ALGEBRAIC APPROXIMATIONS OF COMPACT KÄHLER THREEFOLDS

by

Hsueh-Yung Lin

Abstract. — We prove that every compact Kähler threefold has arbitrarily small deformations to some projective manifolds, thereby solving the Kodaira problem in dimension 3.

1 The Kodaira problem in dimension 3

Let X be a compact Kähler manifold. An *algebraic approximation of* X is a deformation $\mathscr{X} \to \Delta$ of X such that up to shrinking Δ, the subset parameterizing projective manifolds in this family is dense in Δ. The so-called Kodaira problem asks whether a compact Kähler manifold in general admits an algebraic approximation. While Kodaira proved that compact Kähler surfaces always have algebraic approximations in the early 1960s [22, Theorem 16.1], starting from dimension 4 there exist compact Kähler manifolds in each dimension for which the Kodaira problem has a negative answer. Such examples were first constructed by Voisin [38], and the obstruction for them to deform to a projective manifold is in fact homotopical; see also [30, Section 8 of the arXiv version] for other examples constructed by Oguiso. In each even dimension no less than 10, there even exist (uniruled) compact Kähler manifolds X such that none of the smooth compact bimeromorphic models of X has the homotopy type of a projective manifold [39].

The aim of this article is to answer in the affirmative the Kodaira problem in dimension 3, which has been an open problem since these works. Together with Kodaira's and Voisin's work mentioned above, this completes the study of the Kodaira problem in terms of the dimension of the manifolds.

Theorem 1.1. — *Every compact Kähler manifold of dimension 3 has an algebraic approximation.*

So far, non-trivial examples of compact Kähler manifolds admitting algebraic approximations can be found in [22, 4, 34, 10, 17, 11, 26, 12, 27] and the list is rather exhaustive at present. A conjecture due to Th. Peternell states that every *minimal* Kähler variety X has an algebraic approximation. So assuming the Minimal Model Program for compact Kähler manifolds, Peternell's conjecture would imply that the bimeromorphic version of the Kodaira problem has a positive answer for every compact Kähler manifold X of nonnegative Kodaira dimension $\kappa(X)$; namely, every such X has a (compact) bimeromorphic model admitting an algebraic approximation. Prior to our work, the latter was already known to hold for threefolds of Kodaira dimension 0 [17].

Obviously, a positive answer to the Kodaira problem for a compact Kähler manifold X implies that the invariants of X that are preserved under small deformations (such as the fundamental group and the Hodge diamond) can be realized by projective manifolds. The statements of the immediate corollaries of Theorem 1.1 of this sort are left to the readers.

2 An overview of the proof

In this section, we give an overview of the proof of Theorem 1.1.

2.1 A general approach proving the existence of algebraic approximations

Given a compact Kähler manifold X, one way to prove that X has an algebraic approximation is to find a (simpler) bimeromorphic model $\nu : X \dashrightarrow X'$ of X together with an algebraic approximation $\mathscr{X}' \to \Delta$ of X' which induces a deformation of ν. Note that in general, it is not enough to only show that X' has an algebraic approximation, because asking a deformation $\Pi : \mathscr{X}' \to \Delta$ of X' to induce deformations of X usually imposes non-trivial restrictions on Π. For instance, if $\nu : X \to X'$ is the blow-up of X' along a submanifold $Y \subset X'$, then a necessary condition for $\Pi : \mathscr{X}' \to \Delta$ to induce a deformation of X is that the submanifold Y is preserved along the deformation Π [23].

On the other hand, it is a simple observation that if $Y \subset X'$ is a subvariety such that $\nu^{-1}_{|X' \backslash Y}$ is biholomorphic onto its image, then to prove that $\Pi : \mathscr{X}' \to \Delta$ is a deformation of X' which induces deformations of $\nu : X \dashrightarrow X'$, it suffices to show that there exists a neighborhood $U \subset X'$ of Y which deforms trivially along Δ (see Lemma 3.13 for the case where ν is a morphism). If such a neighborhood exists, then we will call $\mathscr{X}' \to \Delta$ a Y-*locally trivial* deformation (see Definition 3.1). This leads to the following proposition.

Proposition 2.1. — *Let X' be a normal compact complex variety. Assume that X' has a Y-locally trivial algebraic approximation for every subvariety $Y \subset X'$ whose irreducible components are of codimension at least* 2. *Then every compact Kähler manifold X bimeromorphic to X' has an algebraic approximation as well.*

We refer to 3.4 for a proof of Proposition 2.1. Thus the problem of finding an algebraic approximation of X is transformed into the problem of finding a normal bimeromorphic model X' of X together with Y'-locally trivial algebraic approximations of X' for every $Y' \subset X'$ such that $\dim Y' \leq \dim X - 2$. This is how we will prove Theorem 1.1 for uniruled threefolds, as well as threefolds X of algebraic dimension $a(X) \leq 1$:

Theorem 2.2. — *Let X be a compact Kähler threefold. If X is uniruled or $a(X) \leq 1$, then X is bimeromorphic to a normal compact complex variety X' such that X' has a Y-locally trivial algebraic approximation for every subvariety $Y \subset X'$ with* $\dim Y \leq 1$.

As for threefolds X of algebraic dimension $a(X) \geq 2$, if X is non-algebraic then $a(X) = 2$. So the rest of the non-algebraic threefolds can be covered by the following theorem that we have proven in [27]:

Theorem 2.3 **([27, Theorem 1.1]).** — *Every compact Kähler manifold X with $a(X) = \dim X - 1$ has an algebraic approximation.*

Remark 2.4. — If the goal of stating Theorem 2.2 is to prove Theorem 1.1 for threefolds which are not covered by Theorem 2.3, then including "uniruled threefolds" in the statement of Theorem 2.2 might seem superfluous. The reason why we make such a statement is that even if we focus on threefolds with $a(X) \leq 1$, it turns out that proof-wisely, it is still more natural to separate uniruled threefolds from the non-uniruled ones. Also the proof of Theorem 2.2 in the uniruled case does not rely on the algebraic dimension of X, so it is more natural to prove Theorem 2.2 (and thus Theorem 1.1) for all uniruled threefolds at once. Note that even in the uniruled case, the Kodaira problem in dimension three was left wide open prior to this work.

Taking the above discussion into account, we will therefore focus on uniruled threefolds and non-uniruled threefolds of algebraic dimension $a \leq 1$ in this text, and prove Theorem 2.2 separately for these two disjoint cases.

2.2 Bimeromorphic description of compact Kähler threefolds

In this paragraph, we describe the bimeromorphic models that we will choose to prove Theorem 2.2. Let us start with non-uniruled threefolds of algebraic dimension $a \leq 1$.

Proposition 2.5. — *A non-uniruled compact Kähler threefold X of algebraic dimension $a(X) \leq 1$ is bimeromorphic to one of the following:*

i) $X' = \tilde{X}/G$ where G is a finite group and $\tilde{X}$ is either a simple 3-torus or the product $S \times B$ of a K-trivial surface S of algebraic dimension 0 with a smooth projective curve B. In the latter case, the G-action on $\tilde{X} = S \times B$ is diagonal.

ii) The total space of a fibration $f : X' \to B$ over a smooth projective curve whose general fiber F is a 2-torus. When F is non-algebraic, we may assume that for some finite Galois cover $\tilde{B} \to B$ (with Galois group G), there exists a G-equivariant smooth isotrivial fibration $\tilde{f} : \tilde{X} \to \tilde{B}$ such that f is the quotient of $\tilde{f}$ by G.

In either case, we can choose X' to be normal.

Here, a K-trivial surface is a (smooth) compact Kähler surface S such that $\omega_S \simeq \mathcal{O}_S$. A compact complex manifold X is called *simple* if X is not covered by proper subvarieties of positive dimension. A complex torus X is simple if and only if X does not contain any proper subvariety of positive dimension, because the translations of a subvariety of X always cover X. Proposition 2.5, which we will prove in 4.1, is essentially based on Campana-Peternell's improvement of Fujiki's classification result [8, Corollary 7.6]. The only non-trivial property that we need to prove is that given an isotrivial fibration $f : X \to B$ over a curve whose general fiber is a non-algebraic 2-torus, up to some finite base change f is bimeromorphic to a smooth fibration (Lemma 4.2).

As for non-algebraic uniruled threefolds, standard arguments show that they are bimeromorphic to $\mathbf{P}^1$-fibrations $X \to S$ over a non-algebraic surface S. Due to Sarkisov [33], these $\mathbf{P}^1$-fibrations are further bimeromorphic to standard conic bundles (see Proposition 4.5). Sarkisov's result was originally stated for algebraic $\mathbf{P}^1$-fibrations but the same argument carries over to the non-algebraic case (at least when the base is a surface); we will review the proof in 4.2 and indicate the minor modifications. As the base S of the standard conic bundle $X \to S$ bimeromorphic to the original uniruled threefold is a non-algebraic surface, its algebraic dimension $a(S)$ is either 0 or 1. We will show that if the base of a standard conic bundle $X \to S$ is a surface with $a(S) = 0$, then $X \to S$ is a *smooth* $\mathbf{P}^1$-fibration (Proposition 4.7). As surfaces of algebraic dimension 1 are elliptic surfaces, we obtain the following bimeromorphic description of non-algebraic uniruled compact Kähler threefolds.

Proposition 2.6. — *A non-algebraic uniruled compact Kähler threefold is bimeromorphic to one of the following:*

i) A $\mathbf{P}^1$-bundle $X \to S$ over a smooth compact Kähler surface S with $a(S) = 0$.
ii) A $\mathbf{P}^1$-fibration $X \to S$ over a non-algebraic compact Kähler elliptic surface with X smooth.

Remark 2.7. — Contrary to what one might think, the Minimal Model Program (MMP) for compact Kähler threefold [19] only plays a minor role in the proof of the above results (and also the proof of Theorem 1.1). In fact, the MMP is only used in the proof of Theorem 1.1 to ruled out simple non-Kummer varieties from Fujiki's classification [8, Corollary 7.6] while we prove Proposition 2.5.

The rest of Section 2 is devoted to an overview of the proof of Theorem 2.2 for each of the bimeromorphic models listed in Proposition 2.5 and 2.6.

2.3 Algebraic approximations of non-uniruled compact Kähler threefolds of algebraic dimension $a \le 1$

We begin with the case of non-uniruled threefolds of algebraic dimension $a \le 1$. For the sake of reference, the following is a reformulation of Theorem 2.2 in this case taking Proposition 2.5 into account.

Theorem 2.8. — *A variety X' as in Proposition 2.5 has a C-locally trivial algebraic approximation for every subvariety $C \subset X'$ with* $\dim C \le 1$.

We will prove Theorem 2.8 in Section 5. If we are in the first case of Proposition 2.5, then Theorem 2.8 is easy and will be treated in 5.1 and 5.2. For the second case, if the fiber of $f : X' \to B$ is an abelian surface, then we will see that Theorem 2.8 is a consequence of the following theorem we proved in [26].

***Theorem 2.9* ([26, Corollary 1.3]).** — *Let $f : X \to B$ be a fibration where X is a compact Kähler manifold, B a smooth projective curve, and a general fiber an abelian variety. Then f has an algebraic approximation*

$$\Pi : \mathscr{X} \to B \times V \to V$$

which is locally trivial over B.

In [26], the deformation Π was constructed explicitly and in the case where f has local sections over every point of B, Π is what we called the *tautological family* associated to f. We will recall the construction of the tautological family in 3.6. (See also 3.5 for the construction of the tautological family associated to a smooth torus fibration).

Now assume that a general fiber of f is non-algebraic, then f is the G-quotient of a G-equivariant smooth isotrivial fibration $\tilde{f}$ in 2-tori. Let S be a fiber of $\tilde{f}$ and $\mathscr{S} \to \Delta$ an algebraic approximation of S. To show that f has an algebraic approximation, we will construct a deformation $\left\{\tilde{f}_{v,t} : \tilde{\mathscr{X}}_{v,t} \to \tilde{B}\right\}_{(v,t)\in V\times\Delta}$ of $\tilde{f}$ such that

- each $\tilde{f}_{t,v}$ is a G-equivariant smooth isotrivial torus fibration with fibers isomorphic to $\mathscr{S}_t$;
- for every $t \in \Delta$, $V \times \{t\}$ contains a dense subset parameterizing fibrations with a multi-section.

If t is fixed, each family $\left\{\tilde{f}_{v,t} : \tilde{\mathscr{X}}_{v,t} \to \tilde{B}\right\}_{v\in V}$ is in fact a tautological family associated to some smooth torus fibration (see 3.5 for the definition and construction). The way we construct $\left\{\tilde{f}_{v,t} : \tilde{\mathscr{X}}_{v,t} \to \tilde{B}\right\}_{(v,t)\in V\times\Delta}$ will be to construct the tautological families *in family*. Once we have such a family, since there exists a dense subset $\Delta_{\mathrm{alg}} \subset \Delta$ such that $\mathscr{S}_t$ is algebraic whenever $t \in \Delta_{\mathrm{alg}}$, the above properties imply that $\left\{\tilde{f}_{v,t} : \tilde{\mathscr{X}}_{v,t} \to \tilde{B}\right\}_{(v,t)\in V\times\Delta}$ is an algebraic approximation of $\tilde{f}$ by Campana's criterion (Corollary 3.10). Therefore the quotient of this family by G will be an algebraic approximation of f. This part of the proof will be carried out in 5.3.

2.4 Algebraic approximations of uniruled threefolds

Now we outline the proof of Theorem 2.2 for uniruled compact Kähler threefolds. We will prove the following result, which implies Theorem 2.2 for uniruled threefolds by Proposition 2.6.

Theorem 2.10. — *Let $f : X \to S$ be one of the $\mathbf{P}^1$-fibrations in Proposition 2.6. There exists a deformation*

$$\Pi : \mathscr{X} \to \mathscr{S} \to \Delta$$

of f such that Π is an $f^{-1}(C)$-locally trivial algebraic approximation of X for every subvariety $C \subsetneq S$.

According to whether we are in the first or the second case of Proposition 2.6, we will prove Theorem 2.10 in Section 6 and 7, which correspond to Corollary 6.6 and Proposition 7.1 respectively. In the first case where $X \to S$ is a $\mathbf{P}^1$-bundle, the main idea consists of regarding X as the projectivization of a twisted vector bundle E of rank 2 and proving Theorem 2.10 by constructing algebraic approximations for the pair (S, E). The deformation theory of twisted vector bundles parallels well that of the untwisted ones, so from the outset, one could argue as if E is a vector bundle in the usual sense.

Finally in the case where $f : X \to S$ is a $\mathbf{P}^1$-fibration over an elliptic surface $p : S \to B$, recall that p has an algebraic approximation $\Pi : \mathscr{S} \to B \times V \to V$ by Theorem 2.9. We will show that the algebraic approximation Π that we constructed to prove Theorem 2.9 can be lifted to an algebraic approximation of $f : X \to B$ which is locally trivial over B. This will imply Theorem 2.10 for the second case.

2.5 Organization of the text and remark on the dependence on [26, 27]

We will first introduce basic terminologies including various definitions of deformations of complex spaces and maps, then recall or prove some general results in Section 3. Section 4 is devoted to a bimeromorphic description of compact Kähler threefolds of algebraic dimension $a \leq 1$ and non-algebraic uniruled compact Kähler threefolds, where Proposition 2.5 and 2.6 will be proven. After that, we will study the existence of algebraic approximations for these threefolds and prove Theorem 2.2. More precisely, we will prove Theorem 2.8 for compact Kähler threefolds of algebraic dimension $a \leq 1$ in Section 5, then Theorem 2.10 for uniruled threefolds in Section 6 and 7. Assembling these results, we will finish the proof of Theorem 1.1 in Section 8.

The proof of Theorem 1.1 depends on the main results of two other articles [26, 27] of the author's (Theorem 2.9 and 2.3). As we can see from the outline of the proof, the arguments proving Theorem 1.1 vary between different threefolds and for certain threefolds, the existence of algebraic approximations turns out to be a more general phenomenon. This is the case for fibrations in abelian surfaces over a curve and threefolds of algebraic dimension 2, and the corresponding statements subsequently evolve into Theorem 2.9 and 2.3 respectively. We decide to separate these two results from this article and prove them in [26] and [27], so as to emphasize that each of them provides a new class of compact Kähler manifolds admitting algebraic approximations with a unifying proof. We regard this article as the one solving the Kodaira problem in dimension 3.

3 Preliminaries and general results

3.1 Basic notions and terminologies

A deformation of a complex space X is a surjective and flat morphism $\Pi : \mathscr{X} \to \Delta$ containing X as a fiber. Let $f : X \to B$ be a holomorphic map. A *deformation of* f is a composition $\Pi : \mathscr{X} \xrightarrow{q} \mathscr{B} \xrightarrow{\pi} \Delta$ where Π and π are deformations of X and B respectively such that $q_{|\mathscr{X}_o} : \mathscr{X}_o \to \mathscr{B}_o$ equals f for some $o \in \Delta$. We say that a deformation of f *preserves* B (or *fixes* B) if in the above definition, the map π is isomorphic to the second projection $B \times \Delta \to \Delta$. Such a deformation will be denoted by

$$\Pi : \mathscr{X} \xrightarrow{q} B \times \Delta \to \Delta.$$

In this text, we are mainly interested in the case where $f : X \to B$ is a *fibration*, namely, a proper holomorphic surjective map with connected fibers. The fiber $f^{-1}(b)$ of f over $b \in B$ will often be denoted by X_b.

Let $\Pi : \mathscr{X} \to \Delta$ be a deformation of a complex variety X and $Y \subset X$ a subvariety of X. We say that Π *preserves* Y if there exists $\mathscr{Y} \subset \mathscr{X}$ such that $\mathscr{Y} \cap X = Y$ and $\mathscr{Y}$ is isomorphic to $Y \times \Delta$ over Δ. Similarly, let $\Pi : \mathscr{X} \xrightarrow{q} B \times \Delta \to \Delta$ be a deformation of $f : X \to B$ fixing the base and Z a subvariety of B. We say that Π *preserves* $Y := f^{-1}(Z) \to Z$ if $q^{-1}(Z \times \Delta)$ is isomorphic to $Y \times \Delta$ over $B \times \Delta$.

Let G be a group and X a complex space endowed with a G-action. We say that a deformation $\Pi : \mathscr{X} \to \Delta$ of X *preserves the* G*-action* (or Π is a G*-equivariant deformation of* X) if there exists a G-action on $\mathscr{X}$ extending the given G-action on X such that $\mathscr{X} \to \Delta$ is G-invariant. Similarly, let $f : X \to B$ be a G-equivariant map. We say that a deformation $\Pi : \mathscr{X} \xrightarrow{q} \mathscr{B} \to \Delta$ of f *preserves the* G*-action* if there exist G-actions on $\mathscr{X}$ and on $\mathscr{B}$ extending the G-action on $f : X \to B$ such that q is G-equivariant and both $\mathscr{X} \to \Delta$ and $\mathscr{B} \to \Delta$ are G-invariant.

***Definition 3.1* (Locally trivial deformations).** —

i) A family of complex varieties $\Pi : \mathscr{X} \to \Delta$ is called *locally trivial* if there exists an open cover $\{U_i\}$ of $\mathscr{X}$ such that the restriction of Π to each U_i is a trivial deformation.
ii) Let X be a complex space and $C \subset X$ a subvariety of X. A deformation $\mathscr{X} \to \Delta$ of X is called *C-locally trivial* if there exists an open subset $\mathscr{U} \subset \mathscr{X}$ (for the Euclidean topology) such that $U := \mathscr{U} \cap X$ is a neighborhood of C and $\mathscr{U} \simeq U \times \Delta$ over Δ.
iii) In ii), let G be a group acting on a X and assume that C is G-stable. A *G-equivariantly C-locally trivial* deformation of X is a C-locally trivial deformation $\mathscr{X} \to \Delta$ of X preserving the G-action with the additional property that $\mathscr{U} \subset \mathscr{X}$ is G-stable and $\mathscr{U} \simeq U \times \Delta$ is G-equivariant.
iv) A deformation $\Pi : \mathscr{X} \xrightarrow{q} B \times \Delta \to \Delta$ of $f : X \to B$ fixing B is said to be *locally trivial over B* if there exists an open cover $\{U_i\}$ of B such that $q^{-1}(U_i \times \Delta) \simeq f^{-1}(U_i) \times \Delta$ over Δ.
v) In iv), let G be a group and $f : X \to B$ a G-equivariant map. We say that Π is *G-equivariantly locally trivial over B* if Π preserves the G-action and the isomorphisms $q^{-1}(U_i \times \Delta) \simeq f^{-1}(U_i) \times \Delta$ above are G-equivariant for some G-invariant open cover $\{U_i\}$ of B.

The following properties of locally trivial deformations holds trivially.

