THE ISOMORPHISM CONJECTURE FOR SOLVABLE GROUPS IN WALDHAUSEN'S A-THEORY

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ABSTRACT. We prove the A-theoretic Isomorphism Conjecture with coefficients and finite wreath products for solvable groups.

1. INTRODUCTION

Recently, there has been a growing interest in applying techniques used to prove the Isomorphism Conjecture in K- and L- theory (see for example [1, 2, 3, 4, 5, 6]) to the A-theory setting. In particular, Ullman–Winges generalized Farrell-Hsiang method [3, 9] to the A-theory setting and used it to verify the A-theory Isomorphism Conjecture for virtually poly-$\mathbb{Z}$-groups [14]. Later, together with Enkelmann– Lück–Pieper, they proved if a group is homotopy transfer reducible, then it satisfies the A-theory Isomorphism Conjecture [8]. In particular, they proved the conjecture for hyperbolic and CAT(0) groups.

Following [14, Conjecture 7.1](see also [8, Conjecture 2.12]), we formulate the Isomorphism Conjecture in A-theory as follows. Let G be a discrete group and W be a connected G-CW-complex with free G-action. Let $\mathbb{A}^{-\infty}(W)$ be the non-connective delooping of Waldhausen's algebraic K-theory of W over the orbit category. The *Isomorphism Conjecture* predicts that the *assembly map*

$$(1.1) \qquad H_n^G(E_{\mathcal{V}Cyc}G; \mathbb{A}_W^{-\infty}) \to H_n^G(G/G; \mathbb{A}_W^{-\infty}) \cong \pi_n \mathbb{A}^{-\infty}(G\backslash W),$$

which is induced by the projection map $E_{\mathcal{V}Cyc}G \to G/G$ from the classifying space for virtually cyclic subgroups to a point, is an isomorphism for all $n \in \mathbb{Z}$. We say the A-theory isomorphism Conjecture with coefficients holds for G if it holds for any such W. We say the A-theory Isomorphism Conjecture with coefficients and finite wreath products holds for G if the conjecture with coefficients holds for any wreath products $G \wr F$, where F is a finite group. Our main theorem can now be stated as follows.

Theorem 1.1. *The Isomorphism Conjecture in A-theory with coefficients and finite wreath products holds for all solvable groups.*

Remark 1.2. *As pointed out in* [8, Remark 1.4]*, our result combined with* [8, Theorem 1.1 + Corollary 6.6] *and arguments in* [12, 13] *implies that the Isomorphism Conjecture in A-theory with coefficients and finite wreath products holds for any (not necessarily co-compact) lattice in a second countable locally compact Hausdorff group with finitely many path components, the groups $GL_n(\mathbb{Q})$ and $GL_n(F(t))$ for $F(t)$ the function field over a finite field F, and all S-arithmetic groups.*

Remark 1.3. *Theorem 1.1 combined with* [8, Theorem 1.1] *and arguments due to Gandini–Meinert–Rüping in* [11] *imply that any fundamental groups of graphs of abelian groups satisfy the Isomorphism Conjecture in A-theory with coefficients and finite wreath products. In particular all Baumslag-Solitar groups satisfy the conjecture.*

Remark 1.4. *Independently, Kasprowski–Ullmann–Wegner–Winges* [7] *also obtained a proof for Theorem 1.1.*

Our proof is based on Wegner's paper [15]. The proof can not be directly taken over due to the fact that one has to change hyper-elementary subgroups in Farrell–Hsiang method (for K- and L- theory [3]) to Dress subgroups (for A-theory [14]). The paper is organized as follows. In Section 2, we generalize Wegner's Farrell-Hsiang-Jones group to the A-theory setting and prove it satisfies the A-theory Isomorphism Conjecture. In section 3, we embed the group $\mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}^*$ (we only need to study these groups due to Lemma 4.1) into a slightly bigger group and study finite quotients of it. We show that the Dress subgroups inside these finite quotients have large index. In the last section, we prove our main theorem based on work of Wegner [15].

In this paper, when we say a group satisfies the full Isomorphism Conjecture in A-theory we mean it satisfies the Isomorphism Conjecture with coefficients and finite wreath products in A-theory.

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2. Dress–Farrell–Hsiang–Jones group

In this section, we generalize Wegner's Farrell–Hsiang–Jones group to Dress-Farrell–Hsiang–Jones group and show that it satisfies the Isomorphism Conjecture in A-theory. Dress–Farrell–Hsiang–Jones group is a combination of the Dress–Farrell–Hsiang group [14] and homotopy transfer reducible group [8, Defintion 6.2].

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*multiplication in $\mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}$ is given by $(x_1, y_1)(x_2, y_2) = (x_1 + w^{y_1} x_2, y_1 + y_2)$, where $(x_i, y_i) \in \mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}$.

2.1. **Dress-Farrell-Hsiang group.** We follow Ullmann and Winges' terminology [14], define Dress group as follows,

Definition 2.1. *A finite group D is a Dress group if there are primes p_1 and p_2 and a normal series $D_1 \lhd D_2 \lhd D$ such that D_1 is a p_1-group, D_2/D_1 is cyclic and D/D_2 is a p_2-group.*

Note that the primes p_1 and p_2 in Definition 2.1 need not be distinct.

Lemma 2.2. *In the definition of Dress group, we can always assume the order satisfies*

$$(|D_1|, |D_2/D_1|) = (|D/D_2|, |D_2/D_1|) = 1.$$

In particular $D_2 \cong D_1 \rtimes \mathbb{Z}/|D_2/D_1|$.

