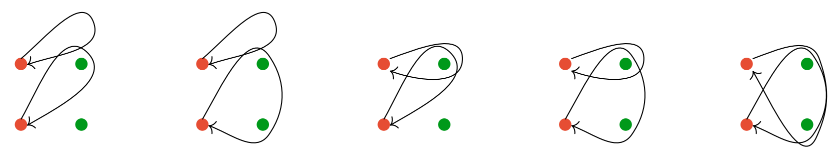
$$\mathcal{V}\binom{k_1, \dots, k_r}{n} := \left\{(Z_1, \dots, Z_r) \in \prod_{i=1}^r \text{Conf}^{\perp}\binom{k_i}{n} ; \, Z_i \cap Z_j = \emptyset \text{ for } i \neq j\right\}.$$

 $\in \mathcal{V}\binom{2,2}{2}$

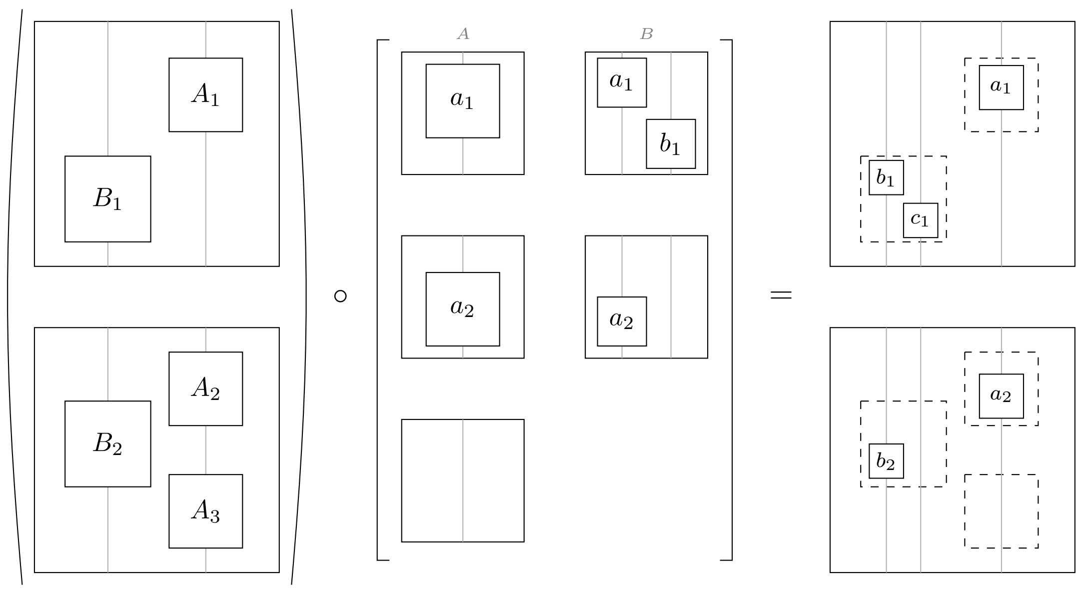
Inspired by [Fuk70] and [Arn69] and using discrete Morse theory, we found the following:

Theorem 1 (Bianchi, K. 2018). *The space $\mathcal{V}\binom{k_1,\dots,k_r}{n}$ has the homotopy type of a finite $(r-1)$-dimensional cell complex and its homology $H_*(\mathcal{V})$ is free.*

For example, $H_1\mathcal{V}\binom{2,2}{1}$ is freely generated by the following 5 loops (the red points move simultaneously):



If we replace each point in a given configuration by a small square, $\mathcal{V}$ obtains the structure of a $\mathbb{Z}_{\geq 1}$-coloured operad. The internal multiplications can be depicted in the following way (note the letter shift):



Note that $\mathcal{V}$ is a suboperad of a larger operad: We have an embedding $\eta : \mathcal{V}\binom{k_1,\dots,k_r}{n} \hookrightarrow \mathcal{C}\binom{k_1,\dots,k_r}{n}$ into the coloured version of the little 2-cube operad by forgetting the vertical coupling.

Applying the monoidal functor $H_* : \textbf{Top} \longrightarrow \textbf{Ab}^{\mathbb{Z}}$ to $\mathcal{V}$, we obtain an coloured operad of graded abelian groups. We were able to prove a structure theorem analogously to the homology of the little cubes $\mathcal{C}$:

Theorem 2 (Bianchi, K. 2019). *The operad $H_*(\mathcal{V})$ is generated in arity at most 2: each homology class in $H_*(\mathcal{V})$ is a sum of iterated classes of arity ≤ 2 and degree ≤ 1.*

Again by implanting slit pictures as in the case $n = 1$, the sequence $\mathfrak{P} := (\mathfrak{P}_n)_{n \geq 1}$ with $\mathfrak{P}_n := \coprod_{g,m} \mathfrak{P}_{g,n}^m$ forms a $\mathbb{Z}_{\geq 1}$-coloured algebra over $\mathcal{V}$. There are two subtleties to keep in mind:

- There are configurations such that the joint slit picture encodes a disconnected Riemann surface, This can be bypassed by considering only those path components of $\mathcal{V}$ which encode "connective configurations". The resulting substructure $\mathcal{V}_c$ forms a suboperad; in the same way $\mathcal{C}_c \hookrightarrow \mathcal{C}$ is a suboperad.
- The multiplication is graded by g and m: The puncture numbers just add up, and for the genus, we see by an Euler characteristic argument $g = \sum_{i=1}^r (g_i + k_i) + (1 - r - n)$.