Lemma 3.2. —

i) A deformation $\Pi : \mathscr{X} \to \Delta$ of X is locally trivial if and only if Π is x-locally trivial for every point $x \in X$.
ii) If $\Pi : \mathscr{X} \to \Delta$ is a smooth family, then it is locally trivial.
iii) If $\Pi : \mathscr{X} \xrightarrow{q} B \times \Delta \to \Delta$ is a deformation of $f : X \to B$ which is locally trivial over B, then $\Pi : \mathscr{X} \to \Delta$ is a locally trivial deformation of X.

Remark 3.3. — A locally trivial family is locally (over the base) topologically trivial by [36, Théorème 4.14].

Another obvious property about locally trivial deformations is that the quotient of a G-equivariantly locally trivial deformation is a locally trivial deformation.

Lemma 3.4. — *Let G be a finite group acting on a complex space X and let $C \subset X$ be a G-stable subvariety. If $\Pi : \mathscr{X} \to \Delta$ is a G-equivariantly C-locally trivial deformation of X, then the quotient $\mathscr{X}/G \to \Delta$ is a C/G-locally trivial deformation of X/G. Similarly, if $\Pi : \mathscr{X} \xrightarrow{q} B \times \Delta \to \Delta$ is a G-equivariantly locally trivial deformation over B of a G-equivariant fibration $f : X \to B$, then the quotient $\mathscr{X}/G \to (B/G) \times \Delta \to \Delta$ is a deformation of $X/G \to B/G$ which is locally trivial over B/G.*

Proof. — We will only prove the first statement since the second one has already been proven in [26, Lemma 2.2]. By assumption, there exists a G-stable open subset $\mathscr{U}$ of $\mathscr{X}$ containing C which is G-equivariantly isomorphic to $U \times \Delta$ over Δ where $U := \mathscr{U} \cap X$. So the open subset $\mathscr{U}/G$ of $\mathscr{X}/G$ is isomorphic to $(U/G) \times \Delta$ over Δ. As C is a G-stable subset of U, the quotient C/G is contained in $\mathscr{U}/G$, which prove that $\mathscr{X}/G \to \Delta$ is a C/G-locally trivial deformation of X/G. □

Now we come to the notion of algebraic approximation. Recall that a compact complex variety X is called *Moishezon* (or *algebraic*) if its algebraic dimension $a(X)$ equals $\dim X$.

***Definition 3.5* (Algebraic approximation).** — Let X be a compact complex variety. An algebraic approximation of X is a deformation $\Pi : \mathscr{X} \to \Delta$ of X such that up to shrinking Δ, the subset of points in Δ parameterizing Moishezon varieties is dense for the Euclidean topology.

Since small deformations of Kähler manifolds remain Kähler, by Moishezon's criterion this definition of algebraic approximation coincides with the one introduced at the beginning of the text for compact Kähler manifolds.

Remark 3.6. — Our definition of algebraic approximations is called "strong algebraic approximations" in [17, Definition 2.10]. We could have defined a weaker version of algebraic approximation of X as a family of complex varieties $\pi : \mathscr{X} \to B$ such that there exists a sequence (t_n) in B parameterizing Moishezon fibers which converge to X. To the author's knowledge, at present there is no known example of compact Kähler manifolds X admitting a weak algebraic approximation and without any algebraic approximation.

3.2 G-equivariant locally trivial deformations

The following lemma shows that if a G-equivariant deformation of a complex manifold is C-locally trivial for some G-stable subvariety C, then it is G-equivariantly C-locally trivial.

Lemma 3.7. — *Let X be a complex manifold and G a finite group acting on X. Let C be a G-stable subvariety of X and let $\Pi : \mathscr{X} \to \Delta$ be a deformation of X which is G-equivariant and C-locally trivial. Then up to shrinking Δ, $\Pi : \mathscr{X} \to \Delta$ is G-equivariantly C-locally trivial.*

The proof of Lemma 3.7 is inspired by the proof of [17, Proposition 6.2]. Before proving Lemma 3.7, let us first prove a technical lemma.

Lemma 3.8. — *Let G be a finite group acting on a complex manifold X and let $\Pi : \mathscr{X} \to \Delta$ be a G-equivariant deformation of X. Let $\mathscr{V} \subset \mathscr{X}$ be an open subset such that there exists an isomorphism $\mathscr{V} \simeq V \times \Delta$ over Δ where $V := \mathscr{V} \cap X$. Let*

$$\mathscr{V}^G := \bigcap_{g \in G} g(\mathscr{V}).$$

Then for every G-stable relatively compact subset $U \subset V^G := \mathscr{V}^G \cap X$, up to shrinking Δ there exists a G-stable open subset $\mathscr{U}$ of $\mathscr{V}^G$ such that $\mathscr{U} \cap X = U$ and that $\mathscr{U}$ is G-equivariantly isomorphic to $U \times \Delta$ over Δ.

Proof. — We may assume that $V^G \neq \emptyset$. Since $\mathscr{V}^G$ is open by finiteness of G, after shrinking Δ we can assume that the restriction of Π to $\mathscr{V}^G$ is surjective. For simplicity, we will only prove Lemma 3.8 for $\dim \Delta = 1$; the same argument can be adapted easily to arbitrary dimension.

Up to shrinking Δ, we assume that Δ is isomorphic to the open unit disc $B(0,1) \subset \mathbf{C}$ such that 0 parameterizes the central fiber X. Fix a generator $\frac{\partial}{\partial t}$ of the space of constant vector fields $\Gamma(\Delta, T_\Delta)_{\mathrm{const}} \simeq \mathbf{C}$ on Δ. For $z \in \mathbf{C}$, let $z\frac{\partial}{\partial t} \in \Gamma(\Delta, T_\Delta)_{\mathrm{const}}$ denote the corresponding vector field.

By identifying $\mathscr{V}^G$ with a subset of $V \times \Delta$ through the isomorphism $\mathscr{V} \simeq V \times \Delta$, we can define the homomorphism of Lie algebras

$$\begin{aligned} \xi : \mathbf{C} &\to \Gamma(\mathscr{V}^G, T_{\mathscr{V}^G}) \\ z &\mapsto \sum_{g \in G} g^* \left(\chi(z)_{|\mathscr{V}^G} \right), \end{aligned} \tag{3.1}$$

where $\chi(z)$ is the vector field on $V \times \Delta$ which projects to $z\frac{\partial}{\partial t}$ in Δ and to 0 in V. By [21, Satz 3] (see also [17, Theorem 4.3]), there exists a local group action

$$\Phi : \Theta \to \mathscr{V}^G$$

of $\mathbf{C}$ on $\mathscr{V}^G$ inducing ξ, where $\Theta \subset \mathbf{C} \times \mathscr{V}^G$ is a neighborhood of $\{0\} \times \mathscr{V}^G$. We recall that the meaning of a local group action is the following:

i) For all $x \in \mathscr{V}^G$, the subset $\Theta \cap (\mathbf{C} \times \{x\})$ is connected.
ii) $\Phi(0, \bullet)$ is the identity map on $\mathscr{V}^G$.
iii) $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ whenever it is well-defined.
iv) The morphism of Lie algebras $\mathbf{C} \to \Gamma(\mathscr{V}^G, T_{\mathscr{V}^G})$ induced by Φ coincides with ξ.

Since the vector field $\xi(z)$ is G-invariant for all $z \in \mathbf{C}$ by construction, the map Φ is also G-equivariant (where G acts trivially on $\mathbf{C}$). Also since G acts on $\mathscr{V}^G \to \Delta$ in a fiber-preserving way, the projection of $\xi(z)$ in $\Gamma(\mathscr{V}^G, \Pi^* T_\Delta)$ is a constant vector field equal to $|G| \cdot z\frac{\partial}{\partial t}$. Hence if Φ_Δ denotes the local group action of $\mathbf{C}$ on Δ defined by

$$\begin{aligned} \Phi_\Delta : (\mathrm{Id}_{\mathbf{C}} \times \Pi)(\Theta) &\to \Delta \\ (x, b) &\mapsto b + |G| \cdot x \end{aligned}$$

then we have the following commutative diagram.

$$\begin{array}{ccc} \Theta & \xrightarrow{\Phi} & \mathscr{V}^G \\ \downarrow & & \downarrow \Pi \\ (\mathrm{Id}_{\mathbf{C}} \times \Pi)(\Theta) & \xrightarrow{\Phi_\Delta} & \Delta \end{array} \tag{3.2}$$

By the relative compactness of U inside V^G, there exists $\varepsilon > 0$ such that

$$\mathfrak{U} := B(0, \varepsilon) \times U \subset \Theta.$$

The restriction of Φ to $\mathfrak{U}$ is isomorphic onto its image. We verify easily with the help of (3.2) and the properties ii) and iii) that the inverse of $\Phi : \mathfrak{U} \to \Phi(\mathfrak{U})$ is

$$\begin{aligned} \Psi : \Phi(\mathfrak{U}) &\to \mathfrak{U} \\ v &\mapsto \left(\frac{\Pi(v)}{|G|}, \Phi\left(-\frac{\Pi(v)}{|G|}, v \right) \right). \end{aligned}$$

Let

$$\mathscr{U} := \Phi\left(B\left(0, \frac{\varepsilon}{|G|}\right) \times U\right) \subset \mathscr{V}^G.$$

We have $U := \mathscr{U} \cap X$ by ii) and up to replacing Δ by $B(0, \varepsilon)$, we have thus by construction an isomorphism

$$\Delta \times U \xrightarrow{\sim} \mathscr{U}$$
$$(t, x) \mapsto \Phi\left(\frac{t}{|G|}, x\right),$$

over Δ, which is moreover G-equivariant since U is G-stable and Φ is G-equivariant. □

Proof of Lemma 3.7. — By the C-local triviality of the deformation $\Pi : \mathscr{X} \to \Delta$, there exists an open subset $\mathscr{V} \subset \mathscr{X}$ isomorphic to $V \times \Delta$ over Δ such that $V := \mathscr{V} \cap X$ contains C. Since C is G-stable and since $\mathscr{V}^G := \bigcap_{g \in G} g(\mathscr{V})$, being a finite intersection, is an open subset of $\mathscr{X}$, it follows that $V^G := \mathscr{V}^G \cap X$ is a G-stable neighborhood of C. Let $U \subset V^G$ be a G-invariant neighborhood of C which is relatively compact in V^G. Applying Lemma 3.8 to $\mathscr{V}$ and to U, we deduce that up to shrinking Δ, there exists a G-stable open subset $\mathscr{U} \subset \mathscr{V}^G$ containing C which is G-equivariantly isomorphic to $U \times \Delta$ over Δ. □

3.3 Campana's criterion

Let X be a complex variety. We say that X is *algebraically connected* if a general pair of points $x, y \in X$ is contained in a compact connected (but not necessarily irreducible) curve of X. By definition, a compact complex variety X is in the Fujiki class $\mathscr{C}$ if it is dominated by a compact Kähler manifold. We have the following criterion due to Campana for such a variety to be Moishezon in terms of algebraic connectedness.

***Theorem 3.9* (Campana [9, Corollaire on p.212]).** — *Let X be a compact complex variety in the Fujiki class $\mathscr{C}$. Then X is Moishezon if and only if X is algebraically connected.*

Together with Moishezon's criterion, Theorem 3.9 implies that a compact complex manifold X is projective if and only if X is Kähler and algebraically connected.

Since we will mainly deal with fibrations $f : X \to B$ with either $\dim B = 1$ or $\dim F = 1$ where F denotes a general fiber of f, we states some variants of Campana's criterion in these particular situations.

***Corollary 3.10* (Special case of Campana's criterion).** — *Let $f : X \to B$ be a fibration over a Moishezon variety (*e.g.* a projective curve). Assume that X is in the Fujiki class $\mathscr{C}$ and a general fiber of f is Moishezon, then X is Moishezon if and only if f has a multi-section.*

Corollary 3.10 implies the following.

Corollary 3.11. — *Let X be a non-algebraic compact complex variety in the Fujiki class $\mathscr{C}$ and $f : X \to B$ a fibration over a curve with Moishezon fibers. Let $\Pi : \mathscr{X} \to B \times \Delta \to \Delta$ be a deformation of f which is locally trivial over B. Then for every subvariety $C \subset X$ with $\dim C \leq 1$, the underlying family $\mathscr{X} \to \Delta$ is a C-locally trivial deformation of X.*

Proof. — Since X is non-algebraic and since the base and the fibers of f are algebraic, every subvariety $C \subset X$ with $\dim C \leq 1$ is contained in a finite number of fibers of f by Corollary 3.10. As Π is locally trivial over B, the underlying deformation $\mathscr{X} \to \Delta$ is necessarily C-locally trivial. □

***Corollary 3.12* (Special case of Campana's criterion).** — *Let $f : X \to B$ be a $\mathbf{P}^1$-fibration. Assume that X is in the Fujiki class $\mathscr{C}$ and B is projective. Then X is Moishezon.*

Proof. — For every curve $C \subset B$, since $X_C := f^{-1}(C)$ is a uniruled surface, X_C is algebraic. As B is algebraically connected, it follows that X is also algebraically connected. Therefore X is Moishezon by Theorem 3.9. □

3.4 General approaches proving the existence of algebraic approximations

In this paragraph we will prove Proposition 2.1, a statement that formulates a general approach to finding algebraic approximations. Let us first prove the following easy lemma.

Lemma 3.13. — *Let $\nu : \tilde{X} \to X$ be a map between complex varieties and assume that there exists a subvariety $Y \subset X$ such that ν maps $\tilde{X} \backslash \nu^{-1}(Y)$ isomorphically to $X \backslash Y$. Then every Y-locally trivial deformation $\pi : \mathscr{X} \to \Delta$ of X induces a deformation $\tilde{\mathscr{X}} \to \mathscr{X} \to \Delta$ of ν.*

Proof. — By assumption, we can find a subset $\mathscr{Y} \subset \mathscr{X}$ with $\mathscr{Y} \cap X = Y$ and a neighborhood $\mathscr{U} \subset \mathscr{X}$ of $\mathscr{Y}$ such that there exists an isomorphism of the pairs

$$(\mathscr{U}, \mathscr{Y}) \simeq (U \times \Delta, Y \times \Delta) \tag{3.3}$$

over Δ where $U := X \cap \mathscr{U}$. So we can write

$$\mathscr{X} \simeq ((\mathscr{X} \backslash \mathscr{Y}) \sqcup (U \times \Delta)) / \sim$$

where $\sim$ glues the two pieces of the union using (3.3). Since ν maps $\tilde{X} \backslash \nu^{-1}(Y)$ isomorphically to $X \backslash Y$, (3.3) also induces an isomorphism

$$\mathscr{U} \backslash \mathscr{Y} \simeq \nu^{-1}(U \backslash Y) \times \Delta \tag{3.4}$$

over Δ. Define

$$\tilde{\mathscr{X}} := \Big((\mathscr{X} \backslash \mathscr{Y}) \sqcup (\nu^{-1}(U) \times \Delta) \Big) \Big/ \sim$$

where $\sim$ glues the two pieces of the union using (3.4). The map $F : \tilde{\mathscr{X}} \to \mathscr{X}$ over Δ obtained by gluing $\mathrm{Id} : \mathscr{X} \backslash \mathscr{Y} \to \mathscr{X} \backslash \mathscr{Y}$ and $\nu \times \mathrm{Id}_\Delta : \nu^{-1}(U) \times \Delta \to U \times \Delta$ is a deformation of ν. □

Proof of Proposition 2.1. — Let $\tau : X' \dashrightarrow X$ be a bimeromorphic map and

$$X' \xleftarrow{\ \nu\ } \tilde{X} \xrightarrow{\ \eta\ } X \tag{3.5}$$

a resolution of τ with $\tilde{X}$ smooth. If $E \subset \tilde{X}$ denotes the exceptional locus of ν, then the normality of X' implies that $\dim \nu(E) \leq \dim X' - 2$. By assumption, X' has a $\nu(E)$-locally trivial algebraic approximation $\pi : \mathscr{X}' \to \Delta$. Since $\nu^{-1}_{|X' \backslash \nu(E)}$ is an isomorphism onto its image, by Lemma 3.13 there exists a deformation $\tilde{\mathscr{X}} \to \Delta$ of $\tilde{X}$ and a map $\tilde{\mathscr{X}} \to \mathscr{X}'$ over Δ whose restriction to the central fiber is $\nu : \tilde{X} \to X'$. As $\eta : \tilde{X} \to X$ is a bimeromorphic morphism of compact complex manifolds, up to shrinking Δ the deformation $\tilde{\mathscr{X}} \to \Delta$ of $\tilde{X}$ induces a deformation $\tilde{\mathscr{X}} \to \mathscr{X}$ of η over Δ [32, Theorem 2.1]. So we have a correspondence $\mathscr{X}' \leftarrow \tilde{\mathscr{X}} \to \mathscr{X}$, which is a deformation of (3.5). Since $\mathscr{X}' \to \Delta$ is an algebraic approximation of X', $\mathscr{X} \to \Delta$ is an algebraic approximation of X. □

Finally, we have the following infinitesimal criterion for the existence of algebraic approximations, which is a consequence of the density criterion [37, Proposition 5.20] due to M. Green.

***Theorem 3.14* (Green [4, Proposition 1]).** — *Let X be a compact Kähler manifold and $\Pi : \mathscr{X} \to B$ a deformation of X over a (connected) complex manifold. If there exists $[\omega] \in H^1(X, \Omega_X^1)$ such that the composition*

$$\mu_{[\omega]} : T_{B,b} \xrightarrow{\text{KS}} H^1(X, T_X) \xrightarrow{\smile[\omega]} H^2(X, T_X \otimes \Omega_X) \longrightarrow H^2(X, \mathscr{O}_X)$$

of the Kodaira-Spencer map at $b := \Pi(X)$ with the contraction by $[\omega]$ is surjective, then $\Pi : \mathscr{X} \to B$ is an algebraic approximation of X.

3.5 Smooth torus fibrations and tautological families (following [11])

Let $f : X \to B$ be a smooth torus fibration and $J \to B$ the Jacobian fibration associated to f. The sheaf $\mathscr{J}_{\mathbf{H}/B}$ of germs of holomorphic sections of $J \to B$ lies in the exact sequence

$$0 \longrightarrow \mathbf{H} \longrightarrow \mathscr{E}_{\mathbf{H}/B} \xrightarrow{\exp} \mathscr{J}_{\mathbf{H}/B} \longrightarrow 0 \tag{3.6}$$

where $\mathbf{H} := R^{2g-1} f_* \mathbf{Z}$ (with g the relative dimension of f) and

$$\mathscr{E}_{\mathbf{H}/B} := (\mathbf{H} \otimes \mathscr{O}_B)/R^{g-1} f_* \Omega^g_{X/B} \simeq R^g f_* \Omega^{g-1}_{X/B}.$$

Every morphism $\phi : B' \to B$ induces a map $\phi^* : \mathscr{J}_{\mathbf{H}/B} \to \phi_* \mathscr{J}_{\phi^{-1}\mathbf{H}/B'}$ by pulling back sections.

Suppose, in addition, that f is G-equivariant for some finite group G, then G acts naturally on (3.6). The fibration f is a G-equivariant J-torsor and to each isomorphism class of G-equivariant J-torsors, we can associate in a biunivocal way an element $\eta_G(f) \in H^1_G(B, \mathscr{J}_{\mathbf{H}/B})$ [11, Section 2.4]. From the construction of $f \mapsto \eta_G(f)$, it is easy to see that for every $\phi : B' \to B$, the base change $f' : X \times_B B' \to B'$ is the smooth torus fibration representing $\phi^* \eta_G(f) \in H^1_G(B', \mathscr{J}_{\phi^{-1}\mathbf{H}/B'})$ where

$$\phi^* : H^1_G(B, \mathscr{J}_{\mathbf{H}/B}) \to H^1_G(B', \mathscr{J}_{\phi^{-1}\mathbf{H}/B'})$$

is the map induced by $\phi^* : \mathscr{J}_{\mathbf{H}/B} \to \phi_* \mathscr{J}_{\phi^{-1}\mathbf{H}/B'}$. Let

$$\exp : H^1(B, \mathscr{E}_{\mathbf{H}/B})^G \to H^1_G(B, \mathscr{J}_{\mathbf{H}/B})$$

denote the morphism induced by $\exp : \mathscr{E}_{\mathbf{H}/B} \to \mathscr{J}_{\mathbf{H}/B}$. For any linear map $\lambda : V \to H^1(B, \mathscr{E}_{\mathbf{H}/B})^G$ from a finite dimensional vector space V, there exists a family

$$\Pi : \mathscr{X} \xrightarrow{q} B \times V \to V \tag{3.7}$$

of G-equivariant J-torsors [11, Proposition 2.10] such that $t \in V$ parameterizes the G-equivariant J-torsor which corresponds to

$$\eta_G(f) + \exp(\lambda(t)) \in H^1_G(B, \mathscr{J}_{\mathbf{H}/B}).$$

The family Π is called the *G-equivariant tautological family associated to f and parameterized by V*, which is (G-equivariantly) locally trivial over B. Tautological families are compatible under base change in the sense that if $f' : X \times_B B' \to B'$ is the base change of $f : X \to B$ by $\phi : B' \to B$ as before, then the G-equivariant tautological family associated to f' and parameterized by V for the composition

$$V \xrightarrow{\lambda} H^1(B, \mathscr{E}_{\mathbf{H}/B})^G \xrightarrow{\phi^*} H^1(B', \mathscr{E}_{\phi^{-1}\mathbf{H}/B'})^G \tag{3.8}$$

is

$$\Pi' : \mathscr{X} \times_{B\times V} (B' \times V) \to B' \times V \to V.$$

When B is compact and $V = H^1(B, \mathscr{E}_{\mathbf{H}/B})^G$, the family (3.7) is called the *G-equivariant tautological family* associated to f.