Proof   We first show we can assume $(|D_1|, |D_2/D_1|) = 1$. Suppose $(|D_1|, |D_2/D_1|) \neq 1$, then it must be a power of p_1 since D_1 is a p_1-group. Assume $(|D_1|, |D_2/D_1|) = p_1^k$ for some $k \geq 1$ and let the projection from D_2 to D_2/D_1 by π_1. Then $D_1' = \pi_1^{-1}(\frac{|D_2/D_1|}{p_1^k} D_2/D_1)$ is a normal subgroup of D_2 such that $(|D_1'|, |D_2/D_1'|) = 1$, D_1' is a p_1-group, D_2/D_1' is cyclic. Since D_2/D_1' is cyclic and $(|D_1'|, |D_2/D_1'|) = 1$, the projection map $D_2 \to D_2/D_1'$ splits and we have $D_2 \cong D_1' \rtimes \mathbb{Z}/|D_2/D_1'|$.

Now we assume $D_2 \cong D_1 \rtimes \mathbb{Z}/|D_2/D_1|$ and $(|D_1|, |D_2/D_1|) = 1$. Note that D_1 is a characteristic subgroup of D_2. Hence it is a normal subgroup of D. We have the following short exact sequence:

$$1 \longrightarrow D_2/D_1 \longrightarrow D/D_1 \longrightarrow D/D_2 \longrightarrow 1$$

Now D_2/D_1 is a cyclic group and D/D_2 is a p_2-group. If $(|D/D_2|, |D_2/D_1|) \neq 1$, then, $(|D/D_2|, |D_2/D_1|) = p_2^l$ for some $l > 0$. This means $|D_2/D_1| = p_2^l s$ where $(p_2, s) = 1$. Now $|D_2/D_1| = \mathbb{Z}/(p_2^l s) \cong \mathbb{Z}/p_2^l \times \mathbb{Z}/s$. Since $\mathbb{Z}/p_2^l$ and $\mathbb{Z}/s$ are characteristic subgroup of $\mathbb{Z}/(p_2^l s)$, we have a characteristic subgroup of order s in D_2/D_1 which is normal in D/D_2. Let the corresponding group in D_2 be D_2', then we would have $(|D/D_2'|, |D_2'/D_1|) = 1$. The new D_2' is a subgroup of D_2 which also has the property that $(D_1', D_2'/D_1') = 1$ for the corresponding D_1' constructed in the first paragraph.

□

Recall the definition of the ℓ^1-*metric* on a simplicial complex. If X is a simplicial complex and $\xi = \sum_x \xi_x \cdot x, \eta = \sum_x \eta_x \cdot x$ are points in X, this metric is given by $d^{\ell^1}(\xi, \eta) = \sum_x |\xi_x - \eta_x|$. All simplicial complexes we consider are equipped with this metric.

We call a generating set S of a group G *symmetric* if $s \in S$ implies $s^{-1} \in S$. The following group is defined by Ullmann and Winges in [14, Section 1].

Definition 2.3. *Let G be a group and S be a symmetric, finite generating set of G. Let $\mathcal{F}$ be a family of subgroups of G.*

Call (G, S) a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$ *if there exists $N \in \mathbb{N}$ such that for every $\epsilon > 0$ there is an epimorphism $\pi \colon G \twoheadrightarrow F$ onto a finite group F such that the following holds: For every Dress subgroup $D \subset F$, there are a $\overline{D} := \pi^{-1}(D)$-simplicial complex E_D of dimension at most N whose isotropy groups lie in $\mathcal{F}$, and a $\overline{D}$-equivariant map $\phi_D \colon G \to E_D$ such that $d^{\ell^1}(\phi_D(g), \phi_D(g')) \leq \epsilon$ whenever $g^{-1}g' \in S$.*

Ullmann and Winges proves the following [14, Theorem 7.4].

Theorem 2.4. *Let G be a discrete group. Suppose that there is a symmetric, finite generating set $S \subset G$ and a family of subgroups $\mathcal{F}$ of G such that (G, S) is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$. Then G satisfies the Isomorphism Conjecture in A-theory with coefficients with respect to $\mathcal{F}$.*

2.2. **Homotopy transfer reducible groups.**

Definition 2.5. [8, Definition 6.1] *A homotopy coherent G-action of a group G on a topological space X is a continuous map*

$$\Psi : \coprod_{j=0}^{\infty} ((G \times [0,1])^j \times G \times X) \to X$$

with the following properties:

$$\Psi(\gamma_k, t_k, \ldots, \gamma_1, t_1, \gamma_0, x) = \begin{cases} \Psi(\ldots, \gamma_j, \Psi(\gamma_{j-1}, \ldots, x)) & t_j = 0 \\ \Psi(\ldots, \gamma_j \gamma_{j-1}, \ldots, x) & t_j = 1 \\ \Psi(\gamma_k, \ldots, \gamma_2, t_2, \gamma_1, x) & \gamma_0 = e \\ \Psi(\gamma_k, \ldots, t_{j+1}t_j, \ldots, \gamma_0, x) & \gamma_j = e, 1 \leq j < k \\ \Psi(\gamma_{k-1}, t_{k-1}, \ldots, t_1, \gamma_0, x) & \gamma_k = e \\ x & \gamma_0 = e, k = 0 \end{cases}$$