Theorem 3 (K. 2019). *The multiplication $\mu : \mathcal{V}_c \times \prod \mathfrak{P}_{*,k_i}^* \longrightarrow \mathfrak{P}_{*,n}^*$ extends up to homotopy as*

$$\begin{array}{ccc} \mathcal{V}_c \times \prod_i \mathfrak{P}_{*,k_i}^* & \xrightarrow{\mu} & \mathfrak{P}_{*,n}^* \\ {\scriptstyle \eta \times \prod p}\downarrow & & \downarrow{\scriptstyle p}\, {\scriptstyle \simeq} \\ \mathcal{C}_c \times \prod_i \mathfrak{M}_{*,k_i}^* & \xrightarrow[\nu]{} & \mathfrak{M}_{*,n}^* \end{array}$$

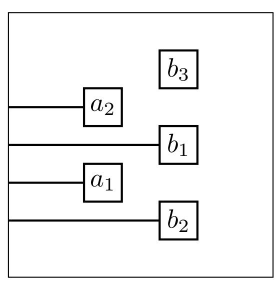
where η is the aforementioned inclusion and ν is a classifying map for the **Grp***-operadic action of the pure braid operad* PBr *on $(\Gamma_{*,n}^*)_{n \geq 1}$ described in* [Mil86]*. Thus, all homology operations coming from $\ker(\eta_*)$ vanish.*

Outlook: The slit operad $\mathcal{S}$

In order to use the generalised models $\mathfrak{P}_g^m[\lambda_1, \dots, \lambda_n]$ for $\mathfrak{M}_{g,n}^m$, we need an operad $\mathcal{S}$ whose colours are tuples $\Lambda := (\lambda_1, \dots, \lambda_n)$, so that the action consists of structure maps of the form

$$\mathcal{S}\binom{\Theta_1, \dots, \Theta_r}{\Lambda} \times \prod_{i=1}^r \mathfrak{P}_*^*[\Theta_i] \longrightarrow \mathfrak{P}_*^*[\Lambda].$$

In order to encode the fact that multiple layers form a common boundary curve, an element $s \in \mathcal{S}$ is a vertical configuration of boxes as before, but now we also allow slits. Here is an element of $\mathcal{S}\binom{(2),(2,1)}{(1)}$:



Each $s \in \mathcal{S}$ also encodes a slit picture by forgetting the box size, the labelling and the "unslitted" boxes, e. g. the above example has genus 1 and puncture number 0. The spaces $\mathcal{S}$ have a much richer structure than the spaces $\mathcal{V}$, but we still have a cellular decomposition similar to those of the models $\mathfrak{P}$ themselves.

Open questions

I am currently working on the following problems:

(1) Can we find a complete presentation of the coloured operad $H_*(\mathcal{V})$? The relations should be a colouring of the usual Poisson and Jacobi identities. We seem to need a "better" system of generators.

(2) How can we describe certain ad-hoc constructions like the T-map $T : H_*(\mathfrak{M}_{g,1}^1) \longrightarrow H_{*+1}(\mathfrak{M}_{g+1,1})$ from [Meh11] as homology operations coming from $\mathcal{V}$ or $\mathcal{S}$? Can we discover new homology operations?

(3) What is the relation between the operad $H_*(\mathcal{S})$ and the algebra $H_*(\mathfrak{M})$, in analogy to the operad $H_*(\mathcal{C})$ and the algebra $H_*(\text{Br})$, the homology of the braid spaces?

References

[ABE08] Jochen Abhau, Carl-Friedrich Bödigheimer, and Ralf Ehrenfried. "Homology of the mapping class group $\Gamma_{2,1}$ for surfaces of genus 2 with a boundary curve". In: *Geometry & Topology Monographs* 14 (2008), pp. 1–25.

[Arn69] Wladimir Igorewitsch Arnol'd. "The cohomology ring of the colored braid group". In: *Matematicheskie Zametki* 5 (1969), pp. 227–231.

[BB19] Felix Boes and Carl-Friedrich Bödigheimer. "A Spectral Sequence for the Homology of the Moduli Spaces of Riemann Surfaces". In preparation. 2019.

[BH14] Felix Jonathan Boes and Anna Hermann. "Moduli Spaces of Riemann Surfaces – Homology Computations and Homology Operations". Master's thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2014.

[Böd90a] Carl-Friedrich Bödigheimer. "On the topology of moduli spaces of Riemann surfaces. Part I: Hilbert Uniformization". In: *Mathematica Gottingensis* (1990).

[Böd90b] Carl-Friedrich Bödigheimer. "On the topology of moduli spaces of Riemann surfaces. Part II: Homology Operations". In: *Mathematica Gottingensis* (1990).

[BT13] Carl-Friedrich Bödigheimer and Ulrike Tillmann. "Symplectic Homology Operations for Moduli Spaces". Preprint. 2013.

[CLM76] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The Homology of Iterated Loop Spaces.* Springer, Lecture Notes in Mathematics 533, 1976.

[Fuk70] Dimitry B. Fuks. "Cohomologies of the braid group mod 2". In: *Funktsional'nyi Analiz i Ego prilozheniya* 4 (1970), pp. 62–73.

[Meh11] Stefan Mehner. "Homologieberechnungen von Modulräumen Riemannscher Flächen durch diskrete Morse-Theorie". Diplomarbeit. Rheinische Friedrich-Wilhelms-Universität Bonn, 2011.

[Mil86] Edward Y. Miller. "The homology of the mapping class group". In: *J. Differential Geometry* 24 (1986), pp. 1–14.

2019 Bonn international graduate school of mathematics
kranhold@math.uni-bonn.de