The tautological families satisfy the following density result.

***Theorem 3.15* (Claudon [11]).** — *Let G be a finite group and $f : X \to B$ a G-equivariant smooth torus fibration. Assume that the total space X is a compact Kähler manifold. Then the base $V := H^1(B, \mathscr{E}_{\mathbf{H}/B})^G$ of the G-equivariant tautological family Π associated to f contains a dense subset parameterizing fibrations with an étale multi-section. In particular, if the fibers of f and B are projective, then Π is an algebraic approximation of f.*

3.6 Tautological families associated to fibrations in abelian varieties over a curve

Let $f : X \to B$ be a fibration over a smooth projective curve and assume that a general fiber of f is an abelian variety. We assume that f has local holomorphic sections at every point of B. As we mentioned before, an algebraic approximation of f as in Theorem 2.9 can be realized by the so-called tautological family associated to it. This paragraph is devoted to a brief review of the construction of the tautological family associated to f following [26, Section 3 and 4]. The same construction will also be used in Section 7 to construct algebraic approximations in Theorem 2.10 when the base of the $\mathbf{P}^1$-fibration is an elliptic surface.

Let us assume, more generally, that there exists a finite group G acting on both X and B such that f is G-equivariant. Let $j : B^\star \hookrightarrow B$ be the inclusion of a nonempty Zariski open subset such that $f^\star : X^\star \to B^\star$ is smooth where $X^\star := f^{-1}(B^\star)$ and $f^\star := f_{|X^\star}$. Let $J \to B^\star$ be the Jacobian fibration associated to $f^\star$. The sheaf $\mathscr{J}_{\mathbf{H}/B} = \mathscr{J}$ of germs of holomorphic sections of $J \to B^\star$ can be identified with the quotient $\mathscr{E}/\mathbf{H}$ where $\mathbf{H} = R^1 f^\star_* \mathbf{Z}$ and $\mathscr{E} := \mathscr{E}_{\mathbf{H}/B} := R^1 f^\star_* \mathscr{O}_{X^\star}$. The sheaf $\mathscr{E}_{\mathbf{H}/B}$ can be extended to a locally free sheaf $\bar{\mathscr{E}} := \bar{\mathscr{E}}_{\mathbf{H}/B}$ over B, called the *Deligne canonical extension* of $\mathscr{E}$. The induced G-action on $\mathscr{E}$ extends to $\bar{\mathscr{E}}$ as well.

Now we construct the *(G-equivariant) tautological family*

$$\Pi : \mathscr{X} \xrightarrow{q} B \times V \to V := H^1(B, \bar{\mathscr{E}}_{\mathbf{H}/B})^G$$

associated to f following the proof of [26, Proposition-Definition 4.10]. This is a family of fibrations $f_t : \mathscr{X}_t \to B$ in abelian varieties parameterized by $t \in V^G$. Let $\{U_i\}_{i\in I}$ be a G-invariant good open cover of B such that $f : X \to B$ has local sections over each U_i and that $U_{ij} \subset B^\star$ for every $i \neq j$. Since V is simply connected, the product $\{U_i \times V\}$ is a good open cover of $B \times V$. Let $\mathrm{pr}_1 : B \times V \to B$ be the first projection. Let

$$\xi \in H^1(B, \bar{\mathscr{E}}_{\mathbf{H}/B}) \otimes H^0(V, \mathscr{O}_V) \simeq H^1\left(B \times V, \mathrm{pr}_1^* \bar{\mathscr{E}}_{\mathbf{H}/B}\right)$$

be the element corresponding to the inclusion $V \hookrightarrow H^1(B, \bar{\mathscr{E}}_{\mathbf{H}/B})$ and let $\left\{\xi_{ij} : V \to \mathscr{E}_{\mathbf{H}/B}(U_{ij})\right\}$ be a G-invariant 1-cocycle representing ξ. We can assume that $\xi_{ij|B\times\{0\}} = 0$ (or equivalently $\xi_{ij}(0) = 0$). Let $X_i = f^{-1}(U_i)$ and $X_{ij} = f^{-1}(U_{ij})$. Then $q : \mathscr{X} \to B \times V$ is obtained by gluing the $X_i \times V \to U_i \times V$ using the translations

$$e_{ij} := \mathrm{tr}\left(\exp(U_{ij} \times V)(\xi_{ij})\right) : X_{ij} \times V \to X_{ij} \times V$$

where $\exp : \mathscr{E}_{\mathrm{pr}_1^{-1}\mathbf{H}/B\times V} \to \mathscr{J}_{\mathrm{pr}_1^{-1}\mathbf{H}/B\times V}$ is the natural quotient map. This is how the tautological family is constructed.

In [26], we proved the following for the tautological families.

Proposition 3.16 **([26, Theorem 1.2 and Proposition-Definition 4.15]).** — *The tautological family*

$$\Pi : \mathscr{X} \xrightarrow{q} B \times V^G \to V^G := H^1(B, \bar{\mathscr{E}}_{\mathbf{H}/B})^G$$

constructed above satisfies the following properties:

i) The central fiber of Π is f.

ii) The family Π preserves the G-action on f and is G-equivariantly locally trivial over B.

iii) The points parameterizing algebraic members of Π form a dense subset of V.

4 Bimeromorphic models of non-algebraic compact Kähler threefolds

The aim of this section is to prove Proposition 2.5 and 2.6, which classify compact Kähler threefolds of algebraic dimension $a \le 1$ and non-algebraic uniruled compact Kähler threefolds.

4.1 Non-uniruled compact Kähler threefolds with $a \le 1$

Let us first recall and prove some results concerning fibrations in non-algebraic K-trivial surfaces, starting with the following result of Campana.

Theorem 4.1 **(Campana [7]).** — *Let $f : X \to B$ be a fibration. Assume that X is a compact Kähler manifold and a general fiber F of f is a K-trivial surface* (i.e. *a surface F with $\omega_F \simeq \mathscr{O}_F$). If F is non-algebraic, then f is isotrivial.*

Given a fibration $f : X \to B$ as in Theorem 4.1, the next lemma shows that up to base changing f by a cyclic cover of B, f is bimeromorphic to a smooth fibration.

Lemma 4.2. — *Let $f : X \to B$ be a fibration from a compact Kähler manifold X to a smooth curve. Assume that a general fiber of f is a non-algebraic K-trivial surface, then there exists a Galois cover $r : \tilde{B} \to B$ and a bimeromorphic modification*

$$\begin{array}{ccc} X \times_B \tilde{B} & \dashleftarrow\dashrightarrow & \tilde{X} \\ & \searrow \quad \swarrow_{\tilde{f}} & \\ & \tilde{B} & \end{array} \tag{4.1}$$

of $X \times_B \tilde{B}$ along the singular fibers of $X \times_B \tilde{B} \to \tilde{B}$ such that $\tilde{f}$ is a smooth isotrivial fibration. Moreover, there exists a $G := \mathrm{Gal}(\tilde{B}/B)$-action on $\tilde{X}$ such that $\tilde{f}$ and the bimeromorphic map $X \times_B \tilde{B} \dashrightarrow \tilde{X}$ are G-equivariant.

We will use Kulikov models in the proof of Lemma 4.2, which we recall now. A degeneration of surfaces is a flat family $\pi : X \to \Delta$ of complex compact surfaces over a disc Δ which is smooth over $\Delta - \{o\}$. By the semi-stable reduction theorem, there exists a finite cyclic cover $r : \Delta \to \Delta$ ramified only at o such that the base change of $\pi : X \to \Delta$ by r is a semi-stable degeneration $\pi' : X' \to \Delta$ up to a bimeromorphic modification of the central fiber. If the canonical line bundles of the smooth fibers of π are trivial, then by Kulikov-Persson-Pinkham's theorem [24, 31] we can further assume that $K_{X'}$ is trivial. Such a degeneration $\pi' : X' \to \Delta$ is also called a Kulikov model of π. The classification of the central fiber of π' under this assumption implies the following result.

***Theorem 4.3* (Persson, Kulikov [13, p.11]).** — *Let $\pi : X \to \Delta$ be a degeneration of K3 surfaces or complex 2-tori. The central fiber of a Kulikov model of π is either a smooth surface or a union of uniruled surfaces.*

Proof of Lemma 4.2. — Let $\Sigma \subset B$ be the finite subset of B parameterizing singular fibers of f and suppose that $\Sigma \neq \emptyset$. For each $p \in \Sigma$, there exists a finite cyclic cover $\tilde{\Delta}_p \to \Delta_p$ of a small disc $\Delta_p \subset B$ around p such that $X \times_B \tilde{\Delta}_p \to \tilde{\Delta}_p$ admits a bimeromorphic modification $X \times_B \tilde{\Delta}_p \dashrightarrow \tilde{X}_p$ along the central fiber to a Kulikov model $f_p : \tilde{X}_p \to \tilde{\Delta}_p$.

Suppose that f_p is not smooth. Then by Theorem 4.3, the central fiber F_p of f_p is a union of uniruled surfaces $\cup_i S_i$. Let F be a general fiber of f_p. Consider the map $\imath : F \hookrightarrow \tilde{X}_p \to X$ which induces by pullback

$$\imath^* : H^2(X, \mathbf{C}) \to H^2(\tilde{X}_p, \mathbf{C}) \to H^2(F, \mathbf{C}). \tag{4.2}$$

Since f_p is proper, by [20, Theorem III.6.2] we have $H^2(\tilde{X}_p, \mathbf{C}) \simeq H^2(F_p, \mathbf{C})$, and morphism (4.2) with $H^2(\tilde{X}_p, \mathbf{C})$ replaced with $H^2(F_p, \mathbf{C})$ is a composition of morphisms of mixed Hodge structures. As F is smooth, we have the factorization

$$\imath^* : H^2(X, \mathbf{C}) \to \mathrm{Gr}_0^W H^2(F_p, \mathbf{C}) \to H^2(F, \mathbf{C})$$

where $\mathrm{Gr}_0^W H^2(F_p, \mathbf{C}) = W_0 H^2(F_p, \mathbf{C}) / W_{-1} H^2(F_p, \mathbf{C})$ denotes the zeroth graded piece of the weight filtration on $H^2(F_p, \mathbf{C})$. Since $\mathrm{Gr}_0^W H^2(F_p, \mathbf{C})$ equals the image of $H^2(F_p, \mathbf{C})$ in $\oplus_i H^2(\tilde{S}_i, \mathbf{C})$ where $\tilde{S}_i$ is a desingularization of S_i and since $p_g(\tilde{S}_i) = 0$, we deduce that the underlying Hodge structure of $W_0 H^2(F_p, \mathbf{C})$, and hence of $\mathrm{Im}(\imath^*)$, has trivial $H^{2,0}$-part. The latter contradicts [7, Proposition 2.1] which implies that $\imath^* : H^0(X, \Omega_X^2) \to H^0(F, \Omega_F^2)$ is non-zero. Therefore f_p is smooth for all $p \in \Sigma$.

Now let r_p be the degree of $\tilde{\Delta}_p \to \Delta_p$ and $d := \mathrm{lcm}_{p \in \Sigma}(r_p)$. Up to adding more points to Σ, the fundamental group $\pi_1(B^\star)$ is a free group where $B^\star := B \backslash \Sigma$ and for each $p \in \Sigma$, the loop around p is non-trivial in $\pi_1(B^\star)$. By Riemann's existence theorem there exists a finite cover $r : \tilde{B} \to B$ of degree d whose restriction to a neighborhood of p is a cyclic cover ramified at p. Replacing the fibers of $X \times_B \tilde{B} \to \tilde{B}$ over $r^{-1}(\Sigma)$ by their Kulikov models, we obtain a smooth fibration $\tilde{f} : \tilde{X} \to \tilde{B}$ which is a bimeromorphic modification of $X \times_B \tilde{B} \to \tilde{B}$ along singular fibers. Up to a further base change, we may assume that $\tilde{B} \to B$ is Galois.

The Galois group $G = \mathrm{Gal}(\tilde{B}/B)$ acts naturally on $X \times_B \tilde{B}$, which induces a bimeromorphic G-action on $\tilde{X}$ *via* the bimeromorphic map $X \times_B \tilde{B} \dashrightarrow \tilde{X}$. The G-action is biholomorphic over $\tilde{B}^\star$, so to show that the G-action on $\tilde{X}$ is biholomorphic, it suffices to show that for each $p \in \Sigma \subset B$ and a simply connected neighborhood Δ of p, the G-action on $\tilde{f}^{-1}(\tilde{\Delta}^\star)$ extends to $\tilde{f}^{-1}(\tilde{\Delta})$ where $\tilde{\Delta} := r^{-1}(\Delta)$ and $\tilde{\Delta}^\star := \tilde{\Delta} \cap \tilde{B}^\star$. As

$\tilde{f}^{-1}(\tilde{\Delta}) \simeq F \times \tilde{\Delta}$, the G-action on $X \times_B \tilde{B}$ induces a meromorphic map

$$\begin{aligned} \Phi : G \times \tilde{\Delta} &\dashrightarrow \mathrm{Aut}^c(F) \\ (g,t) &\mapsto \left(x \mapsto \mathrm{pr}_1(g(x,t))\right), \end{aligned} \tag{4.3}$$

which is well-defined for $t \in \tilde{\Delta}^\star$ where $\mathrm{pr}_1 : F \times \tilde{\Delta} \to F$ is the first projection and $\mathrm{Aut}^c(F)$ is the connected component of $\mathrm{Aut}(F)$ containing $\Phi(g,t)$ for some $t \in \tilde{\Delta}^\star$. According to whether F is a K3 surface of a 2-torus, $\mathrm{Aut}^c(F)$ is either a point or F. So in both case Φ is in fact holomorphic. Therefore the G-action on $\tilde{X}$ is holomorphic, and by construction, the bimeromorphic map $X \times_B \tilde{B} \dashrightarrow \tilde{X}$ and $\tilde{f}$ are G-equivariant. □

Now we can prove Proposition 2.5.

Proof of Proposition 2.5. — Proposition 2.5 will follow as a consequence of Campana-Peternell's improvement of Fujiki's classification of algebraic reductions of compact Kähler threefolds [8, Corollary 7.6]. To show that a compact Kähler threefold X with $a(X) \leq 1$ is bimeromorphic to one of the variety listed in Proposition 2.5, according to [8, Corollary 7.6] we only need to exclude simple non-Kummer compact Kähler threefold and verify that given a fibration $f : X \to B$ whose general fiber is a non-algebraic 2-torus (so f is isotrivial by Theorem 4.1), there exists a finite Galois cover $\tilde{B} \to B$ (with Galois group G) such that the base change $X \times_B \tilde{B} \to \tilde{B}$ is G-equivariantly bimeromorphic to a G-equivariant smooth isotrivial fibration $\tilde{X} \to \tilde{B}$ in 2-tori. The former follows from [19, Theorem 6.2] and the later from Lemma 4.2.

Finally we show that the varieties X' listed in Proposition 2.5 can be chosen normal. If X' is the total space of a fibration $f : X' \to B$ whose general fiber is an abelian surface, then since f is generically smooth, the composition $\tilde{X}' \to X' \to B$ of f with a minimal desingularization of X' is still a fibration whose general fiber is an abelian surface. Thus up to replacing X' with $\tilde{X}'$, we may assume that X' is smooth. In the other cases X' is the quotient of a smooth variety by a finite group, so X' is normal. □

4.2 Uniruled threefolds and standard conic bundles

In this paragraph we will prove Proposition 2.6, which describes non-algebraic uniruled threefolds up to bimeromorphic transformations. We start with the following well-known result.

Lemma 4.4. — *A non-algebraic uniruled compact Kähler threefold X is bimeromorphic to a $\mathbf{P}^1$-fibration.*

Proof. — Let $X' \to S$ be a resolution of the MRC fibration of X where both X' and S are compact Kähler manifolds. We claim that $\dim S = 2$. Indeed, since X is uniruled, we have $\dim S \leq 2$. If $\dim S = 0$, then X is rationally connected, so X would already be algebraic. If $\dim S = 1$, then since a general fiber of $X' \to S$ does not have any global holomorphic 1-form nor 2-form, we have $H^0(X', \Omega^2_{X'}) = 0$. So X would also be algebraic in this case. Therefore $\dim S = 2$, which implies that $X' \to S$ is a $\mathbf{P}^1$-fibration. □

Next we fix some terminology. A *conic bundle* is a $\mathbf{P}^1$-fibration $f : X \to B$ such that $f = \pi_{|X}$ where $\pi : \mathbf{P}(\mathscr{E}) \to B$ is the projectivization of a locally free sheaf $\mathscr{E}$ of rank 3 and X the zero locus of a non-trivial section $\sigma \in H^0\left(\mathbf{P}(\mathscr{E}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(2) \otimes \pi^*\mathscr{L}\right)$ for some invertible sheaf $\mathscr{L}$ on B. The section σ defines a map $\sigma : \mathscr{E} \to \mathscr{E}^\vee \otimes \mathscr{L}$, and induces $\det \sigma : \det(\mathscr{E}) \to \det(\mathscr{E})^\vee \otimes \mathscr{L}^{\otimes 3}$. The divisor D defined by

$\det\sigma \in H^0(\det(\mathscr{E}^\vee)^{\otimes 2} \otimes \mathscr{L}^{\otimes 3})$ is called the *discriminant locus* of the conic bundle f. As a set, this is the locus where the quadratic form defined by the restriction of σ to the fibers of $S^2\mathscr{E}^\vee \otimes \mathscr{L}$ does not have maximal rank. A flat conic bundle $f : X \to B$ is called *standard* if the discriminant locus of f is a simple normal crossing divisor and X and B are compact complex manifolds with $\rho(X) = \rho(B) + 1$.

A flat $\mathbf{P}^1$-fibration $f : X \to B$ with X and B assumed to be smooth is an example of conic bundles: By flatness of f, which implies the vanishing of $R^1 f_*\mathscr{O}_X$ [3, Corollary III.11.2], the sheaf $\mathscr{E} := (f_*\omega_X^\vee)^\vee$ is locally free of rank 3. Since $\omega_X^\vee$ is relatively very ample, $\omega_X^\vee$ defines an embedding $X \hookrightarrow \mathbf{P}(\mathscr{E})$ and $X \cap \mathbf{P}(\mathscr{E}_b)$ is a curve of degree 2 in $\mathbf{P}(\mathscr{E}_b)$. So X is the zero locus of a section $\sigma \in H^0\left(\mathbf{P}(\mathscr{E}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(2) \otimes \pi^*\mathscr{L}\right)$ for some invertible sheaf $\mathscr{L}$ on B.

Given an algebraic $\mathbf{P}^1$-fibration $f : X \to B$, Sarkisov proved that it is always birational to a standard conic bundle [33, Proposition 1.13]. We shall extend Sarkisov's theorem to non-algebraic $\mathbf{P}^1$-fibrations under the assumption that $\dim B = 2$ then prove Proposition 2.6 at the end.

***Proposition 4.5* (*cf.* Miyanishi** [29, Theorem on p.89] **or Sarkisov** [33, Proposition 1.13]**)**

Let $f : X \to B$ be a $\mathbf{P}^1$-fibration over a compact complex surface B. There exists a bimeromorphic modification

$$\begin{array}{ccc} X & \overset{\sim}{\dashrightarrow} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \overset{\sim}{\dashrightarrow} & B' \end{array}$$

of f to a standard conic bundle $f' : X' \to B'$.

Remark 4.6. — When B is a projective surface, Proposition 4.5 was already proven in [29, Theorem on p.89] by Miyanishi and Zagorskih. A similar statement was proven by Sarkisov when B is any complete algebraic variety [33]. Sarkisov's result might continue to hold for any compact complex manifold B, so the assumption $\dim B = 2$ in Proposition 4.5 might be superfluous. This assumption is used in the proof to first find a conic bundle (over a compact base) which is bimeromorphic to the original one. Once we obtain a conic bundle bimeromorphic to $f : X \to B$, the rest of the argument follows *mutatis mutandis* from the proof of Sarkisov's theorem mentioned above.