Definition 2.6. [8, Definition 6.8] *Let (X, d_X) be a metric space, Ψ a homotopy coherent G-action on X, and $S \subset G$ a finite subset containing the trivial element. Let $k \in \mathbb{N}$ and $\Lambda > 0$. Define*

$$d_{S,k,\Lambda}((x, g), (y, h)) \in [0, \infty]$$

to be the infimum over the numbers

$$l + \sum_i \Lambda \cdot d_X(x_i, z_i),$$

where the infimum is taken over all $l \in \mathbb{N}$, $x_0, \ldots, x_l, z_0, \ldots, z_l \in X$ and $a_1, \ldots, a_l, b_1, \ldots, b_l \in S$ such that

(a) *$x_0 = x$ and $z_l = y$;*
(b) *$g a_1^{-1} b_1 \ldots a_l^{-1} b_l = h$;*

(c) *for each $1 \leq i \leq l$ there are elements $r_0, \ldots, r_k, s_0, \ldots, s_k \in S$ such that $a_i = r_k \ldots r_1$, $b_i = s_k \ldots s_1$ and $\Psi(r_k, t_k, \ldots, r_0, z_{i-1}) = \Psi(s_k, u_k, \ldots, s_0, x_i)$ for some $t_1, \ldots, t_k, u_1, \ldots, u_k \in [0, 1]$.*

If no such data exist, take the infimum to be ∞.

Definition 2.7. *Let G be a discrete group. Let $S \subset G$ be a finite, symmetric generating set of G which contains the trivial element and S^n be the ball of radius n around the identity element under the word metric of G with respect to S. Let $\mathcal{F}$ be a family of subgroups of G.*

Then G is homotopy transfer reducible over $\mathcal{F}$ *if there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there are*

  (i) *a compact, contractible metric space (X, d_X) such that for every $\epsilon > 0$ there is an ϵ-controlled domination of X by an at most N-dimensional, finite simplicial complex.*
 (ii) *a homotopy coherent G-action Ψ on X.*
(iii) *a G-simplicial complex Σ of dimension at most N whose isotropy is contained in $\mathcal{F}$.*
 (iv) *a positive real number Λ;*
  (v) *a G-equivariant map $\phi \colon X \times G \to \Sigma$ such that*

$$n \cdot d^{\ell^1}(\phi(x, g), \phi(y, h)) \leq d_{S^n, n, \Lambda}((x, g), (y, h))$$

holds for all $(x, g), (y, h) \in X \times G$.

Remark 2.8. *Note our definition of homotopy transfer reducible differs slightly from the one in* [8, Definition 6.2]*, but as in the proof of* [8, Theorem 6.19] *shows, our definition implies the definition there.*

Theorem 2.9. [8, Theorem 6.14] *Let G be a discrete group and let $\mathcal{F}$ be a family of subgroups of G. If G is homotopy transfer reducible over $\mathcal{F}$, then G satisfies the Isomorphism Conjecture with coefficients in A-theory with respect to $\mathcal{F}$.*

2.3. **Category of controlled retractive G-CW-complexes.** Let G be a discrete group and $\mathcal{F}$ be a family of subgroups of G. In [14, Corollary 6.11] it was shown that the Isomorphism Conjecture with coefficients holds for G iff the algebraic K-theory of $\mathfrak{R}_f^G(W, \mathbb{J}(G, E_{\mathcal{F}}(G)), h)$ vanishes for every free G-CW-complex W, so it plays the role of obstruction category as in the proof of the Isomorphism Conjecture for algebraic K-theory, see for example [5, Section 3]. Most of the material here are directly borrowed from [8, Section 6.2] which in turn come from [14].

Definition 2.10. [14, Definition 2.1] *Let Z be a G-space which is Hausdorff. A set of morphism control conditions $\mathfrak{Z}$ is a collection of G-invariant subsets of $Z \times Z$ with the following properties:*

(C1) *Every $C \in \mathfrak{C}$ contains the diagonal $\Delta(Z) := \{(z, z) \mid z \in Z\}$.*

(C2) *Every $C \in \mathfrak{C}$ is symmetric.*

(C3) *For all $C, C' \in \mathfrak{C}$ there is some $C'' \in \mathfrak{C}$ such that $C \cup C' \subset C''$.*

(C4) *For all $C, C' \in \mathfrak{C}$ there is some $C'' \in \mathfrak{C}$ such that $C' \circ C \subset C''$, where the composition $C' \circ C$ is defined as*

$$C' \circ C := \{(z'', z) \mid \exists z' \colon (z', z) \in C, (z'', z') \in C'\}.$$

A set of object support conditions *$\mathfrak{S}$ is a collection of G-invariant subsets of Z with the following properties:*

(S1) *For all $S, S' \in \mathfrak{S}$ there is some $S'' \in \mathfrak{S}$ such that $S \cup S' \subset S''$.*

The triple $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ is called a coarse structure.

Fix a coarse structure $\mathfrak{Z}$. A *labeled G-CW-complex relative W* [14, Definition 2.3], is a pair (Y, κ), where Y is a free G-CW-complex relative W together with a G-equivariant function $\kappa \colon \diamond Y \to Z$. Here, $\diamond Y$ denotes the (discrete) set of relative cells of Y.

A *$\mathfrak{Z}$-controlled map $f \colon (Y_1, \kappa_1) \to (Y_2, \kappa_2)$* is a G-equivariant, cellular map $f \colon Y_1 \to Y_2$ relative W such that for all $k \in \mathbb{N}$ there is some $C \in \mathfrak{C}$ for which

$$(\kappa_2, \kappa_1)(\{(e_2, e_1) \mid e_1 \in \diamond_k Y_1, e_2 \in \diamond Y_2, \langle f(e_1) \rangle \cap e_2 \neq \emptyset\}) \subseteq C$$

holds, where $\diamond_k Y_1$ denotes the set of relative k-cells of Y.

A *$\mathfrak{Z}$-controlled G-CW-complex relative W* is a labeled G-CW-complex (Y, κ) relative W, such that the identity is a $\mathfrak{Z}$-controlled map and for all $k \in \mathbb{N}$ there is some $S \in \mathfrak{S}$ such that

$$\kappa(\diamond_k Y) \subseteq S.$$

A *$\mathfrak{Z}$-controlled retractive space relative W* is a $\mathfrak{Z}$-controlled G-CW-complex (Y, κ) relative W together with a G-equivariant retraction $r \colon Y \to W$, i.e., a left inverse to the inclusion $W \hookrightarrow Y$. The $\mathfrak{Z}$-controlled retractive spaces relative W form a category $\mathcal{R}^G(W, \mathfrak{Z})$ in which morphisms are $\mathfrak{Z}$-controlled maps which additionally respect the chosen retractions.

The category of controlled G-CW-complexes (relative W) and controlled maps admits a notion of *controlled homotopies*[14, Definition 2.5] via the objects $(Y \diagdown [0, 1], \kappa \circ pr_Y)$, where $Y \diagdown [0, 1]$ denotes the reduced product which identifies $W \times [0, 1] \subseteq Y \times [0, 1]$ to a single copy of W and $pr_Y \colon \diamond Y \diagdown [0, 1] \to \diamond Y$ is the canonical projection. In particular, we obtain a notion of *controlled homotopy equivalence* (or *h-equivalence*).

A $\mathfrak{Z}$-controlled retractive space (Y, κ) is called *finite* [14, Definition 3.3] if it is finite-dimensional, the image of $Y \backslash W$ under the retraction meets the orbits of only finitely many path components of W and for all $z \in Z$ there is some open neighborhood U of z such that $\kappa^{-1}(U)$ is finite.

A $\mathfrak{Z}$-controlled retractive space (Y, κ) is called *finitely dominated*, if there are a finite $\mathfrak{Z}$-controlled, retractive G-CW-complex D relative to W, a morphism $p \colon D \to Y$ and a $\mathfrak{Z}$–controlled map $i \colon Y \to D$ such that $p \circ i \simeq_{\mathfrak{Z}} \mathrm{id}_Y$ as $\mathfrak{Z}$-controlled maps.

The finite, respectively finitely dominated, $\mathfrak{Z}$-controlled retractive spaces form full subcategories $\mathcal{R}_f^G(W, \mathfrak{Z}) \subset \mathcal{R}_{fd}^G(W, \mathfrak{Z}) \subset \mathcal{R}^G(W, \mathfrak{Z})$. All three of these categories support a Waldhausen category structure in which inclusions of G-invariant subcomplexes up to isomorphism are the cofibrations and controlled homotopy equivalences are the weak equivalences [14, Corollary 3.22].

Let $\mathfrak{Z}_1 = (Z_1, \mathfrak{C}_1, \mathfrak{S}_1)$, $\mathfrak{Z}_2 = (Z_2, \mathfrak{C}_2, \mathfrak{S}_2)$ be two coarse structures. A *morphism of coarse structures* [14, Definition 3.23] $z \colon \mathfrak{Z}_1 \to \mathfrak{Z}_2$ is a G-equivariant map of sets $z \colon Z_1 \to Z_2$ satisfying the following properties:

(a) For every $S_1 \in \mathfrak{S}_1$, there is some $S_2 \in \mathfrak{S}_2$ such that $z(S_1) \subset S_2$.

(b) For every $S \in \mathfrak{S}_1$ and $C_1 \in \mathfrak{C}_1$, there is some $C_2 \in \mathfrak{C}_2$ such that $(z \times z)((S \times S) \cap C_1) \subset C_2$.

(c) For every $S \in \mathfrak{S}_1$ and all subsets $A \subset S$ which are locally finite in Z_1, the set $z(A)$ is locally finite in Z_2 and for all $x \in z(A)$, the set $z^{-1}(x) \cap A$ is finite.

Morphism of coarse structures induce morphism of corresponding categories of controlled G-CW-complexes (relative to W), see [14, Proposition 3.24] for more details.

Let X be a G-CW-complex and let M be a metric space with free, isometric G-action. Define $\mathfrak{C}_{bdd}(M)$ to be the collection of all subsets $C \subset M \times M$ which are of the form

$$C = \{(m, m') \in M \times M \mid d(m, m') \leq \alpha\}$$

for some $\alpha \geq 0$. Define further $\mathfrak{C}_{Gcc}(X)$ to be the collection of all $C \subset (X \times [1, \infty[) \times (X \times [1, \infty[)$ which satisfy the following:

(a) For every $x \in X$ and every G_x-invariant open neighborhood U of (x, ∞) in $X \times [1, \infty]$, there exists a G_x–invariant open neighborhood $V \subset U$ of (x, ∞) such that $(((X \times [1, \infty[) \setminus U) \times V) \cap C = \emptyset$.

(b) Let $p_{[1,\infty[} \colon X \times [1, \infty[\to [1, \infty[$ be the projection map. Equip $[1, \infty[$ with the Euclidean metric. Then there exists some $B \in \mathfrak{C}_{bdd}([1, \infty[)$ such that $C \subset p_{[1,\infty[}^{-1}(B)$.

(c) C is symmetric, G–invariant and contains the diagonal.