Proof of Proposition 4.5. — Up to base-changing f with a desingularization of B and resolving the singularities, we can assume that both X and B are smooth. Let $\mathscr{E} := (f_*\omega_X^\vee)^{\vee\vee}$. Since B is a smooth surface and $\mathscr{E}$ is reflexive, $\mathscr{E}$ is in fact locally free. Let $B^\circ \subset B$ be a nonempty Zariski open over which the fibration f is flat, then $(f_*\omega_X^\vee)_{|B^\circ}$ is already locally free so $X^\circ := f^{-1}(B^\circ)$ is the zero locus of a section

$$\sigma^\circ \in H^0\left(\mathbf{P}(\mathscr{E}_{|B^\circ}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(2) \otimes \pi^*\mathscr{L}\right)$$

having a meromorphic extension to $\mathbf{P}(\mathscr{E})$, where $\pi : \mathbf{P}(\mathscr{E}) \to B$ is the projection and $\mathscr{L}$ is some invertible sheaf over B. Therefore up to replacing $\mathscr{L}$ by $\mathscr{L} \otimes \mathscr{O}(D)$ for some divisor D supported on $B \backslash B^\circ$, the closure $\overline{X^\circ}$ of X° in $\mathbf{P}(\mathscr{E})$ is the zero locus of some section

$$\sigma \in H^0\left(\mathbf{P}(\mathscr{E}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(2) \otimes \pi^*\mathscr{L}\right).$$

As $\overline{X^\circ}$ is bimeromorphic to X over B, up to replacing X by $\overline{X^\circ}$ we can assume that $f : X \to B$ is a conic bundle embedded into $\mathbf{P}(\mathscr{E})$.

Starting from the conic bundle $f : X \to B$ which is embedded in $\mathbf{P}(\mathscr{E})$, the rest of the proof is almost the same as the proof of [33, Proposition 1.13] or [29, Theorem on p.89]. We shall only provide an outline and refer to *loc. cit.* (1) for the details.

For each point $o \in B$, we can find a neighborhood $U \subset B$ of o and local coordinates

$$(x \in U; [X_0 : X_1 : X_2] \in \mathbf{P}(\mathscr{E}_x))$$

of $\mathbf{P}(\mathscr{E}_{|U})$ such that the restriction of σ to U viewed as a family of quadratic forms is of the form $a_0(x)X_0^2 + a_1(x)X_1^2 + a_2(x)X_2^2$. Let $Z_i \subset B$ denote the locus where σ is of rank i. Since X is irreducible, if $Z_0 \neq \emptyset$ then $\dim Z_0 = 0$. Up to replacing X with the strict transformation of X in $\mathbf{P}(\mathscr{E}')$ for some vector bundle $\mathscr{E}'$ over the blow-up $\nu : B' \to B$ of B along Z_0 such that $\mathscr{E}'_{|B'\backslash\nu^{-1}(Z_0)} \simeq \mathscr{E}_{|B\backslash Z_0}$, we can assume that $Z_0 = \emptyset$ (*cf.* [29, page 90]).

Let $C \subset B$ be the discriminant divisor of f. Up to base-changing $f : X \to B$ with a log-resolution of the pair (B, C), we can assume that C is a normal crossing divisor. By performing elementary transformations of $f : X \to B$ along the non-reduced irreducible components of C, we can further assume that C is reduced (*cf.* [33, Lemma 1.14] or [29, page 90 and 91]). Therefore locally around a point $o \in C\backslash\mathrm{Sing}(C)$ (resp. $o \in \mathrm{Sing}(C)$), there exist local coordinates (u, v) in which σ is of the form $X_0^2 + uX_1^2 + X_2^2$ (resp. $X_0^2 + uX_1^2 + vX_2^2$). Accordingly $Z_1 = \mathrm{Sing}(C)$ and $Z_2 = C\backslash\mathrm{Sing}(C)$, so we can conclude by [33, Corollary 1.11] that f is flat and X is smooth.

It remains to show that $f : X \to B$ can be contracted to a standard conic bundle. Suppose that $C' \subset C$ is an irreducible component of C such that $X_{C'} := f^{-1}(C')$ is reducible, then $C' \cap \mathrm{Sing}(C) = \emptyset$. Indeed, if $o \in C' \cap \mathrm{Sing}(C)$, then as we can see from the local expression of σ above, the monodromy action around $o \in C'$ on the double cover of $C'\backslash\mathrm{Sing}(C)$ parameterizing lines in the fibers of f exchanges the two lines in $f^{-1}(p)$ for every $p \in C'$, so the total space $X_{C'}$ would be irreducible. The divisor $X_{C'}$ has two irreducible components E_1 and E_2: both E_1 and E_2 are ruled surfaces over C', and $E_1 \cap E_2$ is a section of both $E_1 \to C'$ and $E_2 \to C'$. It follows that $F \cdot E_1 = -1$ where F is a fiber of $E_1 \to C'$. So one can blow down the divisor E_1 in X to a curve isomorphic to C' [14, Theorem 2] and obtain a new conic bundle which is smooth along the smooth curve $C' \subset B$ as $C' \cap \mathrm{Sing}(C) = \emptyset$. After contracting all such ruled surfaces in the same way, we will obtain a conic bundle $f : X \to B$ satisfying $\rho(X) = \rho(B) + 1$. □

Over a compact Kähler surface of algebraic dimension 0, a standard conic bundle is always smooth.

Proposition 4.7. — *If $f : X \to S$ is a standard conic bundle over a smooth compact Kähler surface of algebraic dimension $a(S) = 0$, then it is a $\mathbf{P}^1$-bundle (namely, a smooth $\mathbf{P}^1$-fibration).*

Proof. — Let $S_{\min}$ be the minimal model of S. As $a(S_{\min}) = 0$, there exists only finitely many curves in $S_{\min}$ and the union of curves in $S_{\min}$ has normal crossings and is a disjoint union of trees of (-2)-curves. Since S

1. N.B. The terminology used in [33] is different from ours. In [33], $\mathbf{P}^1$-fibrations are called *conic bundles* and conic bundles are called *embedded conic bundles*.

is a sequence of blow-ups of $S_{\min}$, the union of curves in S is also a normal crossing divisor and the dual graph of which is still a disjoint union of trees. So the connected components of the discriminant locus C of $f : X \to S$ are also trees. If C is irreducible, then $C \simeq \mathbf{P}^1$ and every fiber of f over C is a union of two $\mathbf{P}^1$ meeting transversally at one point. So $f^{-1}(C)$ would be irreducible, which contradicts [29, Lemma 4.5]. Therefore if $C \neq \emptyset$, then C is reducible. In this case, since C is a disjoint union of trees, it would contain an irreducible component C' meeting the closure of $C \backslash C'$ in C in less than one point, which is in contradiction with [34, Proposition 3.1]. Hence $C = \emptyset$. □

Finally we prove Proposition 2.6.

Proof of Proposition 2.6. — By Lemma 4.4, a non-algebraic uniruled compact Kähler threefold is bimeromorphic to a $\mathbf{P}^1$-fibration $X \to S$. Since X is non-algebraic, by Corollary 3.12 S is non-algebraic as well (so $a(S) = 0$ or 1). If $a(S) = 0$, then since we can further assume that $f : X \to S$ is a standard conic bundle by Proposition 4.5, we can conclude by Proposition 4.7 that X is bimeromorphic to a $\mathbf{P}^1$-bundle. If $a(S) = 1$, then S is an elliptic surface and up to replacing X with a minimal desingularization, X is smooth. □

5 Smooth isotrivial fibrations in K-trivial surfaces and 3-tori

In this section we will prove Theorem 2.8, which implies Theorem 2.2 for non-uniruled threefolds of algebraic dimension $a \leq 1$. Let us start with deformations of K-trivial surfaces.

5.1 Some precisions on the algebraic approximations of K-trivial surfaces

Let $\Pi : \mathscr{S} \to \Delta$ be a smooth deformation of a compact Kähler manifold S. We say that Π preserves the Néron-Severi group $\mathrm{NS}(S)$ if the parallel transport of every class in $\mathrm{NS}(S)$ under the Gauss-Manin connection remains of type $(1,1)$ along Δ.

Lemma 5.1. — *Let S be a K-trivial surface with $a(S) = 0$ and G a finite group acting on S. Then S has a G-equivariant algebraic approximation which preserves* $\mathrm{NS}(S)$ *and is C-locally trivial for every subvariety $C \subsetneq S$.*

Proof. — Since S is smooth, every small deformation of S is locally trivial (Lemma 3.2). So it suffices to prove the existence of a G-equivariant algebraic approximation of S which is C-locally trivial for every $C \subset S$ of pure dimension 1. As $a(S) = 0$, there are only finitely many curves in S, so we can assume that C is the (finite) union of the irreducible curves in S. Connected components of C are A-D-E curves.

Note that since $H^2(S,\mathbf{Z})^G$ is a sub-Hodge structure of $H^2(S,\mathbf{Z})$ of weight 2, if the G-action does not preserve the holomorphic symplectic form, then $H^2(S,\mathbf{Z})^G$ is concentrated in bi-degree $(1,1)$. In particular $H^2(S,\mathbf{Q})^G$ is dense in $H^{1,1}(S)^G \cap H^2(S,\mathbf{R})$. As the intersection of $H^{1,1}(S)^G$ with the Kähler cone $\mathscr{K}_S \subset H^2(S,\mathbf{C})$ is not 0, we can find a Kähler class $H^2(S,\mathbf{Z})^G$, which is in contradiction with the hypothesis that S is non-algebraic. So the G-action preserves the holomorphic symplectic forms of S, and the contraction $T_S \to \Omega_S^1$ with one of these forms yields an isomorphism

$$\sigma : H^1(S,T_S)^G \xrightarrow{\sim} H^1(S,\Omega_S^1)^G.$$

Since the universal deformation space $\mathscr{U}$ of S is smooth, its locus preserving the G-action can be identified with an open subset of $H^1(S,T_S)^G$ [17, Proposition 6.1]. By the local Torelli theorem, the period map $\mathscr{P}$ is a local isomorphism, so we can further identify $\mathscr{U}$ with an open subset of $H^1(S,\Omega_S)^G$ through $d\mathscr{P} = \sigma : H^1(S,T_S)^G \xrightarrow{\sim} H^1(S,\Omega_S)^G$. Under this identification, the universal deformation space of S preserving the G-action and $\mathrm{NS}(S)$ is identified with an open subset Δ of

$$V := H^1(S,\Omega_S^1)^G \cap \mathrm{NS}(S)^\perp$$

where the orthogonal complement is defined with respect to the cup product on $H^1(S,\Omega_S^1)$. Also, given a Kähler class $[\omega]$, the map $\mu_{[\omega]}$ defined in Theorem 3.14 for the family $\mathscr{S} \to \Delta$ has the following factorization

$$\mu_{[\omega]} : T_{\Delta,o} \subset H^1(S,T_S)^G \xrightarrow{\quad\sigma\quad} H^1(S,\Omega_S^1)^G \xrightarrow{\smile[\omega]} H^2(S,\Omega_S^2) \simeq H^2(S,\mathscr{O}_S). \tag{5.1}$$

with $\sigma(T_{\Delta,o}) = V$.

As $\mathrm{NS}(S)$ is negative with respect to the cup-product, by the Hodge index theorem there exists $v \in \mathrm{NS}(S)^\perp$ such that $v^2 > 0$. Up to replacing v with $\sum_{g\in G} g^*v$, we can assume that v is G-invariant, so $v \in V$. If $[\omega]$ is a Kähler class, then again by the Hodge index theorem we have $v \cdot [\omega] \neq 0$. Using (5.1), it follows that since $h^2(S,\mathscr{O}_S) = 1$, the map $\mu_{[\omega]}$ defined in Theorem 3.14 for the deformation $\mathscr{S} \to \Delta$ of S is surjective. Therefore by Theorem 3.14, $\mathscr{S} \to \Delta$ is an algebraic approximation of S.

It remains to show that $\mathscr{S} \to \Delta$ is a C-locally trivial deformation of S. Since $\mathscr{S} \to \Delta$ preserves $\mathrm{NS}(S)$, the curve class $[C_i] \in H^2(S,\mathbf{C})$ of each irreducible component C_i of C remains of type $(1,1)$ under the parallel transports by the Gauss-Manin connection along Δ. So $\mathscr{S} \to \Delta$ induces for each i, a deformation $(\mathscr{S},\mathscr{C}_i) \to \Delta$ of the pair (S,C_i). Let us decompose $C = \sqcup_{i=1}^m C_i'$ into its connected components and let $\mathscr{C}_i' \subset \mathscr{C} \to \Delta$ be the deformation of C_i'. Since C_i' is an A-D-E curve, up to shrinking Δ each fiber of $\mathscr{C}_i' \to \Delta$ is still an A-D-E curve with the same configuration. As the holomorphic structure of a neighborhood of an A-D-E curve depends only on the A-D-E configuration, it follows that $\mathscr{S} \to \Delta$ is a C-locally trivial deformation. □

Lemma 5.2. — *Let S be a 2-torus and G a finite group acting on S. There exists a G-equivariant algebraic approximation of S which is C-locally trivial for every subvariety $C \subsetneq S$.*

Proof. — We may assume that S is non-algebraic, so either $a(S) = 0$ or 1. If $a(S) = 0$, then Lemma 5.2 follows from Lemma 5.1. If $a(S) = 1$, then the algebraic reduction of S is a smooth elliptic fibration $S \to E$ over an elliptic curve E. So in this case, Lemma 5.1 follows from Theorem 3.15 and Corollary 3.11. □

For later use (in Section 6), let us prove a related result about deformations of compact Kähler surfaces of algebraic dimension 0.

Lemma 5.3. — *Let $\tilde{S}$ be a smooth compact Kähler surface with $a(\tilde{S}) = 0$ and let $\eta : \tilde{S} \to S$ be the map from $\tilde{S}$ to its minimal model. Let C (resp. $\tilde{C}$) be the union of all the curves in S (resp. $\tilde{S}$), which consists of only finitely many irreducible components. Then every C-locally trivial deformation $\Pi : \mathscr{S} \to \Delta$ of S can be lifted to a deformation*

$$\tilde{\Pi} : \tilde{\mathscr{S}} \xrightarrow{\tilde{\eta}} \mathscr{S} \to \Delta$$

of η satisfying the following property: For every subvariety $Y \subsetneq S$, there exist a neighborhood $U \subset S$ of Y and a trivial deformation $\mathscr{U} \subset \mathscr{S} \to \Delta$ of U such that $\tilde{\eta}^{-1}(\mathscr{U}) \subset \tilde{\mathscr{S}} \to \Delta$ is a trivial deformation of $\eta^{-1}(U)$.

In particular, $\tilde{\Pi}$ is a $\tilde{C}$-locally trivial deformation of $\tilde{S}$.

Proof. — Since $\Pi : \mathscr{S} \to \Delta$ is a smooth family, it is locally trivial. As $\eta(\tilde{C})$ is the union of C with some finite subset of S, by the local triviality and the C-local triviality of Π, there exists an open subset $\mathscr{U}_0 \subset \mathscr{S}$ such that $U_0 := \mathscr{U}_0 \cap S$ is a neighborhood of $\eta(\tilde{C})$ and $\mathscr{U}_0 \simeq U_0 \times \Delta$ over Δ. Let

$$\eta : \tilde{S} = S_n \to \cdots \to S_1 \xrightarrow{\eta_1} S_0 = S$$

be decomposed as a sequence of contractions of (-1)-curves. Let $p \in S$ be the blow-up center of $\eta_1 : S_1 \to S$. Since $p \in \eta(\tilde{C}) \subset U_0$, we can find a deformation $\mathscr{P} \subset \mathscr{S}$ of p over Δ such that

$$(\mathscr{P} \subset \mathscr{U}_0) \simeq (\{p\} \subset U_0) \times \Delta \ \text{ over } \ \Delta.$$

So if $\tilde{\eta}_1 : \mathscr{S}_1 \to \mathscr{S}$ is the blow-up of $\mathscr{S}$ along $\mathscr{P}$, then $\tilde{\eta}_1^{-1}(\mathscr{U}_0) \simeq \eta_1^{-1}(U_0) \times \Delta$ over Δ. From this, it is easy to see that for every subvariety $Y \subsetneq S$, there exist a neighborhood $U \subset S$ of Y and a trivial deformation $\mathscr{U} \subset \mathscr{S} \to \Delta$ of U such that $\tilde{\eta}_1^{-1}(\mathscr{U}) \simeq \eta_1^{-1}(U) \times \Delta$ over Δ. Iterating the same construction for each blow-up $S_{i+1} \to S_i$, we obtain a sequence of blow-ups

$$\tilde{\eta} : \tilde{\mathscr{S}} = \mathscr{S}_n \to \cdots \to \mathscr{S}_0 = \mathscr{S}$$

such that $\tilde{\mathscr{S}} \xrightarrow{\tilde{\eta}} \mathscr{S} \to \Delta$ satisfies the conclusion of Lemma 5.3. □

5.2 Simple 3-tori and product of a K-trivial surface with a curve

The results in this short paragraph will allow us to prove Theorem 2.8 when X' is in the first case of Proposition 2.5. We first deal with the case where X' is a finite quotient of the product of a K-trivial surface with a curve.

Lemma 5.4. — *Let G be a finite group acting on a K-trivial surface S and on a smooth projective curve B. Assume that $a(S) = 0$, then for every subvariety $C \subset S \times B$ such that $\dim C \leq 1$, the threefold $S \times B$ has a G-equivariant and C-locally trivial algebraic approximation.*

Proof. — Let $\mathrm{pr}_1 : S \times B \to S$ be the first projection and let $C' = \mathrm{pr}_1(C)$. By Lemma 5.1, there exists a G-equivariant algebraic approximation $\mathscr{S} \to \Delta$ of S which is C'-locally trivial. So the product $\Pi : \mathscr{S} \times B \to \Delta$ is a G-equivariant algebraic approximation of $S \times B$. Let $\mathscr{U} \subset \mathscr{S}$ be an open subset containing C' such that $\mathscr{U} \simeq U \times \Delta$ over Δ where $U := \mathscr{U} \cap S$. Then the open subset $\mathscr{U} \times B \subset \mathscr{S} \times B$ contains C and is isomorphic to $(U \times B) \times \Delta$ over Δ, which shows that Π is C-locally trivial. □

It is well-known that complex tori have algebraic approximations. The same proof works also in the G-equivariant setting.

Lemma 5.5. — *A complex torus X endowed with a finite group action G has an algebraic approximation preserving the G-action.*

Proof. — As X is a torus, it has no deformation obstructions. Consider the universal deformation $\mathscr{X} \to \Delta$. Let $[\omega] \in H^1(X, \Omega_X^1)$ be a G-invariant Kähler class. Since $\omega_X \simeq \mathscr{O}_X$, the map $\mu_{[\omega]}$ in Theorem 3.14 can be identified with

$$\smile[\omega] : H^1(X, \Omega_X^{n-1}) \to H^2(X, \Omega_X^n)$$

which is surjective by the hard Lefschetz theorem. We conclude by [17, Theorem 1.11] that X has an algebraic approximation preserving the G-action. □

5.3 Smooth isotrivial fibrations in 2-tori

Next we study algebraic approximations for smooth isotrivial fibrations in 2-tori. First we show that if such a fibration has a multi-section, then it becomes trivial after some finite base change.

Lemma 5.6. — *Let $f : X \to B$ be a smooth isotrivial fibration in complex tori over a smooth curve. Assume that X is Kähler and f has a multi-section, then there exists a finite cover $\tilde{B} \to B$ of B (with $\tilde{B}$ smooth) such that the base change $X \times_B \tilde{B} \to \tilde{B}$ is a trivial fibration.*

Proof. — As f has a multi-section, there exists a finite cover $\tilde{B} \to B$ of B such that $\tilde{B}$ is smooth and the base change $\tilde{f} : \tilde{X} = X \times_B \tilde{B} \to \tilde{B}$ has a section, namely $\tilde{f}$ is a Jacobian fibration. Since the restriction of a Kähler class on $\tilde{X}$ to a fiber F of $\tilde{f}$ is a Kähler class fixed by the induced $\pi_1(\tilde{B})$-action on $H^2(F, \mathbf{R})$, the map $\pi_1(\tilde{B}) \to \mathrm{Aut}(F)/\mathrm{Aut}_0(F)$ has finite image by [25, Proposition 2.2] where $\mathrm{Aut}_0(F)$ denotes the identity component of $\mathrm{Aut}(F)$. So after a further finite étale base change of $\tilde{f} : \tilde{X} \to \tilde{B}$, we can assume that the monodromy action of $\pi_1(\tilde{B})$ on $H^1(F, \mathbf{Z})$ is trivial. Therefore since $\tilde{f} : \tilde{X} \to \tilde{B}$ is a Jacobian fibration, $\tilde{f}$ is isomorphic to the projection $F \times \tilde{B} \to \tilde{B}$. □

The following lemma rephrases Theorem 2.8 for $X' = \tilde{X}/G$ where G is a finite group and $\tilde{X}$ a G-equivariant smooth isotrivial fibration in 2-tori.