Next define $\mathfrak{C}(M, X)$: Let $p_M \colon M \times X \times [1, \infty[\to M$ and $p_{X \times [1,\infty[} \colon M \times X \times [1, \infty[\to X \times [1, \infty[$ denote the projection maps. Then $\mathfrak{C}(M, X)$ is the collection of all subsets $C \subset (M \times X \times [1, \infty[)^2$ which are of the form

$$C = p_M^{-1}(B) \cap p_{X \times [1,\infty[}^{-1}(C')$$

for some $B \in \mathfrak{C}_{bdd}(M)$ and $C' \in \mathfrak{C}_{Gcc}(X)$.

Finally, define $\mathfrak{S}(M, X)$ to be the collection of all subsets $S \subset M \times X \times [1, \infty[$ which are of the form $S = K \times [1, \infty[$ for some G-compact subset $K \subset M \times X$.

All these data combine to a coarse structure

$$\mathbb{J}(M, X) := (M \times X \times [1, \infty[, \mathfrak{C}(M, X), \mathfrak{S}(M, X))$$

which serves to define the "obstruction category" $\mathfrak{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}(G)), h)$ [14, Example 2.2 and Definition 6.1], where $E_{\mathcal{F}}(G)$ is the classifying space for G respect to the family of subgroups $\mathcal{F}$. We will consider the non-connective K-theory spectrum of $\mathfrak{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}(G)))$ with respect to the h-equivalences[14, Section 5].

Proposition 2.11. [14, Corollary 6.11] *A group G satisfies the Isomorphism Conjecture with coefficients in A-theory with respect to $\mathcal{F}$ if and only if the algebraic K-theory of $\mathfrak{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}(G)), h)$ vanishes for every free G-CW-complex W.*

Suppose that $(M_n)_n$ is a sequence of metric spaces with a free, isometric G-action. Let X be a G-CW-complex. Following [14, Section 7], define the coarse structure

$$\mathbb{J}((M_n)_n, X) := (\coprod_n M_n \times X \times [1, \infty[, \mathfrak{C}((M_n)_n, X), \mathfrak{S}((M_n)_n, X))$$

as follows: Members of $\mathfrak{C}((M_n)_n, X)$ are of the form $C = \coprod_n C_n$ with $C_n \in \mathfrak{C}(M_n, X)$, and we additionally require that C satisfies the *uniform metric control condition*: There is some $\alpha > 0$, independent of n, such that for all $((m, x, t), (m', x', t')) \in C_n$ we have $d(m, m') < \alpha$. Members of $\mathfrak{S}((M_n)_n, X)$ are sets of the form $S = \coprod_n S_n$ with $S_n \in \mathfrak{S}(M_n, X)$. The resulting category $\mathfrak{R}^G(W, \mathbb{J}((M_n)_n, X))$ is canonically a subcategory of the product category $\prod_n \mathfrak{R}^G(W, \mathbb{J}(M_n, X))$.

We need also another class of weak equivalences on $\mathfrak{R}^G(W, \mathbb{J}((M_n)_n, E))$ to describe the target of the transfer in the proof of Theorem 2.15. These h^{fin}-equivalences were introduced in the proof of [14, Theorem 10.1]. Let $(M_n)_n$ be a sequence of metric spaces with free, isometric G-action. Let $(Y_n)_n$ be an object of $\mathfrak{R}^G(W, \mathbb{J}((M_n)_n, E))$. For $\nu \in \mathbb{N}$, we denote by $(-)_{n > \nu}$ the endofunctor which sends $(Y_n)_n$ to the sequence $(\widetilde{Y}_n)_n$ with $\widetilde{Y}_n = *$ for $n \le \nu$ and $\widetilde{Y}_n = Y_n$ for $n > \nu$. A morphism $(f_n)_n : (Y_n)_n \to (Y'_n)_n$ is an h^{fin}-*equivalence* if there is some $\nu \in \mathbb{N}$, such that $(f_n)_{n > \nu} : (Y_n)_{n > \nu} \to (Y'_n)_{n > \nu}$ is an h-equivalence. To distinguish them, we will denote the one with h-equivalence by $\mathfrak{R}^G(W, \mathbb{J}((M_n)_n, E), h)$ and the new one with h^{fin}-equivalence by $\mathfrak{R}^G(W, \mathbb{J}((M_n)_n, E), h^{fin})$.

2.4. **Dress–Farrell–Hsiang–Jones group.** In this subsection we introduce and study Dress-Farrell-Hsiang-Jones groups. They arise from a combination of the Farrell-Hsiang method and transfer reducibility.

Definition 2.12 (DFHJ group). *Let G be a finitely generated group and let $\mathcal{F}$ be a family of subgroups. Let $S \subseteq G$ be a finite symmetric subset which generates G and contains the trivial element $e \in G$. We call G a* Dress–Farrell–Hsiang–Jones group *with respect to the family $\mathcal{F}$ if there exist a natural number N and surjective homomorphisms $\alpha_n : G \to F_n$ $(n \in \mathbb{N})$ onto finite groups F_n such that the following condition is satisfied. For any Dress subgroup D of F_n there exist*

- *a compact, contractible, controlled N-dominated metric space $X_{n,D}$,*
- *a homotopy coherent G-action $\Psi_{n,D}$ on $X_{n,D}$,*

- *a positive real number $\Lambda_{n,D}$,*
- *a simplicial complex $E_{n,D}$ of dimension at most N with a simplicial $\alpha_n^{-1}(D)$-action whose stabilizers belong to $\mathcal{F}$,*
- *and an $\alpha_n^{-1}(D)$-equivariant map $f_{n,D} \colon G \times X_{n,D} \to E_{n,D}$*

such that

$$n \cdot d_{E_{n,D}}^1 (f_{n,D}(g,x), f_{n,D}(h,y)) \leq d_{\Psi_{n,D},S^n,n,\Lambda_{n,D}}((g,x),(h,y))$$

for all $(g,x),(h,y) \in G \times X_{n,D}$ with $h^{-1}g \in S^n$, where $d_{\Psi_{n,D},S^n,n,\Lambda_{n,D}}$ is the metric on $G \times X_{n,D}$ induced from the homotopy coherent G-action $\Psi_{n,D}$ on $X_{n,D}$ (Definition 2.6).

Example 2.13.   (a) *Every Dress-Farrell-Hsiang group is a DFHJ group (Choose $X_{n,H}$ as a point).*

(b) *If a group G is homotopy transfer reducible over $\mathcal{F}$ then G is a DFHJ group with respect to $\mathcal{F}$ (Choose F_n as the trivial group).*

Lemma 2.14. *Let G be a DFHJ group with respect to the family $\mathcal{F}_G := \{H < G \mid H$ satisfies the full Isomorphism Conjecture in A-theory$\}$. Let F be a finite group. Then the wreath product $G \wr F$ is a DFHJ group with respect to the family $\mathcal{F}_{G \wr F} := \{H < G \wr F \mid H$ satisfies satisfies the full Isomorphism Conjecture in A-theory $\}$.*

Proof   The proof follows exactly the same as proof of [15, Lemma 4.3] since Dress group is also closed under taking subgroups and quotients.   □

Theorem 2.15. *Let G be a DFHJ group with respect to the family of subgroups $\mathcal{F}$, then G satisfies the A-theoretic Isomorphism Conjecture with coefficients with respect to the family $\mathcal{F}$.*

Proof   The proof follows closely to [15, Proposition 4.9]. We fix a finite symmetric generating subset $S \subseteq G$ which contains the trivial element $e \in G$. We denote by d_G the word metric with respect to $S \setminus \{e\}$. Since G is a DFHJ group with respect to $\mathcal{F}$, there exist a natural number N and surjective homomorphisms $\alpha_n \colon G \to F_n$ ($n \in \mathbb{N}$) with F_n a finite group such that the following condition is satisfied. For any Dress subgroup D of F_n there exist

- a compact, contractible, controlled N-dominated metric space $X_{n,D}$,
- a homotopy coherent G-action $\Psi_{n,D}$ on $X_{n,D}$,
- a positive real number $\Lambda_{n,D}$,
- a simplicial complex $E_{n,D}$ of dimension at most N with a simplicial $\alpha_n^{-1}(D)$-action whose stabilizers belong to $\mathcal{F}$,
- and an $\alpha_n^{-1}(D)$-equivariant map $f_{n,D} \colon G \times X_{n,D} \to E_{n,D}$

such that

$$n \cdot d_{E_{n,D}}^1 (f_{n,D}(g,x), f_{n,D}(h,y)) \leq d_{\Psi_{n,D},S^n,n,\Lambda_{n,D}}((g,x),(h,y))$$

for all $(g, x), (h, y) \in G \times X_{n,D}$ with $h^{-1}g \in S^n$. We denote by $\mathcal{D}_n$ the family of Dress subgroups of F_n and $\bar{D} := \alpha_n^{-1}(D)$. We set

$$S_n := \coprod_{D \in \mathcal{D}_n} G/\bar{D} \times G = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G$$

$$Y_n := \coprod_{D \in \mathcal{D}_n} G/\bar{D} \times G \times X_{n,D} = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G \times X_{n,D}$$

$$\Sigma_n := \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} E_{n,D} \times G$$

Here for example $G \times_{\bar{D}} G$ means $G \times G$ module out the relation $(g, h) = (gr^{-1}, rh)$, $r \in \bar{D}$ and we can identify $G \times_{\bar{D}} G$ with $G/\bar{D} \times G$ via sending (g, g') to $(g\bar{D}, gg')$ (inverse given by mapping (gD, g') to $(g, g^{-1}g')$). We will use the quasi-metrics on $S_n = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G$ and $Y_n = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G \times X_{n,D}$ given by

$$d_G((g, h)_D, (g', h')_{D'}) \quad := \quad d_G(gh, g'h'),$$
$$d_{Y_n}((g, h, x)_D, (g', h', x')_{D'}) \quad := \quad d_{\Psi_{n,D}, S^n, n, \Lambda_{n,D}}((gh, x), (g'h', x')),$$

if $D = D'$, $g\bar{D} = g'\bar{D}'$. Otherwise, we set

$$d_{S_n}((g, h)_D, (g', h')_{D'}) := \infty \quad \text{and} \quad d_{Y_n}((g, h, x)_D, (g', h', x')_{D'}) := \infty.$$

If we define the quasi-metric on $\coprod_{D \in \mathcal{D}_n} G/D$ by assigning distance ∞ to any points that are not equal. Then $d_{S_n} = d_{\coprod_{D \in \mathcal{D}_n} G/D} + d_G$ and $d_{Y_n} = d_{\coprod_{D \in \mathcal{D}_n} G/D} + d_{\Psi_{n,D}, S^n, n, \Lambda_{n,D}}$. We define the action of G on $S_n = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G$ via $r(g, h) = (rg, h)$ and the action on $Y_n = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G \times X_{n,D}$ by $r(g, h, x) = (rg, h, x)$ for any $r \in G$.