Lemma 5.7. — *Let G be a finite group and $f : X \to B$ a G-equivariant smooth isotrivial fibration in non-algebraic 2-tori such that X is a compact Kähler manifold, B a smooth projective curve, and G acts faithfully on B. Assume that $a(X) \le 1$, then for every subvariety $C \subset X$ such that $\dim C \le 1$, the threefold X has a G-equivariant and C-locally trivial algebraic approximation.*

Proof. — First we assume that f has a multi-section. In this case, the proof of Lemma 5.7 is similar to that of Lemma 5.4. By Lemma 5.6, there exists a finite cover $\tilde{B} \to B$ such that the base change $\tilde{X} := X \times_B \tilde{B} \to \tilde{B}$ is isomorphic to the second projection $S \times \tilde{B} \to \tilde{B}$ where S is a fiber of f. Up to making a further base change, we can assume that $\tilde{B}$ is Galois over B/G.

Claim. — *The induced $\tilde{G} := \mathrm{Gal}\left(\tilde{B}/(B/G)\right)$-action on $S \times \tilde{B}$ is diagonal.*

Proof. — Since S is non-algebraic, we have $a(S) = 0$ or 1. If $a(S) = 1$, then S is a smooth isotrivial elliptic fibration $S \to E$ over an elliptic curve. So there is a surjective map $S \times B \to E \times B$ and thus $a(S \times B) \ge 2$. Since $a(X) = a(S \times B)$ [35, Theorem 3.8], this contradicts the assumption that $a(X) \le 1$. Therefore $a(S) = 0$, so the only subvariety of S of positive dimension is S. For every $g \in \tilde{G}$, consider the map $\Phi_g : \tilde{B} \to \mathrm{Aut}(S)$ defined by

$$\Phi_g(t)(s) := \mathrm{pr}_1(g(s,t)).$$

Since $\Phi_g(\tilde{B})$ is a subvariety of a connected component of $\mathrm{Aut}(S)$, which is isomorphic to S, and since $\dim \Phi_g(\tilde{B}) < \dim S$, the image $\Phi_g(\tilde{B})$ is a point. Therefore Φ_g is constant, which shows that the $\tilde{G}$-action on $S \times \tilde{B}$ is diagonal. □

Let $\tilde{C}$ be the pre-image of C in $\tilde{X}$, which is $\mathrm{Gal}(\tilde{B}/B)$-stable by assumption. By Lemma 3.7 and Lemma 3.4, it suffices to show that $\tilde{X}$ has a $\tilde{G}$-equivariant and $\tilde{C}$-locally trivial algebraic approximation. Let $p_1 : S \times \tilde{B} \to S$ be the first projection and $C' := p_1(\tilde{C})$. By Lemma 5.2, there exists a $\tilde{G}$-equivariant and C'-locally trivial algebraic approximation $\mathscr{S} \to \Delta$ of S. Repeating the same argument as in the proof of Lemma 5.4, we deduce that $\mathscr{S} \times \tilde{B} \to \Delta$ is a $\tilde{G}$-equivariant and $\tilde{C}$-locally trivial algebraic approximation of $\tilde{X}$.

Now assume that f does not have any multi-section. Since f is a G-equivariant smooth isotrivial fibration, the Jacobian fibration associated to f is a trivial fibration $S \times B \to B$ and the induced G-action on $S \times B$ is diagonal, which further induces a G-action on S. By Lemma 5.2, there exists a G-equivariant algebraic approximation $\pi : \mathscr{S} \to \Delta$ of S which is D-locally trivial for every subvariety $D \subsetneq S$. We can assume that Δ is simply connected.

By [11, Proposition 3.5], there exists a deformation

$$\Pi : \mathscr{X} \xrightarrow{q} B \times \Delta \to \Delta \tag{5.2}$$

of $f : X \to B$ (fixing the base) such that

- the G-action is preserved;
- each $t \in \Delta$ parameterizes a G-equivariant torus fibration $f_t : \mathscr{X}_t \to B$ whose associated Jacobian fibration is G-equivariantly isomorphic to $\mathscr{S}_t \times B \to B$.

In what follows, we will construct a deformation

$$\tilde{\Pi} : \tilde{\mathscr{X}} \xrightarrow{\tilde{q}} B \times \Delta \times V \to \Delta \times V$$

of $f : X \to B$ (fixing the base) over $\Delta \times V$ such that the restriction

$$\tilde{\Pi}_t : \tilde{\mathscr{X}}_t \xrightarrow{\tilde{q}} \tilde{B} \times \{t\} \times V \to \{t\} \times V$$

of $\tilde{\Pi}$ to $\tilde{\mathscr{X}}_t = \tilde{\Pi}^{-1}(\{t\} \times V)$ for each $t \in \Delta$ is a G-equivariant tautological family associated to $f_t : \mathscr{X}_t \to B$. We will then show that $\tilde{\Pi}$ is a desired algebraic approximation of $\tilde{f}$.

Let $p = (\mathrm{Id}_B, \pi) : B \times \mathscr{S} \to B \times \Delta$. Let $\mathbf{H} = R^3 p_* \mathbf{Z}$ and $\mathscr{E}_{\mathbf{H}/B\times\Delta} = R^2 p_* \Omega^1_{B\times\mathscr{S}/B\times\Delta}$. For every $t \in \Delta$, we have

$$\dim H^1\left(B \times \{t\}, \mathscr{E}_{\mathbf{H}/B\times\Delta}\right) = \dim H^1\left(B, H^2(\mathscr{S}_t, \Omega^1_{\mathscr{S}_t}) \otimes \mathscr{O}_B\right) = h^{0,1}(B) \cdot h^{1,2}(\mathscr{S}_t),$$

which is constant in t. So the higher pushforward $R^1 \mathrm{pr}_{1*} \mathscr{E}_{\mathbf{H}/B\times\Delta}$ of the locally free sheaf $\mathscr{E}_{\mathbf{H}/B\times\Delta}$ by the projection $\mathrm{pr}_1 : B \times \Delta \to \Delta$ is locally free. Up to shrinking Δ, we can assume that $R^1 \mathrm{pr}_{1*} \mathscr{E}_{\mathbf{H}/B\times\Delta}$ is free, so there exists a finite dimensional subspace

$$V \subset H^0(\Delta, R^1{\mathrm{pr}_1}_*\mathscr{E}_{\mathbf{H}/B\times\Delta})^G \simeq H^1(B\times\Delta, \mathscr{E}_{\mathbf{H}/B\times\Delta})^G$$

such that the restriction map

$$\rho_t : V \subset H^1(B\times\Delta, \mathscr{E}_{\mathbf{H}/B\times\Delta})^G \to H^1(B\times\{t\}, \mathscr{E}_{\mathbf{H}/B\times\Delta})^G \tag{5.3}$$

is surjective for every $t \in \Delta$. Let

$$\tilde{\Pi}' : \tilde{\mathscr{X}} \xrightarrow{\tilde{q}} (B\times\Delta)\times V \to V \tag{5.4}$$

be the G-equivariant tautological family associated to $q : \mathscr{X} \to B\times\Delta$ and parameterized by V (see 3.5). Let

$$\tilde{\Pi} : \tilde{\mathscr{X}} \xrightarrow{\tilde{q}} (B\times\Delta)\times V \to \Delta\times V$$

be the composition of $\tilde{q}$ with the projection $(B\times\Delta)\times V \to \Delta\times V$. By construction, the fibration $f : X \to B$ is a member of the family $\tilde{\Pi}$. Over each $t\in\Delta$, the restriction

$$\tilde{\Pi}_t : \tilde{\mathscr{X}_t} \to B\times V \to V$$

of $\tilde{\Pi}$ to $\tilde{\mathscr{X}_t} := \tilde{q}^{-1}((B\times\{t\})\times V)$ is the G-equivariant tautological family associated to $f_t : \mathscr{X}_t \to B$ and parameterized by V for the map

$$\rho_t : V \hookrightarrow H^1(B\times\Delta, \mathscr{E}_{\mathbf{H}/B\times\Delta})^G \to H^1\left(B\times\{t\}, \mathscr{E}_{\mathbf{H}/B\times\Delta}\right)^G = H^1(B, \mathscr{E}_{\mathbf{H}_t/B})^G$$

where $\mathscr{E}_{\mathbf{H}_t/B} = R^2{p_t}_*\Omega_{B\times\mathscr{S}_t/B}$ and $p_t : B\times\mathscr{S}_t \to B$ is the projection. Since ρ_t is surjective for every $t\in\Delta$, Theorem 3.15 implies that in the family $\tilde{\Pi}$, there exists a dense subset of $\{t\}\times V$ parameterizing fibrations in $\mathscr{S}_t$ over B having a multi-section. By Corollary 3.10, such a fibration is algebraic if $\mathscr{S}_t$ is algebraic. Therefore, as $\mathscr{S} \to \Delta$ is an algebraic approximation of S, $\tilde{\Pi}$ is a (G-equivariant) algebraic approximation of f.

Finally we prove that $\tilde{\Pi}$ is a C-locally trivial deformation of X. As f does not have any multi-section, the subvariety $C \subset X$ is contained in a finite union of fibers of f. We may assume that C is contained in one fiber $f^{-1}(t)$. Since (5.2) is a family of smooth isotrivial fibrations, there exists a neighborhood $U \subset B$ of t such that

$$q^{-1}(U\times\Delta) \simeq U\times\mathscr{S} \tag{5.5}$$

over $U\times\Delta$. Let $C' \subset S \subset U\times\mathscr{S}$ be the image of $C \subset q^{-1}(U\times\Delta)$ under (5.5). As (5.4) is a tautological family associated to $q : \mathscr{X} \to B\times\Delta$, it is locally trivial over $B\times\Delta$. Therefore up to shrinking U and Δ, we have

$$\tilde{q}^{-1}(U\times\Delta\times V) \simeq q^{-1}(U\times\Delta)\times V \simeq U\times\mathscr{S}\times V. \tag{5.6}$$

over $U\times\Delta\times V$. The deformation $\mathscr{S}\to\Delta$ of S is C'-locally trivial for $C'\subset S$, in other words, there exists an open subset $\mathscr{U}' \subset \mathscr{S}$ containing C' such that $\mathscr{U}' \simeq U'\times\Delta$ over Δ where $U' = \mathscr{U}'\cap S$. Let $\mathscr{V} \subset \tilde{q}^{-1}(U\times\Delta\times V)$ be the pre-image of $U\times\mathscr{U}'\times V$ under (5.6), which is a neighborhood of C in $\tilde{\mathscr{X}}$. It follows that

$$\mathscr{V} \simeq U\times U'\times\Delta\times V$$

over $\Delta\times V$, hence $\tilde{\Pi}$ is a C-locally trivial deformation of X. □

5.4 Proof of Theorem 2.8

We finish Section 5 with the proof of Theorem 2.8.

Proof of Theorem 2.8. — Let X' be a variety as in Proposition 2.5. If X' is the total space of a fibration $f : X' \to B$ in abelian surfaces, then Theorem 2.8 follows from Theorem 2.9 and Corollary 3.11. Otherwise X' is a quotient $\tilde{X}/G$ and it suffices by Lemma 3.4 and 3.7 to show that $\tilde{X}$ has an algebraic approximation which is G-equivariant and $\tilde{C}$-locally trivial where $\tilde{C}$ is the pre-image of $C \subset X'$ in $\tilde{X}$. If we are in the first case of Proposition 2.5, then we use either Lemma 5.4 or Lemma 5.5 together with Lemma 3.2 to conclude. If $\tilde{X}$ is the total space of a G-equivariant smooth isotrivial fibration $\tilde{X} \to \tilde{B}$ in non-algebraic 2-tori with G acting faithfully on $\tilde{B}$, then Theorem 2.9 follows from Lemma 5.7. □

6 Proof of Theorem 2.10 for $\mathbf{P}^1$-bundles

In Section 6, we will prove Theorem 2.10 for $\mathbf{P}^1$-bundles over a compact Kähler surface of algebraic dimension 0. Since $\mathbf{P}^1$-bundles are the projectivization of twisted vector bundles of rank 2, their deformation problem is directly related to the deformation-obstruction theory of twisted vector bundles that we shall study now.

6.1 Semi-regularity maps and deformations of twisted vector bundles

We will first recall the definition of a twisted vector bundle. The reader is referred to [6, Chapter 1] for their basic properties.

Let X be a complex space. Given a Čech 2-cocycle $\alpha = \{\alpha_{ijk}\}$ with coefficients in $\mathcal{O}_X^\times$ with respect to an open cover $\{U_i\}_{i\in I}$ of X, an *α-twisted vector bundle* (E, α) of rank r is a collection of vector bundles E_i over each U_i together with isomorphisms $g_{ij} : E_{i|U_i\cap U_j} \to E_{j|U_i\cap U_j}$ such that $g_{ii} = \mathrm{Id}$, $g_{ij} = g_{ji}^{-1}$, and $g_{ki} \circ g_{jk} \circ g_{ij}$ is the multiplication by α_{ijk} on $E_{i|U_i\cap U_j\cap U_k}$. Similarly and more generally we can define α-twisted coherent sheaves, which form an abelian category $\mathrm{Coh}(X, \alpha)$ [(2)]. Up to equivalence of categories, $\mathrm{Coh}(X, \alpha)$ does not depend on the representative α of $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$.

Let g_{ij} be given as above and let $\bar{g}_{ij} : \mathbf{P}(E_{i|U_i\cap U_j}) \to \mathbf{P}(E_{j|U_i\cap U_j})$ be the induced map on the projectivizations. Since $g_{ki} \circ g_{jk} \circ g_{ij}$ is a scalar multiplication, the collection $\{\bar{g}_{ij}\}$ is a closed 1-cocycle and thus defines a $\mathbf{P}^{r-1}$-bundle over X denoted by $\mathbf{P}(E, \alpha) \to X$, or simply $\mathbf{P}(E)$. Conversely, every $\mathbf{P}^{r-1}$-bundle over X is isomorphic to some $\mathbf{P}(E, \alpha) \to X$ and (E, α) can be chosen so that α is with coefficients in $\mu_r \subset \mathcal{O}_X^\times$ [6, p.9 and Example 1.1.1]; in particular, $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$ is torsion.

Let $\mathcal{X} \to B$ be a holomorphic map and $\mathcal{E}$ an α-twisted locally free sheaf on $\mathcal{X}$. As in the untwisted case, we define the Atiyah class $\mathrm{At}(\mathcal{E})$ as follows. Let $\Delta_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X}$ denote the relative diagonal of $\mathcal{X}$ and $I_{\Delta_{\mathcal{X}}}$ its ideal sheaf. Let $\mathrm{pr}_i : \mathcal{X} \times_B \mathcal{X} \to \mathcal{X}$ be the i-th projection. Regarding $\mathcal{O}_{\mathcal{X}\times_B\mathcal{X}}/I_{\Delta_{\mathcal{X}}}^2$ as an $(\alpha^{-1} \boxtimes \alpha)$-twisted

2. This category depends tacitly on the Čech cover $\{U_i\}_{i\in I}$. It is only up to equivalence of categories that $\mathrm{Coh}(X, \alpha)$ is independent of $\{U_i\}_{i\in I}$.

sheaf, we define the first jet bundle of $\mathscr{E}$ to be the α-twisted sheaf

$$J^1(\mathscr{E}) := \mathrm{pr}_{2*}\left(\mathrm{pr}_1^*\mathscr{E} \otimes \left(\mathscr{O}_{\mathscr{X}\times_B\mathscr{X}}/I^2_{\Delta_{\mathscr{X}}}\right)\right).$$

This sheaf sits in the middle of the short exact sequence

$$0 \longrightarrow \mathscr{E} \otimes \Omega^1_{\mathscr{X}/B} \longrightarrow J^1(\mathscr{E}) \longrightarrow \mathscr{E} \longrightarrow 0 \tag{6.1}$$

and the corresponding element $\mathrm{At}(\mathscr{E}) \in \mathrm{Ext}^1(\mathscr{E}, \mathscr{E} \otimes \Omega^1_{\mathscr{X}/B})$ is called the *Atiyah class* of $\mathscr{E}$.

The following result is classical in the untwisted case (see for example [5, Theorem 5.1]) and can be proven with a similar argument.

Proposition 6.1. — *Let X be a compact Kähler manifold and α a Čech 2-cocycle with coefficients in $\mu_r \subset \mathscr{O}_X^\times$. Let E be an α-twisted locally free sheaf over X. Assume that the trace map $\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(X, \mathscr{O}_X)$ is injective. Then for every smooth deformation $\Pi : \mathscr{X} \to B$ of X over a complex manifold B, whenever the class*

$$\mathrm{tr}\,(\mathrm{At}(E)) \in H^1(X, \Omega^1_X) \subset H^2(X, \mathbf{C})$$

remains of type $(1,1)$ along B under the parallel transport with respect to the Gauss-Manin connection, up to shrinking B there exists a twisted locally free sheaf $\mathscr{E}$ on $\mathscr{X}$ such that $\mathscr{E}_{|X} \simeq E$.

Proof. — Let $o \in B$ be the point parameterizing the central fiber X. By Artin's approximation theorem [1], it suffices to show that E extends to a twisted vector bundle over the formal neighborhood of $o \in B$. To this end, we will prove by induction on n starting from $\mathscr{E}_0 := E$ that if

$$\mathscr{X}_n \to \Delta_n := \mathrm{Spec}\,(\mathbf{C}[\varepsilon_1, \ldots, \varepsilon_{\dim B}]/(\varepsilon_1, \ldots, \varepsilon_{\dim B})^{n+1})$$

denotes the base change of $\Pi : \mathscr{X} \to B$ with the inclusion $\Delta_n \hookrightarrow B$ of the n-th infinitesimal neighborhood of o, then an α_n-twisted locally free sheaf $\mathscr{E}_n$ on $\mathscr{X}_n$ extends to an α_{n+1}-twisted locally free sheaf $\mathscr{E}_{n+1}$ on $\mathscr{X}_{n+1}$. Here α_n is the unique 2-cocycle with coefficients in the locally constant subsheaf $\mu_r \subset \mathscr{O}^\times_{\mathscr{X}_n}$ extending α.

Lemma 6.2. — *Let $\mathscr{X}_n \to \Delta_n$ be a smooth morphism whose special fiber is isomorphic to a compact Kähler manifold X. Let $\mathscr{X}_{n+1} \to \Delta_{n+1}$ be a square-zero extension of $\mathscr{X}_n \to \Delta_n$ and let $\theta \in \mathrm{Ext}^1(\Omega_{\mathscr{X}_n/\Delta_n}, I)$ be the corresponding element where I is the ideal sheaf of $\mathscr{X}_n$ in $\mathscr{X}_{n+1}$ but considered as an $\mathscr{O}_{\mathscr{X}_n}$-sheaf. Let $\mathscr{E}_n$ be an α_n-twisted locally free sheaf on $\mathscr{X}_n$ extending E. Set*

$$\mathrm{ob}_{\mathscr{E}_n} : \mathrm{Ext}^1(\Omega_{\mathscr{X}_n/\Delta_n}, I) \xrightarrow{\mathrm{Id}_{\mathscr{E}_n}\otimes} \mathrm{Ext}^1(\mathscr{E}_n \otimes \Omega_{\mathscr{X}_n/\Delta_n}, \mathscr{E}_n \otimes I) \longrightarrow \mathrm{Ext}^2(\mathscr{E}_n, \mathscr{E}_n \otimes I)$$

where the last arrow is the Yoneda product with $\mathrm{At}(\mathscr{E}_n)$. Then $\mathscr{E}_n$ can further be extended to an α_{n+1}-twisted locally free sheaf $\mathscr{E}_{n+1}$ on $\mathscr{X}_{n+1}$ if and only if $\mathrm{ob}_{\mathscr{E}_n}(\theta) = 0$.

Proof. — The vanishing $\mathrm{ob}_{\mathscr{E}_n}(\theta) = 0$ is equivalent by [16, Exercise III.5.3] to the existence of an α_n-twisted sheaf $\mathscr{F}$ sitting in the middle of the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{E}_n \otimes I & = & \mathscr{E}_n \otimes I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{E}_n \otimes \Omega_{\mathscr{X}_{n+1}/\Delta_{n+1}|\mathscr{X}_n} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{E}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow r & & \| & & \\
0 & \longrightarrow & \mathscr{E}_n \otimes \Omega_{\mathscr{X}_n/\Delta_n} & \longrightarrow & J^1(\mathscr{E}_n) & \longrightarrow & \mathscr{E}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{6.2}$$

If $\mathscr{E}_n$ can be extended to an α_{n+1}-twisted locally free sheaf $\mathscr{E}_{n+1}$ on $\mathscr{X}_{n+1}$, then $\mathscr{F} := J^1(\mathscr{E}_{n+1})_{|\mathscr{X}_n}$ fits in (6.2) as in the untwisted case, where the horizontal exact sequence in the middle is the restriction to $\mathscr{X}_n$ of the analogue extension (6.1) for $J^1(\mathscr{E}_{n+1})$ and $r : J^1(\mathscr{E}_{n+1})_{|\mathscr{X}_n} \to J^1(\mathscr{E}_n)$ is the map induced by the restriction $\mathscr{E}_{n+1} \to \mathscr{E}_n$.

Conversely, assume that (6.2) exists. Let $j : \mathscr{E}_n \to J^1(\mathscr{E}_n)$ be the map sending each local section of $\mathscr{E}_n$ to its first-order jet, which is compatible with the structures of α_n-twisted sheaves because given a good open cover $\{U_i\}$ of $\mathscr{X}_n$, the maps

$$\alpha_{ijk} : U_i \cap U_j \cap U_k \to \mu_r \subset \mathbf{C}^\times$$

defining the 2-cocycles α_n are constant. Instead of being $\mathscr{O}_{\mathscr{X}_n}$-linear, j satisfies the "Leibniz rule": given local sections σ and f of $\mathscr{E}_n$ and $\mathscr{O}_{\mathscr{X}_n}$ respectively, we have $j(f\sigma) = f \cdot j(\sigma) + df \cdot \sigma$. Define

$$\mathscr{E}_{n+1} := \ker\Big(r \circ \mathrm{pr}_1 - j \circ \mathrm{pr}_2 : \mathscr{F} \oplus \mathscr{E}_n \to J^1(\mathscr{E}_n)\Big).$$

As $\mathscr{E}_n$ and $\mathscr{E}_n \otimes \Omega_{\mathscr{X}_{n+1}/\Delta_{n+1}|\mathscr{X}_n}$ are locally free, their extension $\mathscr{F}$ in (6.2) is also locally free. Let us fix a good open cover $\{U_i\}$ of $\mathscr{X}_n$ and gluing isomorphisms

$$\theta_{ij} : \mathscr{F}_{i|U_i\cap U_j} \xrightarrow{\sim} \mathscr{F}_{j|U_i\cap U_j} \quad \text{and} \quad \theta'_{ij} : (\mathscr{E}_n)_{i|U_i\cap U_j} \xrightarrow{\sim} (\mathscr{E}_n)_{j|U_i\cap U_j}$$

defining $\mathscr{F}$ and $\mathscr{E}_n$ such that they are compatible with the morphisms in (6.2) and that both $\theta_{ki} \circ \theta_{jk} \circ \theta_{ij}$ and $\theta'_{ki} \circ \theta'_{jk} \circ \theta'_{ij}$ are multiplications by α_{ijk}. For every i, recall that $(\mathscr{E}_{n+1})_i := \mathscr{E}_{n+1|U_i}$ can be endowed with a structure of $\mathscr{O}_{\mathscr{X}_{n+1}|U_i}$-sheaf as follows. Regarding $\mathscr{O}_{\mathscr{X}_{n+1}}$ as the kernel

$$\mathscr{O}_{\mathscr{X}_{n+1}} = \ker\Big(r' \circ \mathrm{pr}_1 - d \circ \mathrm{pr}_2 : \Omega_{\mathscr{X}_{n+1}/\Delta_{n+1}|\mathscr{X}_n} \oplus \mathscr{O}_{\mathscr{X}_n} \to \Omega_{\mathscr{X}_n/\Delta_n}\Big)$$

where $r' : \Omega_{\mathscr{X}_{n+1}/\Delta_{n+1}|\mathscr{X}_n} \to \Omega_{\mathscr{X}_n/\Delta_n}$ is the restriction map, given local sections $(\alpha, f) \in \mathscr{O}_{\mathscr{X}_{n+1}}(U)$ and $(\beta, \sigma) \in \mathscr{E}_{n+1}(U)$ for some open subset $U \subset U_i$ where α, f, β, σ are elements of $\Omega_{\mathscr{X}_{n+1}/\Delta_{n+1}|\mathscr{X}_n}(U)$, $\mathscr{O}_{\mathscr{X}_n}(U)$, $\mathscr{F}(U)$, and $\mathscr{E}_n(U)$ respectively, we define

$$(\alpha, f) \cdot (\beta, \sigma) := (f\beta + \sigma \otimes \alpha, f\sigma). \tag{6.3}$$

Then (6.3) defines a structure of $\mathscr{O}_{\mathscr{X}_{n+1}|U_i}$-sheaf on $(\mathscr{E}_{n+1})_i$ and $(\mathscr{E}_{n+1})_i$ is locally free of the same rank r of $\mathscr{E}_n$. Finally let

$$\Theta_{ij} : (\mathscr{E}_{n+1})_{i|U_i\cap U_j} \to (\mathscr{E}_{n+1})_{j|U_i\cap U_j}$$

be the restriction of $\theta_{ij} \oplus \theta'_{ij}$ to $(\mathscr{E}_{n+1})_{i|U_i\cap U_j}$. By construction, these maps are $\mathscr{O}_{\mathscr{X}_{n+1}}$-linear and so is the composition $\Theta_{ki} \circ \Theta_{jk} \circ \Theta_{ij}$, which is the restriction of the multiplication by α_{ijk} to $(\mathscr{E}_{n+1})_{i|U_i\cap U_j \cap U_k}$. Therefore $\{\Theta_{ij}\}$ is a collection of gluing isomorphisms which defines $\mathscr{E}_{n+1}$ as an α_{n+1}-twisted sheaf on $\mathscr{X}_{n+1}$. □

To finish the proof, it suffices by Lemma 6.2 to verify that $\mathrm{ob}_{\mathscr{E}_n}(\theta) = 0$. We can compose $\mathrm{ob}_{\mathscr{E}_n}$ with the trace map $\mathrm{tr} : \mathrm{Ext}^2(\mathscr{E}_n, \mathscr{E}_n \otimes I) \to H^2(\mathscr{X}_n, I)$. As the trace maps commute with the Yoneda products, we have

$$\mathrm{tr} \circ \mathrm{ob}_{\mathscr{E}_n}(\theta) = \theta \smile \mathrm{tr}(\mathrm{At}(\mathscr{E}_n)) \in H^2(\mathscr{X}_n, I).$$

If the parallel transports of $\mathrm{tr}(\mathrm{At}(E))$ by the Gauss-Manin connection remain of type $(1,1)$, then by [5, Lemma 5.7 and 5.8] we have $\theta \smile \mathrm{tr}(\mathrm{At}(\mathscr{E}_n)) = 0$. Since $\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(X, \mathscr{O}_X)$ is injective, by [5, Lemma 5.10] with the same proof adapted for twisted sheaves, the trace map $\mathrm{tr} : \mathrm{Ext}^2(\mathscr{E}_n, \mathscr{E}_n \otimes I) \to H^2(\mathscr{X}_n, I)$ is also injective. Hence $\mathrm{ob}_{\mathscr{E}_n}(\theta) = 0$. □

6.2 Deformations of $\mathbf{P}^1$-bundles

Let $\tilde{S}$ be a (smooth) compact Kähler surface with $a(\tilde{S}) = 0$ and $f : X \to \tilde{S}$ a $\mathbf{P}^1$-bundle over $\tilde{S}$. Let E be an α-twisted locally free sheaf of rank 2 over $\tilde{S}$ whose projectivization $\mathbf{P}(E) \to \tilde{S}$ is isomorphic to the $\mathbf{P}^1$-bundle f. As we mentioned before, we can assume that α is with coefficients in $\mu_2 \subset \mathscr{O}_{\tilde{S}}^\times$. The following proposition shows that if S is the minimal model of $\tilde{S}$ and $C \subset S$ the union of all the curves of S, then every C-locally trivial deformation of S preserving $\mathrm{NS}(S)$ can be lifted to a deformation of f.

Proposition 6.3. — *Let S, $\tilde{S}$, E, and $C \subset S$ be as above and $\Pi : \mathscr{S} \to \Delta$ a C-locally trivial deformation of S preserving* $\mathrm{NS}(S)$. *Let*

$$\tilde{\Pi} : \tilde{\mathscr{S}} \xrightarrow{\eta} \mathscr{S} \to \Delta$$

be the deformation of $\tilde{S}$ constructed in Lemma 5.3 lifting Π. Then up to shrinking Δ, $\tilde{\Pi}$ can be lifted to a deformation of the pair $(\tilde{S}, E)$.

Remark 6.4. — When $\mathrm{NS}(S) = 0$, it follows from Proposition 6.3 that there is no obstruction to deforming a $\mathbf{P}^1$-bundle over S along any small deformation of S.

Proof. — First we assume that E is the extension of two torsion free α-twisted sheaves of rank 1. As torsion free twisted sheaves of rank 1 are isomorphic to untwisted ones, the α-twisted sheaf E is also isomorphic to an untwisted sheaf (still denoted by E) and can be considered as an extension

$$0 \longrightarrow \tilde{L} \longrightarrow E \longrightarrow \tilde{L}' \otimes I_Y \longrightarrow 0 \tag{6.4}$$

for some invertible sheaves $\tilde{L}$ and $\tilde{L}'$ on $\tilde{S}$ and some 0-dimensional subscheme $Y \subset \tilde{S}$. The sheaves $\tilde{L}$ and $\tilde{L}'$ deform along $\tilde{\mathscr{S}} \to \Delta$. Indeed, let L and L' be the invertible sheaves on S such that $\tilde{L} = \eta^*_{|\tilde{S}} L(D)$ and $\tilde{L}' = \eta^*_{|\tilde{S}} L'(D')$ for some divisors D and D' with supports in the exceptional locus of $\eta_{|\tilde{S}} : \tilde{S} \to S$. Since $\Pi : \mathscr{S} \to \Delta$ preserves $\mathrm{NS}(S)$, the classes $c_1(L)$ and $c_1(L')$ remain of type $(1,1)$ under the parallel transport by the Gauss-Manin connection along Δ, so there exist invertible sheaves $\mathscr{L}$ and $\mathscr{L}'$ on $\mathscr{S}$ which are deformations of L and L'. For later use, we can even choose $\mathscr{L}$ and $\mathscr{L}'$ so that $h^i(\mathscr{S},(\mathscr{L}'\otimes\mathscr{L}^\vee)_{|\mathscr{S}_t})$ is constant in t by [34, Proposition 4.3 and 4.4]. By the construction of $\eta:\tilde{\mathscr{S}}\to\mathscr{S}$, there exist divisors $\mathscr{D}$ and $\mathscr{D}'$ on $\tilde{\mathscr{S}}$ with supports in the exceptional locus of the blow-up $\tilde{\mathscr{S}}\to\mathscr{S}$ which are deformations of D and D'. The invertible sheaves $\tilde{\mathscr{L}}:=\eta^*\mathscr{L}(\mathscr{D})$ and $\tilde{\mathscr{L}}':=\eta^*\mathscr{L}'(\mathscr{D}')$ are deformations of $\tilde{L}$ and $\tilde{L}'$.

Let $\tilde{C}$ be the union of all the curves in $\tilde{S}$, which consists of only finitely many irreducible components because $a(\tilde{S})=0$. By Lemma 5.3, there exists a neighborhood $U\subset S$ of $\eta(\tilde{C}\cup Y)$ and a trivial deformation $\mathscr{U}\subset\mathscr{S}\to\Delta$ of U such that $\tilde{\mathscr{U}}:=\eta^{-1}(\mathscr{U})\simeq\tilde{U}\times\Delta$ over Δ where $\tilde{U}:=\eta^{-1}(U)$. So there exists a deformation $\mathscr{Y}\subset\tilde{\mathscr{S}}$ of the 0-dimensional subscheme $Y\subset\tilde{S}$ such that

$$(\mathscr{Y}\subset\tilde{\mathscr{U}})\simeq(Y\subset\tilde{U})\times\Delta \ \text{ over } \ \Delta. \tag{6.5}$$

To show that E deforms with $\tilde{S}$ over Δ up to shrinking Δ, by (6.4) it suffices to show that

$$t\mapsto\dim\operatorname{Ext}^1(\tilde{\mathscr{L}}_t'\otimes I_{\mathscr{Y}_t},\tilde{\mathscr{L}}_t)$$

is constant, which we will deduce from the next lemma.

Lemma 6.5. — *Let $\mathcal{L}$ be an invertible sheaf on $\mathscr{S}$ and $\mathcal{D}$ a divisor on $\tilde{\mathscr{S}}$ supported on the exceptional divisor of $\eta:\tilde{\mathscr{S}}\to\mathscr{S}$. Let $\tilde{\mathcal{L}}=\eta^*\mathcal{L}(\mathcal{D})$. Suppose that $t\mapsto h^i(\mathscr{S},\mathcal{L}_{|\mathscr{S}_t})$ is constant for every i, then up to shrinking Δ, $h^1(\tilde{\mathscr{S}},(\tilde{\mathcal{L}}\otimes I_{\mathscr{Y}})_{|\tilde{\mathscr{S}}_t})$ is constant in t.*

Proof. — By the exact sequence

$$\begin{aligned}0\longrightarrow H^0(\tilde{\mathscr{S}},(\tilde{\mathcal{L}}\otimes I_{\mathscr{Y}})_{|\tilde{\mathscr{S}}_t})&\longrightarrow H^0(\tilde{\mathscr{S}},\tilde{\mathcal{L}}_{|\tilde{\mathscr{S}}_t})\longrightarrow H^0(\tilde{\mathscr{S}},\tilde{\mathcal{L}}_{|\mathscr{Y}_t})\\ \longrightarrow H^1(\tilde{\mathscr{S}},(\tilde{\mathcal{L}}\otimes I_{\mathscr{Y}})_{|\tilde{\mathscr{S}}_t})&\longrightarrow H^1(\tilde{\mathscr{S}},\tilde{\mathcal{L}}_{|\tilde{\mathscr{S}}_t})\longrightarrow H^1(\tilde{\mathscr{S}},\tilde{\mathcal{L}}_{|\mathscr{Y}_t})=0,\end{aligned}$$

it suffices to show that $h^0(\tilde{\mathscr{S}},(\tilde{\mathcal{L}}\otimes I_{\mathscr{Y}})_{|\tilde{\mathscr{S}}_t})$ and $h^i(\tilde{\mathscr{S}},\tilde{\mathcal{L}}_{|\tilde{\mathscr{S}}_t})$ are constant in t.

Since $\mathcal{D}$ is supported on the exceptional locus of $\eta:\tilde{\mathscr{S}}\to\mathscr{S}$, which deforms trivially along Δ, we have

$$(\mathcal{D}\subset\tilde{\mathscr{U}})\simeq\big((\mathcal{D}\cap\tilde{S})\subset\tilde{U}\big)\times\Delta \ \text{ over } \ \Delta.$$

Since $(\mathscr{Y}\subset\tilde{\mathscr{U}})\to\Delta$ is a trivial deformation of $(Y\subset\tilde{U})$ by (6.5), it follows that there exists a 0-dimensional subscheme $Z\subset U$ together with a trivial deformation $(\mathscr{Z}\subset\mathscr{U})\to\Delta$ of $(Z\subset U)$ such that $\eta_*(\mathscr{O}_{\tilde{\mathscr{S}}}(\mathcal{D})\otimes I_{\mathscr{Y}})\simeq I_{\mathscr{Z}}$, and thus

$$\eta_*(\tilde{\mathcal{L}}\otimes I_{\mathscr{Y}})\simeq\mathcal{L}\otimes I_{\mathscr{Z}}.$$

If $h^0(S,\mathcal{L}_{|S}\otimes I_{\mathscr{Z}\cap S})=0$, then

$$t\mapsto h^0(\tilde{\mathscr{S}},(\tilde{\mathcal{L}}\otimes I_{\mathscr{Y}})_{|\tilde{\mathscr{S}}_t})=h^0(\mathscr{S}_t,\mathcal{L}_{|\mathscr{S}_t}\otimes I_{\mathscr{Z}\cap\mathscr{S}_t})$$

is constant by upper semi-continuity. If $h^0(S,\mathcal{L}_{|S}\otimes I_{\mathscr{Z}\cap S})>0$, then $h^0(S,\mathcal{L}_{|S}\otimes I_{\mathscr{Z}\cap S})=h^0(S,\mathcal{L}_{|S})=1$ by [3, Proposition IV.8.1]. Since $h^0(\mathscr{S},\mathcal{L}_{|\mathscr{S}_t})$ is constant in t, we have $\Pi_*\mathcal{L}\simeq\mathscr{O}_\Delta$. Let $\sigma\in H^0(\mathscr{S},\mathcal{L})\simeq H^0(\Delta,\mathscr{O}_\Delta)$ be a section which does not vanish. As $\operatorname{Div}(\sigma)$ is supported on $\mathscr{C}$ where $\mathscr{C}\subset\mathscr{S}\to\Delta$ is the trivial deformation of C over Δ, it follows that $(\operatorname{Div}(\sigma)\subset\mathscr{U})$ is a trivial deformation of $(\operatorname{Div}(\sigma_{|S})\subset U)$. In particular, $(\mathscr{Z},\operatorname{Div}(\sigma))\to\Delta$ is a trivial deformation of the pair $(Z,\operatorname{Div}(\sigma_{|S}))$. Since $h^0(S,\mathcal{L}_{|S}\otimes I_{\mathscr{Z}\cap S})=h^0(S,\mathcal{L}_{|S})$, the section $\sigma_{|S}$ vanishes on $\mathscr{Z} \cap S$. So $\mathscr{Z}_t \subset \mathrm{Div}(\sigma_{|\mathscr{S}_t})$ for every $t \in \Delta$ and thus

$$1 \le h^0\left(\mathscr{S}_t, \mathcal{L}_{|\mathscr{S}_t} \otimes I_{\mathscr{Z}\cap\mathscr{S}_t}\right) \le h^0\left(\mathscr{S}_t, \mathcal{L}_{|\mathscr{S}_t}\right) = 1.$$

Therefore $h^0(\tilde{\mathscr{S}}, (\tilde{\mathcal{L}} \otimes I_{\mathscr{Y}})_{|\tilde{\mathscr{S}}_t})$ is also constant in t when $h^0\left(S, \mathcal{L}_{|S} \otimes I_{\mathscr{Z}\cap S}\right) > 0$.

By the same argument, $t \mapsto h^0(\tilde{\mathscr{S}}, \tilde{\mathcal{L}}_{|\tilde{\mathscr{S}}_t})$ as well as $t \mapsto h^0(\tilde{\mathscr{S}}, \tilde{\mathcal{L}}^{\vee}_{|\tilde{\mathscr{S}}_t} \otimes \omega_{\tilde{\mathscr{S}}_t}) = h^2(\tilde{\mathscr{S}}, \tilde{\mathcal{L}}_{|\tilde{\mathscr{S}}_t})$ are also constant. Since $\tilde{\mathcal{L}}$ is flat over Δ, it follows that $h^1(\tilde{\mathscr{S}}, \tilde{\mathcal{L}}_{|\tilde{\mathscr{S}}_t})$ is constant in t as well. □

Since $t \mapsto h^i(\mathscr{S}, (\mathscr{L}' \otimes \mathscr{L}^{\vee})_{|\mathscr{S}_t})$ is assumed to be constant and $K_{\tilde{\mathscr{S}}/\Delta}$ is supported on the exceptional divisor of $\eta : \tilde{\mathscr{S}} \to \mathscr{S}$, Lemma 6.5 together with Serre duality implies that up to shrinking Δ,

$$t \mapsto \dim \mathrm{Ext}^1(\tilde{\mathscr{L}}'_t \otimes I_{\mathscr{Y}_t}, \tilde{\mathscr{L}}_t) = h^1\left(\tilde{\mathscr{S}}, (\mathscr{L}' \otimes \mathscr{L}^{\vee} \otimes I_{\mathscr{Y}} \otimes \omega_{\tilde{\mathscr{S}}/\Delta})_{|\mathscr{S}_t}\right)$$

is constant. Thus there exists a locally free sheaf $\mathscr{E}$ of rank 2 over $\tilde{\mathscr{S}}$ such that $\mathscr{E}_{|\tilde{S}} = E$.

Now we turn to the case where E is not the extension of any pair of twisted sheaves of rank 1. As $[\alpha] \in H^2(\tilde{S}, \mathscr{O}^{\times}_{\tilde{S}})$ is torsion, there exists $n \in \mathbf{Z}_{>0}$ such that $E^{\otimes n}$ is isomorphic to an untwisted vector bundle. So

$$n \cdot \mathrm{tr}(\mathrm{At}(E)) = \mathrm{tr}(\mathrm{At}(E^{\otimes n})) = -c_1(E^{\otimes n}) \in \mathrm{NS}(\tilde{S}).$$

Since $\tilde{\mathscr{S}} \to \Delta$ preserves $\eta^*_{|\tilde{S}}\mathrm{NS}(S)$ and $\tilde{C}$ is the union of all the curves in $\tilde{S}$, the $\tilde{C}$-local triviality of the deformation $\tilde{\mathscr{S}} \to \Delta$ of $\tilde{S}$ implies that $\tilde{\Pi} : \tilde{\mathscr{S}} \to \Delta$ preserves $\mathrm{NS}(\tilde{S})$. In particular, $\mathrm{tr}(\mathrm{At}(E))$ remains of type $(1,1)$ under the parallel transport by the Gauss-Manin connection along Δ. Thus by Proposition 6.1, it suffices to show that $\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(\tilde{S}, \mathscr{O}_{\tilde{S}})$ is injective.