Similarly, we define the quasi-metric on Σ_n to be $nd^{l^1}_{G \times_{\bar{D}} E_{n,D}} + d_G$ and G acts on $\Sigma_n = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} E_{n,D} \times G$ by $r(g, x, h) = (rg, x, rh)$. We define the map

$$f_n := \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G \times X_{n,D} \to \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} E_{n,D} \times G : (g, h, x) \to (g, f_{n,D}(h, x), gh)$$

By Proposition 2.11, it suffices to show the K-theory of the obstruction category

$$(\mathfrak{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}G)), h)$$

vanishes. This is a direct consequence of the following commuting diagram (abbreviate $E_{\mathcal{F}} := E_{\mathcal{F}}G$)

$$
\begin{array}{ccc}
& \xleftarrow{\quad tr_1 \quad} & \\
(\mathfrak{R}_f^G(W,\mathbb{J}(G,E_{\mathcal{F}})),h) & \xleftarrow{\quad P_j \quad} & (\mathfrak{R}_f^G(W,\mathbb{J}((S_n)_n,E_{\mathcal{F}})),h) \\
\downarrow{\scriptstyle \Delta} & \nearrow{\scriptstyle P_{S_n\to G}^f} & \downarrow{\scriptstyle incl_2} \\
(\mathfrak{R}_f^G(W,\mathbb{J}((G)_n,E_{\mathcal{F}})),h) & & (\mathfrak{R}_{fd}^G(W,\mathbb{J}((S_n)_n,E_{\mathcal{F}})),h) \\
\downarrow{\scriptstyle incl_1} & & \downarrow{\scriptstyle i_2} \\
(\mathfrak{R}_{fd}^G(W,\mathbb{J}((G)_n,E_{\mathcal{F}})),h) & \xrightarrow{\quad P_{S_n\to G}\quad} & (\mathfrak{R}_{fd}^G(W,\mathbb{J}((S_n)_n,E_{\mathcal{F}})),h^{fin}) \\
\downarrow{\scriptstyle i_1} & \nearrow{\scriptstyle P_{Y_n\to G}} & \\
(\mathfrak{R}_{fd}^G(W,\mathbb{J}((G)_n,E_{\mathcal{F}})),h^{fin}) & \xleftarrow{\quad} & (\mathfrak{R}_{fd}^G(W,\mathbb{J}((Y_n)_n,E_{\mathcal{F}})),h^{fin}) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle F} \\
(\mathfrak{R}_{fd}^G(W,\mathbb{J}((G)_n,E_{\mathcal{F}})),h^{fin}) & \xleftarrow{\quad P_{\Sigma_n\to G}\quad} & (\mathfrak{R}_{fd}^G(W,\mathbb{J}((\Sigma_n)_n,E_{\mathcal{F}})),h^{fin})
\end{array}
$$

Here the map $incl_1$, $incl_2$, i_1 and i_2 are all inclusions of categories. Δ is the diagonal map. The maps $P_{*\to G}$ (with or without f decoration) are induced by the obvious projections. The map P_j is defined to be the projection functor which takes the inclusion into the full product category, projects onto the j-th component, and then then apply the function induced by the projections $\coprod_{D\in\mathcal{D}_j} G/\bar{D} \times G \times E_{\mathcal{F}} \times [0,\infty[\to G \times E_{\mathcal{F}} \times [0,\infty[$.

The theorem now follows once we show the following

(i) In map $incl_1$ and $incl_2$ induces isomorphism after applying K-theory;

(ii) The composition $i_1 \circ incl_1 \circ \Delta$ induces injective map on K-theory;

(iii) The K-theory of $(\mathfrak{R}_{fd}^G(W,\mathbb{J}((\Sigma_n)_n,E_{\mathcal{F}})),h^{fin})$ vanishes.

(iv) The map tr_1 exists after applying K-theory and the composition $P_j \circ tr_1$ induces identity in K-theory for any j, in particular tr_1 induces injective maps in K-theory. Moreover, the composition $P_{S_n\to G}^f \circ tr_1$ and Δ induces the same map in K-theory;

(v) The functor F, defined as a restriction of

$$\prod_{n\in\mathbb{N}} f_n : (\prod_{n\in\mathbb{N}}\mathfrak{R}_{fd}^G(W,\mathbb{J}((Y_n)_n,E_{\mathcal{F}})),h^{fin}) \to (\prod_{n\in\mathbb{N}}\mathfrak{R}_{fd}^G(W,\mathbb{J}((\Sigma_n)_n,E_{\mathcal{F}})),h^{fin}),$$

is well defined.

(vi) The map tr_2 exists after applying K-theory. Moreover, the composition $P_{Y_n\to G} \circ tr_2$ and $P_{S_n\to G} \circ i_2 \circ incl_2$ induces the same map in K-theory.

We will prove these statements in positive degrees, for non-positive ones, one has to consider further deloopings of the category, but the proof works in the same way, see [14, section 5] and [8, Section 6.6] for more details. We proceed to prove the positive degree case as follows

(i) This is stated in [14, Remark 5.5];

(ii) This is proved in [8, Lemma 6.18] notice that Δ and $incl_1$ commutes;

(iii) This is proved in [8, Proposition 6.15(iii)], which follows from the "squeezing Theorem" in A-theory [14, Theorem 10.1];

(iv) The first part is proved in [14, Corollary 9.3]. For the second part, one can argue the same as in [14, Proposition 9.2(2)] note that the map $P^f_{S_n \to G}$ only changes the controlled conditions on the object.