By Serre duality, the dual $\mathrm{tr}^{\vee}$ of the trace map is isomorphic to

$$H^0(\tilde{S}, \omega_{\tilde{S}}) \to H^0(\tilde{S}, E^{\vee} \otimes E \otimes \omega_{\tilde{S}}) = \mathrm{Hom}(E, E \otimes \omega_{\tilde{S}})$$

induced by $\mathscr{O}_{\tilde{S}} \hookrightarrow E^{\vee} \otimes E$. So $\mathrm{tr}^{\vee}$ is injective. Therefore to show that $\mathrm{tr}^{\vee}$ is surjective, it suffices to have $h^0(\tilde{S}, \omega_{\tilde{S}}) \ge \dim \mathrm{Hom}(E, E \otimes \omega_{\tilde{S}})$. On the one hand since $\tilde{S}$ is a sequence of blow-ups of a K-trivial surface, we have $h^0(\tilde{S}, \omega_{\tilde{S}}^{\otimes i}) = 1$ for every $i \ge 0$. On the other hand since E has no non-trivial torsion-free quotient, the determinant

$$\det : \mathrm{Hom}(E, E \otimes \omega_{\tilde{S}}) \to H^0(\tilde{S}, \omega_{\tilde{S}}^{\otimes 2})$$

satisfies $\det^{-1}(0) = \{0\}$. Therefore, as $H^0(\tilde{S}, \omega_{\tilde{S}}^{\otimes 2}) \simeq \mathbf{C}$ and det is a polynomial function, we have

$$\dim \mathrm{Hom}(E, E \otimes \omega_{\tilde{S}}) \le 1 = h^0(\tilde{S}, \omega_{\tilde{S}}).$$

□

Now we can prove Theorem 2.10 for $\mathbf{P}^1$-bundles $f : X \to S$ with $a(S) = 0$.

Corollary 6.6. — *Let $f : X \to S$ be a $\mathbf{P}^1$-bundle over a smooth compact Kähler surface S of algebraic dimension 0. There exists a deformation*

$$\Pi : \mathscr{X} \to \mathscr{S} \to \Delta$$

of f such that Π is an $f^{-1}(C)$-locally trivial algebraic approximation of X for every subvariety $C \subsetneq S$.

Proof. — Since f is a $\mathbf{P}^1$-bundle over a smooth base, if $Z \subset S$ is a finite subset, then every small deformation of f is an $f^{-1}(Z)$-locally trivial deformation of X. Therefore we can assume that C is a union of curves. Since $a(S) = 0$, there are only finitely many curves in S. So we can further assume that C is the union of all the curves in S. Let $\nu : S \to S_{\text{min}}$ be the map from S to its minimal model S_{min}, which is a surface satisfying $\omega_{S_{\text{min}}} \simeq \mathscr{O}_{S_{\text{min}}}$. If C_{min} denotes the union of all the curves in S_{min}, then by Lemma 5.1 there exists a C_{min}-locally trivial algebraic approximation $\mathscr{S}_{\text{min}} \to \Delta$ of S_{min} preserving $\text{NS}(S_{\text{min}})$. By Lemma 5.3 and Proposition 6.3, the family $\mathscr{S}_{\text{min}} \to \Delta$ can be lifted to a C-locally trivial deformation $\mathscr{S} \to \Delta$ of S which can be further lifted to a deformation $\mathscr{X}_0 \xrightarrow{\tilde{f}} \mathscr{S} \to \Delta$ of the $\mathbf{P}^1$-bundle $f : X \to S$.

Let $\eta : S \xrightarrow{\nu} S_{\text{min}} \to S_{\text{can}}$ be the composition of ν with the contraction of the (-2)-curves (so all the curves) of S_{min}. Let $U_{\text{can}} \subset S_{\text{can}}$ be a Stein neighborhood of the finite subset $\eta(C)$. Since $\mathscr{S} \to \Delta$ is C-locally trivial, up to shrinking U_{can} we can assume that there exists a neighborhood $\mathscr{U} \subset \mathscr{S}$ of the trivial deformation $\mathscr{C} \subset \mathscr{S} \to \Delta$ of C such that $\mathscr{U} \cap S = U := \eta^{-1}(U_{\text{can}})$ and $\mathscr{U} \simeq U \times \Delta$ over Δ. We claim that $H^2(\mathscr{U}, \mathscr{O}_{\mathscr{U}}^{\times}) = 0$. Indeed, it suffices to show that $H^3(\mathscr{U}, \mathbf{Z}) = 0$ and $H^2(\mathscr{U}, \mathscr{O}_{\mathscr{U}}) = 0$. Since $\mathscr{U}$ deformation retracts to C, we have $H^3(\mathscr{U}, \mathbf{Z}) = H^3(C, \mathbf{Z}) = 0$. As $U_{\text{can}} \times \Delta$ is Stein and the fibers of the composition $\phi : \mathscr{U} \to U_{\text{can}} \times \Delta$ of the isomorphism $\mathscr{U} \xrightarrow{\sim} U \times \Delta$ with $\eta \times \text{Id}_{\Delta}$ are of dimension ≤ 1, it follows from the Leray spectral sequence that $H^2(\mathscr{U}, \mathscr{O}_{\mathscr{U}}) = 0$.

As $H^2(\mathscr{U}, \mathscr{O}_{\mathscr{U}}^{\times}) = 0$, there exists a locally free sheaf $\mathscr{F}$ of rank 2 on $\mathscr{U}$ such that $\tilde{f}^{-1}(\mathscr{U}) \simeq \mathbf{P}(\mathscr{F})$ over $\tilde{\mathscr{U}}$ where $\tilde{f}$ is the map $\tilde{f} : \mathscr{X}_0 \to \mathscr{S}$ defining the deformation of f. If we consider $(\phi_*\mathscr{F})^{\vee\vee}$ as a family of reflexive sheaves $\mathscr{G}_t$ on U_{can} parameterized by $t \in \Delta$, then since U_{can} has at worst rational double points as singularity, Artin-Verdier's theorem shows that $t \mapsto \mathscr{G}_t$ is isomorphic to a constant family [2, Theorem 1.11]. In particular, $\mathscr{F}_{|\mathscr{U}\backslash\mathscr{C}}$ is isomorphic to $\text{pr}_1^*(\mathscr{F}_{|U\backslash C})$ where $\text{pr}_1 : \mathscr{U}\backslash\mathscr{C} \to U\backslash C$ is the composition of the isomorphism $\mathscr{U}\backslash\mathscr{C} \simeq (U\backslash C) \times \Delta$ with the projection $(U\backslash C) \times \Delta \to U\backslash C$. Therefore we can glue $\mathscr{X}_0\backslash\tilde{f}^{-1}(\mathscr{C})$ with $f^{-1}(U) \times \Delta$ along $\tilde{f}^{-1}(\mathscr{U}\backslash\mathscr{C}) \subset \mathscr{X}_0\backslash\tilde{f}^{-1}(\mathscr{C})$ and $f^{-1}(U\backslash C) \times \Delta \subset f^{-1}(\tilde{U}) \times \Delta$ using the isomorphisms

$$\tilde{f}^{-1}(\mathscr{U}\backslash\mathscr{C}) \simeq \mathbf{P}\left(\mathscr{F}_{|\mathscr{U}\backslash\mathscr{C}}\right) \simeq \mathbf{P}\left(\text{pr}_1^*(\mathscr{F}_{|U\backslash C})\right) \simeq f^{-1}(U\backslash C) \times \Delta$$

to obtain a new deformation $\mathscr{X} \to \mathscr{S} \to \Delta$ of the $\mathbf{P}^1$-fibration $f : X \to S$, which is now $f^{-1}(C)$-locally trivial.

As $\mathscr{S}_{\text{min}} \to \Delta$ is an algebraic approximation of S_{min}, the family $\mathscr{S} \to \Delta$ is an algebraic approximation of S. It follows from Corollary 3.12 that $\mathscr{X} \to \mathscr{S} \to \Delta$ is an algebraic approximation of f. □

7 Proof of Theorem 2.10: Case $a(S) = 1$

Now we prove Theorem 2.10 for $\mathbf{P}^1$-fibrations $X \to S$ over an elliptic surface S. Recall from 3.6 that an elliptic surface has itself an algebraic approximation realized by its associated tautological family. We shall prove that given a $\mathbf{P}^1$-fibration $X \to S$ over a non-algebraic Kähler elliptic surface S, the tautological family associated to S can always be lifted to a deformation of $X \to S$.

Proposition 7.1. — *Let $f : X \to S$ be a $\mathbf{P}^1$-fibration over a non-algebraic compact Kähler elliptic surface $p : S \to B$. There exists an algebraic approximation $\Pi : \mathscr{S} \to B \times V \to V$ of p which can be lifted to an algebraic approximation*

$$\mathscr{X} \to \mathscr{S} \to B \times V \to V$$

of f such that the underlying deformation of $\pi := p \circ f : X \to B$ is locally trivial over B. In particular, Π is an $f^{-1}(C)$-locally trivial algebraic approximation of X for every subvariety $C \subsetneq S$

The schema of proof of Proposition 7.1 is simple. For simplicity, let us assume that p is smooth. Let $\mathscr{J}$ be the sheaf of germs of local sections of the Jacobian fibration associated to p, which we consider as a subsheaf of $\mathscr{A}ut_{S/B}$, the sheaf (on B) of germs of automorphisms of S over B. Based on the assumption that S is non-algebraic, we will first describe the ruled surfaces $f_b : X_b \to S_b$ that can appear as a general fiber of $f : X \to B$ (see Lemma 7.3). These descriptions allow us to lift locally sections of $\mathscr{J}$ to sections of $\mathscr{A}ut_{X/B}$ using results of [28]. By Propoition 3.16, the tautological family

$$\Pi : \mathscr{S} \xrightarrow{q} B \times V \to V$$

associated to p is an algebraic approximation of p. Recall from 3.6 that Π corresponds to the deformation of 1-cocycles $t \mapsto \left\{\exp(\xi_{ij}(t))\right\}$ with coefficients in $\mathscr{J}$ (with respect to some open cover $\{U_i\}$ of B), so it suffices to show that $t \mapsto \left\{\exp(\xi_{ij}(t))\right\}$ can be lifted to a deformation of 1-cocycles with coefficients in $\mathscr{A}ut_{X/B}$, so as to glue the local fibrations $f^{-1}(U_i) := X_i \to U_i$ using these 1-cocycles and obtain a deformation of f lifting Π.

Proof of Proposition 7.1. — Since S is not algebraic, every subvariety of S is contained in a finite union of fibers of $p : S \to B$. In particular by generic smoothness of f, the image of the non-smooth locus of f in S is supported on a finite union of fibers of p. Let $B^\star \subsetneq B$ be a nonempty Zariski open subset such that the restrictions $f^\star : X^\star \to S^\star$ and $p^\star : S^\star \to B^\star$ of f and p respectively are smooth where $S^\star := p^{-1}(B^\star)$ and $X^\star := \pi^{-1}(B^\star)$. It follows that $\pi^\star := p^\star \circ f^\star$ is a smooth family of ruled surfaces $X_b \to S_b$.

Lemma 7.2. — *Let $f : \mathcal{S} \to \mathcal{B}$ be a smooth elliptic fibration over an affine curve. Every $\mathbf{P}^n$-bundle over $\mathcal{S}$ is the projectivization of some vector bundle.*

Proof. — Since the obstruction to lifting a projective bundle to a vector bundle is a torsion element in $H^2(\mathcal{S}, \mathscr{O}_{\mathcal{S}}^\times)$, it suffices to show that $H^2(\mathcal{S}, \mathscr{O}_{\mathcal{S}}^\times)$ is torsion-free.

Since $R^i f_* \mathscr{O}_{\mathcal{S}} = 0$ for every $i \geq 2$ and $H^p(\mathcal{B}, R^q f_* \mathscr{O}_{\mathcal{S}}) = 0$ for every $p \geq 1$ because $\mathcal{B}$ is affine, using the Leray spectral sequence we obtain $H^i(\mathcal{S}, \mathscr{O}_{\mathcal{S}}) = 0$ for $i = 2$ or 3. It follows from the exponential sheaf sequence that $H^2(\mathcal{S}, \mathscr{O}_{\mathcal{S}}^\times) \simeq H^3(\mathcal{S}, \mathbf{Z})$. As the fibers of f are curves and $\mathcal{B}$ is an affine curve, whenever $p \geq 2$ or $q \geq 3$ we have the vanishing $E_2^{p,q} = H^p(\mathcal{B}, R^q f_* \mathbf{Z}) = 0$ of the E_2-terms in the Leray spectral sequence computing $H^3(\mathcal{S}, \mathbf{Z})$. It follows that

$$H^3(\mathcal{S}, \mathbf{Z}) \simeq E_2^{1,2} = H^1(\mathcal{B}, R^2 f_* \mathbf{Z}) \simeq H^1(\mathcal{B}, \mathbf{Z}),$$

which is torsion-free. □

By Lemma 7.2, there exists a locally free sheaf $\mathscr{E}$ of rank 2 on $S^\star$ such that $X^\star = \mathbf{P}(\mathscr{E})$. For every $\mathscr{E}_b := \mathscr{E}_{|S_b}$, let $e(\mathscr{E}_b) = 2\deg(\mathscr{M}_b) - \deg(\mathscr{E}_b)$ where $\mathscr{M}_b$ is a maximal invertible subsheaf of $\mathscr{E}_b$. The invariant $e(\mathscr{E}_b)$ is well-defined by virtue of [18, Proposition 2.8].

Lemma 7.3. — *If b is a general point of $B^\star$, then either $\mathscr{E}_b$ is indecomposable with $e(\mathscr{E}_b) \le 0$, or $\mathscr{E}_b \simeq \mathscr{L}_b \oplus \mathscr{L}_b'$ for some invertible sheaves $\mathscr{L}_b$ and $\mathscr{L}_b'$ on S_b with $\deg \mathscr{L}_b = \deg \mathscr{L}_b'$.*

Proof. — Let us assume the contrary. If $b \in B^\star$ is a very general point, then by [18, Theorem V.2.15] $\mathscr{E}_b \simeq \mathscr{L}_b \oplus \mathscr{L}_b'$ for some invertible sheaves $\mathscr{L}_b$ and $\mathscr{L}_b'$ on S_b with $\deg \mathscr{L}_b \neq \deg \mathscr{L}_b'$. By [18, Example V.2.11.3], there is exist a unique section $C_b \subset X_b$ of $X_b \to S_b$ such that in X_b, we have $[C_b]^2 < 0$. So $H^0(C_b, N_{C_b/X_b}) = 0$ and thus the projection $\mathscr{D}_{X/B} \to B$ from the relative Douady space $\mathscr{D}_{X/B}$ of $\pi : X \to B$ is unramified at C_b. As B is irreducible, C_b is contained in a unique irreducible component $\mathscr{C}$ of $\mathscr{D}_{X/B}$, which also contains $C_{b'}$ for every other very general point $b' \in B$. So over a countable intersection $B^\circ \subset B^\star$ of nonempty Zariski open subsets of $B^\star$, there is a section $B^\circ \to \mathscr{C}$ of $\mathscr{C} \to B$ associating b to C_b, which shows that $\mathscr{C} \to B$ is generically injective. Since $\mathscr{C} \to B$ is proper [15, Theorem 5.2], the image in X of the total space of the universal family $\mathscr{D} \to \mathscr{C}$ contains an irreducible closed surface $D \subset X$ such that $D \cap X_b = C_b$.

Let $\tau : \tilde{D} \to D$ be a desingularization of D. We have

$$c_1(\tau^* N_{D/X}) \cdot [\tau^{-1}(C_b)] = [D]^2 \cdot [X_b] = [C_b]^2 \ \text{(in } X_b\text{)} \ < 0.$$

So $c_1(\tau^* N_{D/X}(mC_b))^2 > 0$ for some $m \ll 0$, and therefore $\tilde{D}$ is algebraic [3, Proposition 6.2]. As $\tilde{D}$ dominates S and S is non-algebraic, there is a contradiction. □

Let $b \in B^\star$ be a general point. By Lemma 7.3 and [18, Theorem V.2.15], there are four types of $\mathscr{E}_b$:

1. $\mathscr{E}_b \simeq \mathscr{L}_b \oplus \mathscr{L}_b$ for some invertible sheaf $\mathscr{L}_b$;
2. $\mathscr{E}_b \simeq \mathscr{L}_b \oplus \mathscr{L}_b'$ where $\mathscr{L}_b \not\simeq \mathscr{L}_b'$ are invertible sheaves of the same degree;
3. $\mathscr{E}_b$ is the non-trivial extension of $\mathscr{L}_b$ by $\mathscr{L}_b$;
4. $\mathscr{E}_b$ is indecomposable with $e < 0$.

When $\mathscr{E}_b$ is of type 1, Proposition 7.1 already follows from [34, Theorem 1.6] (one can easily check that the algebraic approximation produced in the proof of [34, Theorem 1.6] is locally trivial over B). From now on we assume that $\mathscr{E}_b$ is of type 2, 3, or 4. Up to replacing $B^\star$ with a smaller Zariski open subset, we can assume that $\mathscr{E}_b$ is purely of type 2, 3, or 4 for every $b \in B^\star$.

By Lemma [12, Proposition 3.11], there exists a Galois cover $r : \tilde{B} \to B$ of B such that the elliptic fibration $\tilde{p} : \tilde{S} := S \times_B \tilde{B} \to \tilde{B}$ has local sections at every point of $\tilde{B}$. Let $G := \mathrm{Gal}(\tilde{B}/B)$. By Proposition 3.16, the G-equivariant tautological family

$$\Pi : \tilde{\mathscr{S}} \xrightarrow{q} \tilde{B} \times V \to V := H^1(\tilde{B}, \bar{\mathscr{E}}_{\tilde{\mathbf{H}}/\tilde{B}})^G \tag{7.1}$$

associated to $\tilde{p}$ is an algebraic approximation $\tilde{p}$ which is G-equivariantly locally trivial over $\tilde{B}$. Here, $j : \tilde{B}^\star \to \tilde{B}$ is the inclusion and $\bar{\mathscr{E}}_{\tilde{\mathbf{H}}/\tilde{B}}$ is the Deligne canonical extension of $\mathscr{E}_{\tilde{\mathbf{H}}/\tilde{B}} = R^1\tilde{p}^\star_* \mathscr{O}_{\tilde{S}^\star}$ where $\tilde{p}^\star : \tilde{S}^\star \to \tilde{B}^\star$ is the restriction of $\tilde{p}$ to $\tilde{S}^\star := \tilde{p}^{-1}(\tilde{B}^\star)$.

Lemma 7.4. — *The tautological family* Π *lifts to a deformation*

$$\Pi' : \tilde{\mathscr{X}} \to \tilde{B} \times V \to V$$

of $\tilde{\pi} : \tilde{X} := X \times_S \tilde{S} \to \tilde{B}$ *which is G-equivariantly locally trivial over* $\tilde{B}$.

Proof. — Let $\{U_i\}_{i\in I}$ be a G-invariant good open cover of $\tilde{B}$ such that $\tilde{p} : \tilde{S} \to \tilde{B}$ has local sections over each U_i and for each pair of indices $i, j \in I$ such that $i \neq j$, we have $U_{ij} := U_i \cap U_j \subset \tilde{B}^\star$ where $\tilde{B}^\star := r^{-1}(B^\star)$. We can also assume that for every $g \in G$, either $g(U_{ij}) = U_{ij}$ or $g(U_{ij}) \cap U_{ij} = \emptyset$. Let $\tilde{S}^\star = \tilde{p}^{-1}(\tilde{B}^\star)$, $\tilde{S}_i := \tilde{p}^{-1}(U_i)$, $\tilde{S}_{ij} := \tilde{p}^{-1}(U_{ij})$, $\tilde{X}_i := \tilde{\pi}^{-1}(U_i)$, and $\tilde{X}_{ij} := \tilde{\pi}^{-1}(U_{ij})$. Let $\mathscr{J}$ be the sheaf of germs of sections of the Jacobian fibration $J \times V \to \tilde{B}^\star \times V$ associated to $\tilde{S}^\star \times V \to \tilde{B}^\star \times V$. Recall from 3.6 that $q : \tilde{\mathscr{S}} \to \tilde{B} \times V$ is obtained by gluing the $\tilde{S}_i \times V \to U_i \times V$ using the 1-cocycle of translations $\left\{e_{ij} : \tilde{S}_{ij} \times V \to \tilde{S}_{ij} \times V\right\}$ defined by

$$e_{ij} = \mathrm{tr}\left(\exp(U_{ij} \times V)(\xi_{ij})\right),$$

where $\exp : \mathscr{E}_{\mathrm{pr}^{-1}\tilde{\mathbf{H}}/\tilde{B}\times V} \to \mathscr{J}$ is the exponential map and $\left\{\xi_{ij} : V \to \mathscr{E}_{\tilde{\mathbf{H}}/\tilde{B}}(U_{ij})\right\}$ is a G-invariant 1-cocycle with coefficients in $\bar{\mathscr{E}}_{\tilde{\mathbf{H}}/\tilde{B}}$ representing the inclusion $V \hookrightarrow H^1(\tilde{B}, \bar{\mathscr{E}}_{\tilde{\mathbf{H}}/\tilde{B}})$, which we can assume that $\xi_{ij}(0) = 0$. In particular, $\{e_{ij}\}$ is G-invariant and $e_{ij|\tilde{S}_{ij}\times\{0\}} = \mathrm{Id}_{\tilde{S}_{ij}}$.

Let $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ denote the sheaf (on $\tilde{B}^\star \times V$) of germs of automorphisms of $\tilde{X}^\star \times V$ over $\tilde{S}^\star \times V$ and $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ its subsheaf of identity components. According to whether $\mathscr{E}_b$ is of type 2, 3, or 4, by [28, Theorem 2] (or rather by the explicit description of $\mathrm{Aut}(\mathscr{E}_b)$ in [28, p.91 and 92]) the automorphism group $\mathrm{Aut}^0_{\tilde{S}_b}(\tilde{X}_b)$ is canonically isomorphic to $\mathbf{C}^\times$, $\mathbf{C}$, or 0, so

$$\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V} \simeq \mathscr{O}^\times_{\tilde{B}^\star\times V} \quad \text{or} \quad \mathscr{O}_{\tilde{B}^\star\times V} \quad \text{or} \quad 0. \tag{7.2}$$

Let $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}$ be the sheaf on $\tilde{B}^\star \times V$ defined as follows: for every open subset $U \subset \tilde{B}^\star \times V$, let $\tilde{S}^\star_U \to U$ be the restriction of the smooth elliptic fibration $\tilde{S}^\star \times V \to \tilde{B}^\star \times V$ to $\tilde{S}^\star_U := (\tilde{p} \times \mathrm{Id}_V)^{-1}(U)$. We define $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}(U)$ to be the group of biholomorphic maps $\phi : \tilde{X}^\star_U \to \tilde{X}^\star_U := (\tilde{\pi} \times \mathrm{Id}_V)^{-1}(U)$ over U which descend to a translation $\tilde{S}^\star_U \to \tilde{S}^\star_U$. By definition, there is a descent morphism $\Psi : \mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V} \to \mathscr{J}$ and $\ker(\Psi) = \mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$. Moreover since $\mathscr{E}_b$ is of type 2, 3, or 4, Ψ is surjective [28, Lemma 8], so we have a short exact sequence

$$0 \longrightarrow \mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{S}^\star\times V} \longrightarrow \mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V} \xrightarrow{\Psi} \mathscr{J} \longrightarrow 0. \tag{7.3}$$

Since $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ is a sheaf of abelian groups, the $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}$-action on by conjugation on $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ descends to a $\mathscr{J}$-action.

Claim. — *The induced* $\mathscr{J}$*-action on* $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ *is trivial.*

Proof. — Let $t \in \tilde{B}^\star \times V$ and let $\tilde{S}_t$ (resp. $\tilde{X}_t$) be the fiber of $\tilde{S}^\star \times V \to \tilde{B}^\star \times V$ (resp. $\tilde{X}^\star \times V \to \tilde{B}^\star \times V$) over t. It suffices to show that for every $x \in J_t$ where J_t is the fiber of the Jacobian fibration $J \times V \to \tilde{B}^\star \times V$ over t, the automorphism $\Phi_x : \mathrm{Aut}_{\tilde{S}_t}(\tilde{X}_t) \to \mathrm{Aut}_{\tilde{S}_t}(\tilde{X}_t)$ induced by the $\mathscr{J}$-action is the identity. For every $g \in \mathrm{Aut}_{\tilde{S}_t}(\tilde{X}_t)$, since $x \mapsto \Phi_x(g)$ is a holomorphic map from the elliptic curve J_t to either $\mathbf{C}^\times$, $\mathbf{C}$, or 0 according to (7.2), it is constant. Hence $\Phi_x(g) = \Phi_0(g) = g$. □

By the surjectivity of Ψ in (7.3), there exists a collection of biholomorphic maps

$$\left\{\tilde{e}_{ij} \in \mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}(U_{ij}\times V)\right\}$$

lifting the 1-cochain $\{e_{ij}\}$ such that $\tilde{e}_{ij|\tilde{X}_{ij}\times\{0\}} = \mathrm{Id}_{\tilde{X}_{ij}}$. We can assume $\{\tilde{e}_{ij}\}$ to be G-invariant. Indeed, since $\{U_{ij}\}$ is G-invariant and since we have either $g(U_{ij}) = U_{ij}$ or $g(U_{ij}) \cap U_{ij} = \emptyset$, it suffices to pick one U_{ij} in each G-orbit of $\{U_{ij}\}$ and construct an H_{ij}-invariant lifting $\tilde{e}_{ij}$ of e_{ij} where $H_{ij} \subset G$ is the subgroup of elements $h \in G$ such that $h(U_{ij}) = U_{ij}$. Since H_{ij} is finite, the image of $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}(U_{ij}\times V)^{H_{ij}}$ in $\mathscr{J}(U_{ij}\times V)^{H_{ij}}$ contains the identity component of $\mathscr{J}(U_{ij}\times V)^{H_{ij}}$. As e_{ij} is connected to the identity in $\mathscr{J}(U_{ij}\times V)^{H_{ij}}$ *via*

$$t \mapsto e_{ij}(t) := \mathrm{tr}\left(\exp(U_{ij}\times V)(t\xi_{ij})\right),$$

e_{ij} has a H_{ij}-invariant lifting $\tilde{e}_{ij} \in \mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}(U_{ij}\times V)^{H_{ij}}$. To be more concrete, there exists a path $t \mapsto \tilde{e}_{ij}(t)$ in $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}(U_{ij}\times V)^{H_{ij}}$ lifting $t \mapsto e_{ij}(t)$ such that $\tilde{e}_{ij}(0) = \mathrm{Id}$ and $\tilde{e}_{ij}(t)_{|\tilde{X}_{ij}\times\{0\}} = \mathrm{Id}_{\tilde{X}_{ij}}$ and we can set $\tilde{e}_{ij} = \tilde{e}_{ij}(1)$.

As $\{e_{ij}\}$ is a 1-cocycle, by (7.3) the collection of compositions $\{\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij}\}$ is a 2-cocycle with coefficients in $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$. Since $t \mapsto \{e_{ij}(t)\}$ is a path of 1-cocycles connecting $\{e_{ij}\}$ and Id, which lifts to a path $t \mapsto \{\tilde{e}_{ij}(t)\}$ connecting $\{\tilde{e}_{ij}\}$ and Id, the 2-cocycle $\{\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij}\}$ is with coefficients in the identity component $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$. As $H^2(\tilde{B}\times V, \mathscr{O}^\times_{\tilde{B}\times V}) = 0$ and $H^2(\tilde{B}\times V, \mathscr{O}_{\tilde{B}\times V}) = 0$, by (7.2) $\{\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij}\}$ is the coboundary of a 1-cochain $\{\beta_{ij}\}$ with respect to the open cover $U_i\times V$ with coefficients in $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ such that $\beta_{ij|\tilde{X}_{ij}\times\{0\}} = \mathrm{Id}_{\tilde{X}_{ij}}$. So

$$\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij} = \beta_{ki}\circ\beta_{jk}\circ\beta_{ij}.$$

Since the $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{B}^\star\times V}$-action on $\mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}$ by conjugation is trivial by the above claim, we have

$$(\tilde{e}_{ki}\circ\beta_{ki}^{-1})\circ(\tilde{e}_{jk}\circ\beta_{jk}^{-1})\circ(\tilde{e}_{ij}\circ\beta_{ij}^{-1}) = \mathrm{Id}.$$

By a similar argument as we did before for $\{\tilde{e}_{ij}\}$, we can also assume that $\{\beta_{ij}\}$ is G-invariant. Indeed, as G is finite, the image of the G-invariant part of the coboundary map

$$\delta^G : C^1\left(\{U_i\times V\}, \mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}\right)^G \to B^2\left(\{U_i\times V\}, \mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}\right)^G$$

contains the identity component of $B^2\left(\{U_i\times V\}, \mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}\right)^G$. Since $\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij}$ is connected to the identity in $\mathscr{A}ut_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}(U_{ijk}\times V)^{H_{ijk}}$ *via* $t \mapsto \tilde{e}_{ki}(t)\circ\tilde{e}_{jk}(t)\circ\tilde{e}_{ij}(t)$ where $H_{ijk} \subset G$ is the stabilizer of U_{ijk}, it follows that the 2-cocycle $\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij}$ is in the identity component of $B^2\left(\{U_i\times V\}, \mathscr{A}ut^0_{\tilde{X}^\star\times V/\tilde{S}^\star\times V}\right)^G$. Therefore $\tilde{e}_{ki}\circ\tilde{e}_{jk}\circ\tilde{e}_{ij}$ is the coboundary of some G-invariant 1-cochain $\{\beta_{ij}\}$.

The G-invariant 1-cocycle of biholomorphic maps

$$\left\{\tilde{e}_{ij}\circ\beta_{ij}^{-1} : \tilde{X}_{ij}\times V \to \tilde{X}_{ij}\times V\right\}$$

is a lifting of $\{e_{ij}\}$, so we can glue the $\tilde{X}_i\times V \to \tilde{S}_i\times V$ to a map $\tilde{\mathscr{X}} \to \tilde{\mathscr{S}}$ and the composition

$$\Pi' : \tilde{\mathscr{X}} \to \tilde{\mathscr{S}} \to \tilde{B}\times V \to V$$

is a lifting of Π which is G-equivariantly locally trivial over $\tilde{B}$. Since $(\tilde{e}_{ij} \circ \beta_{ij}^{-1})_{|\tilde{X}_{ij}\times\{0\}} = \mathrm{Id}_{\tilde{X}_{ij}}$, the central fiber of Π' is $\tilde{\pi} : \tilde{X} \to \tilde{B}$. In other words Π' is a deformation of $\tilde{\pi}$, which finishes the proof of Lemma 7.4. □

By Lemma 3.4, the quotient

$$\mathscr{X} := \tilde{\mathscr{X}}/G \to \mathscr{S} := \tilde{\mathscr{S}}/G \to B \times V \to V$$

of Π' is a deformation of $\pi : X \to B$ which is locally trivial over B. As the fiber $\mathscr{X}_t$ of $\mathscr{X} \to V$ over t is a $\mathbf{P}^1$-fibration over $\mathscr{S}_t$, by Corollary 3.12 $\mathscr{X}_t$ is algebraic if $\mathscr{S}_t$ is algebraic. Since $\mathscr{S} \to V$ is an algebraic approximation, it follows that $\mathscr{X} \to V$ is also an algebraic approximation, which concludes the proof of the main statement of Proposition 7.1. For the last statement, as S is non-algebraic, every subvariety $C \subsetneq S$ is contained in a finite union of fibers of p. Therefore since Π is locally trivial over B, it is $f^{-1}(C)$-locally trivial for every $C \subsetneq S$. □

Now Theorem 2.10 follows easily from the results we proved in Section 6 and 7.

Proof of Theorem 2.10. — Let $f : X \to S$ be a $\mathbf{P}^1$-fibration as in Proposition 2.6. According to whether f is in the first or the second case of Proposition 2.6, we use either Corollary 6.6 and Proposition 7.1. □

8 Conclusion

8.1 Proofs of the main theorems

Assembling the results proven previously in this article, now we can prove Theorem 2.2 and 1.1 as we outlined in Section 2.

Proof of Theorem 2.2. — For non-uniruled compact Kähler threefold of algebraic dimension $a \leq 1$, Theorem 2.2 follows from Proposition 2.5 and Theorem 2.8. For uniruled compact Kähler threefold, we use Proposition 2.6 and Theorem 2.10 to conclude. □

Proof of Theorem 1.1. — Let X be a compact Kähler threefold and $a(X)$ its algebraic dimension, which is an integer between 0 and 3. If $a(X) = 3$, then X is already projective by Moishezon's criterion. If $a(X) = 2$, then Theorem 1.1 is covered by Theorem 2.3. Finally if $a(X) \leq 1$, then we apply Theorem 2.2 to X and conclude by Proposition 2.1 that X has an algebraic approximation. □

Remark 8.1. — The same proof of [27, Corollary 1.4] shows that more generally, as an immediate corollary of Theorem 1.1, every compact complex threefold X in the Fujiki class $\mathscr{C}$ with at worst rational singularities admits an algebraic approximation.

8.2 Some open problems

Now we know that compact Kähler manifolds X with either $\dim X = 3$ or $a(X) \geq \dim X - 1$ have algebraic approximations (Theorem 1.1 and [27, Theorem 1.1]). As current known examples of compact Kähler manifolds X answering negatively the Kodaira problem all satisfy $a(X) \leq \dim X - 4$, the following question remains open.

Question 8.2. — *Does there exist a compact Kähler manifold X of algebraic dimension $a(X) \geq \dim X - 3$ which does not have any algebraic approximation?*

Due to Voisin's examples [38], uniruled compact Kähler manifolds can fail to admit algebraic approximations in dimension at least 5. Indeed, let X be one of the Voisin's examples of dimension 4 and take the product $X \times \mathbf{P}^1$. Since every deformation of $X \times \mathbf{P}^1$ induces a deformation of the projection $X \times \mathbf{P}^1 \to X$ [32, Theorem 2.1], $X \times \mathbf{P}^1$ cannot have any algebraic approximation. For uniruled fourfolds, the Kodaira problem is still open.

Question 8.3. — *Does there exist a uniruled compact Kähler fourfold X which does not have any algebraic approximation, or even the homotopy type of a projective manifold?*

Acknowledgement

The author is supported by the SFB/TR 45 "Periods, Moduli Spaces and Arithmetic of Algebraic Varieties" of the DFG (German Research Foundation) at the University of Bonn. Part of the work was carried out during his visit to the Academia Sinica, whose hospitality and financial support are greatly acknowledged.

I am grateful to D.Morrison and U. Persson for valuable discussions on the Kulikov models, to D. Huybrechts on twisted sheaves, to Y. Prokhorov and S. Schreieder on conic bundles, and to F. Gounelas, K. Oguiso, A. Soldatenkov, and C. Voisin for general discussions and e-mail correspondences related to the subject.

References

[1] Michael Artin. On the solutions of analytic equations. *Invent. Math.*, 5:277–291, 1968. 26

[2] Michael Artin and Jean-Louis Verdier. Reflexive modules over rational double points. *Math. Ann.*, 270(1):79–82, 1985. 31

[3] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van De Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2 edition, 2004. 17, 29, 33

[4] Nicholas Buchdahl. Algebraic deformations of compact Kähler surfaces. II. *Math. Z.*, 258(3):493–498, 2008. 1, 12

[5] Ragnar-Olaf Buchweitz and Hubert Flenner. The Atiyah-Chern character yields the semiregularity map as well as the infinitesimal Abel-Jacobi map. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 24 of *CRM Proc. Lecture Notes*, pages 33–46. Amer. Math. Soc., Providence, RI, 2000. 26, 28

[6] Andrei. H. Caldararu. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Cornell University. 25

[7] Frédéric Campana. Isotrivialité de certaines familles kählériennes de variétés non projectives. *Math. Z.*, 252:147–156, 2006. 14, 15

[8] Frédéric Campana and Thomas Peternell. Complex threefolds with non-trivial holomorphic 2-forms. *J. Algebraic Geom.*, 9(2):223–264, 2000. 3, 4, 16

[9] Frédéric Campana. Coréduction algébrique d'un espace analytique faiblement kählérien compact. *Invent. Math.*, 63(2):187–223, 1981. 10

[10] Junyan Cao. On the approximation of Kähler manifolds by algebraic varieties. *Math. Ann.*, 363(1-2):393–422, 2015. 1

[11] Benoît Claudon. Smooth family of tori and linear Kähler groups. To appear in Annales de la Faculté des Sciences de Toulouse (arXiv:1604.03367v2), 2016. 1, 12, 13, 23
[12] Benoît Claudon, Andreas Höring, and Hsueh-Yung Lin. The fundamental group of compact Kähler threefolds. arXiv:1612.04224, 2018. 1, 33
[13] Robert Friedman and David R. Morrison. The birational geometry of degenerations: an overview. In *The birational geometry of degenerations (Cambridge, Mass., 1981)*, volume 29 of *Progr. Math.*, pages 1–32. Birkhäuser, Boston, Mass., 1983. 15
[14] Akira Fujiki. On the blowing down of analytic spaces. *Publ. Res. Inst. Math. Sci.*, 10:473–507, 1974/75. 18
[15] Akira Fujiki. Closedness of the Douady spaces of compact Kähler spaces. *Publ. Res. Inst. Math. Sci.*, 14(1):1–52, 1978/79. 33
[16] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. 27
[17] Patrick Graf. Algebraic approximation of Kähler threefolds of Kodaira dimension zero. *Mathematische Annalen*, Aug 2017. 1, 2, 8, 9, 20, 22
[18] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Math.* Springer, 1977. 33
[19] Andreas Höring and Thomas Peternell. Bimeromorphic geometry of Kähler threefolds. arxiv:1701.01653, 2017. to appear in "Proceedings of 2015 Summer Institute on Algebraic Geometry". 4, 16
[20] Birger Iversen. *Cohomology of sheaves.* Berlin: Springer, 1986. 15
[21] Wilhelm Kaup. Infinitesimale Transformationsgruppen komplexer Räume. *Math. Ann.*, 160:72–92, 1965. 9
[22] Kunihiko Kodaira. On compact analytic surfaces. II, III. *Ann. of Math. (2) 77 (1963), 563–626; ibid.*, 78:1–40, 1963. 1
[23] Kunihiko Kodaira. On stability of compact submanifolds of complex manifolds. *Amer. J. Math.*, 85:79–94, 1963. 2
[24] Vik. S. Kulikov. Degenerations of $K3$ surfaces and Enriques surfaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(5):1008–1042, 1199, 1977. 15
[25] David I. Lieberman. Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds. In *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, volume 670 of *Lecture Notes in Math.*, pages 140–186. Springer, Berlin, 1978. 22
[26] Hsueh-Yung Lin. Algebraic approximations of fibrations in abelian varieties over a curve. *ArXiv:1612.09271*, 2016. 1, 4, 5, 6, 7, 13, 14
[27] Hsueh-Yung Lin. Algebraic approximations of compact Kähler manifolds of algebraic codimension 1. *ArXiv:1809.03344*, 2018. 1, 2, 3, 5, 6, 36
[28] Masaki Maruyama. On automorphism groups of ruled surfaces. *J. Math. Kyoto Univ.*, 11:89–112, 1971. 32, 34
[29] Masayoshi Miyanishi. Algebraic methods in the theory of algebraic threefolds—surrounding the works of Iskovskikh, Mori and Sarkisov. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 69–99. North-Holland, Amsterdam, 1983. 17, 18, 19
[30] Keiji Oguiso. Bimeromorphic automorphism groups of non-projective hyperkähler manifolds—a note inspired by C. T. McMullen. *J. Differential Geom.*, 78(1):163–191, 2008. 1
[31] Ulf Persson and Henry Pinkham. Degeneration of surfaces with trivial canonical bundle. *Ann. of Math. (2)*, 113(1):45–66, 1981. 15
[32] Ziv Ran. Stability of certain holomorphic maps. *J. Differential Geom.*, 34(1):37–47, 1991. 12, 37
[33] V. G. Sarkisov. On conic bundle structures. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(2):371–408, 432, 1982. 3, 17, 18
[34] Florian Schrack. Algebraic approximation of Kähler threefolds. *Math. Nachr.*, 285(11-12):1486–1499, 2012. 1, 19, 29, 33
[35] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. Notes written in collaboration with P. Cherenack. 22
[36] Jean-Louis Verdier. Stratifications de Whitney et théorème de Bertini-Sard. *Invent. Math.*, 36:295–312, 1976. 7
[37] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003. 12
[38] Claire Voisin. On the homotopy types of compact Kähler and complex projective manifolds. *Invent. Math.*, 157:329–343, 2004. 1, 37
[39] Claire Voisin. On the homotopy types of Kähler manifolds and the birational Kodaira problem. *J. Differential Geom.*, 72(1):43–71, 2006. 1

Hsueh-Yung Lin, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, Office 303, 53115 Bonn, Germany
E-mail : linhsueh@math.uni-bonn.de