(v) This is [8, Proposition 6.15(ii)];

(vi) The map tr_2 can be defined as in [8, Section 7] with some modifications. We explain now how this should proceed based on the terminology and proof there. Firstly by [8, Remark 7.3], it suffices to define a transfer functor

$$tr_2^{\alpha,d} : (\mathfrak{R}_f^G(W, \mathbb{J}((S_n)_n, E_{\mathcal{F}}))_{\alpha,d}, h) \to (\mathfrak{R}_{fd}^G(W, \mathbb{J}((Y_n)_n, E_{\mathcal{F}})), h^{fin})$$

such that the corresponding induced diagram is homotopy commutative on K-theory. Now note that $S_n = \coprod_{D \in \mathcal{D}_n} G \times_{\bar{D}} G = \coprod_{D \in \mathcal{D}_n} G/\bar{D} \times G$. For each $D \in \mathcal{D}_n$, we have homotopy coherent G-action $\Psi_{n,D}$ on a space $X_{n,D}$ satisfies conditions of [8, Definition 6.2] by Remark 2.8. Thus we have a transfer map [8, Lemma 7.12] $tr_2^{\alpha,d}|_{n,D}$ such that $P_{G \times X_{n,D} \to G} \circ tr_2^{\alpha,d}|_{n,D}$ and $incl_2$ coincides after applying K-theory [8, Proposition 7.24]

$$
\begin{array}{c}
(\mathfrak{R}_f^G(W, \mathbb{J}((g\bar{D} \times G), E_{\mathcal{F}}))_{\alpha,d}, h) \\
\downarrow {\scriptstyle incl_2} \qquad\qquad \overset{tr_2^{\alpha,d}|_{n,D}}{\dashrightarrow} \\
(\mathfrak{R}_{fd}^G(W, \mathbb{J}((g\bar{D} \times G), E_{\mathcal{F}})), h) \xleftarrow{\;P_{G \times X_{n,D} \to G}\;} (\mathfrak{R}_{fd}^G(W, \mathbb{J}((g\bar{D} \times G \times X_{n,D}), E_{\mathcal{F}})), h)
\end{array}
$$

Put all these maps together, we get a transfer map (in fact an exact functor by [8, Proposition 7.25] with a little modification)

$$tr_2^{\alpha,D} : K(\mathfrak{R}_f^G(W, \mathbb{J}((\coprod_{D \in \mathcal{D}_n} G/\bar{D} \times G \times X_{n,D})_n, E_{\mathcal{F}})), h)_{\alpha,D}$$

$$\longrightarrow K(\mathfrak{R}_{fd}^G(W, \mathbb{J}((\coprod_{D \in \mathcal{D}_n} G/\bar{D} \times G)_n, E_{\mathcal{F}})), h^{fin})$$

The rest of [8, Section 7.10] tells us how to get tr_2 from $tr_2^{\alpha,D}$.

3. Dress subgroups

In this section, we embed our group $\mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}$ into a slightly bigger group $O_w \rtimes_w \mathbb{Z}$ and study Dress subgroups in its finite quotients. Part of the ideas in this section are from [10, Section 4] and [15, Section 5.6].

3.1. **The ring O_w.** We review some algebraic number theory background and notions related to the group $\mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}$ for the reader's convenience, for more details see [15, Section 5.1].

Let $w \in \bar{\mathbb{Q}}^\times$ be a non-zero algebraic number. Let O be the ring of integers in the albebraic number field $\mathbb{Q}(w)$, i.e. $O \subset \mathbb{Q}(w)$ is the subring consisting of all elements which are integral. Recall that an algebraic number is integral iff it is a root of a monic polynomial with rational integer coefficients. Note that O is a Dedekind domain, in particular every nonzero prime ideal is maximal.

For a prime ideal $\mathfrak{p} \subset O$ we denote by

$$O_\mathfrak{p} := \{\frac{x}{y} \mid x, y \in O, y \notin \mathfrak{p} \subset \mathbb{Q}(w)\}$$

the localization of O at $\mathfrak{p}$. Let $V_\mathfrak{p} : \mathbb{Q}(w)^\times \to \mathbb{Z}$ be the corresponding valuation, i.e., $xO = \mathfrak{p}^{V_\mathfrak{p}(x)} O_\mathfrak{p}$. We extend the valuation $V_\mathfrak{p}$ to $\mathbb{Q}(w)$ by the convention $V_\mathfrak{p}(0) = \infty$. Note that $O_\mathfrak{p} = \{x \in \mathbb{Q}(w) \mid V_\mathfrak{p}(x) \geq 0\}$.

A fractional ideal is a finitely generated O-submodule $\mathfrak{a} \neq 0$ of $\mathbb{Q}(w)$. Every fractional ideal possesses a unique factorization $\mathfrak{a} = \prod_\mathfrak{p} \mathfrak{p}^{\nu_\mathfrak{p}}$ with $\nu_\mathfrak{p} \in \mathbb{Z}$ and $\nu_\mathfrak{p} = 0$ for almost all prime ideals $\mathfrak{p}$. We define M_α to be the set of prime ideals appeared in the prime factorization of α. For $\mathfrak{a} = xO$ with $x \in \mathbb{Q}(w)^\times$ we have $\nu_\mathfrak{p} = V_\mathfrak{p}(x)$. We conclude that

$$M_x := M_{xO} = \{\mathfrak{p} \subset O \text{ prime ideal} \mid V_\mathfrak{p}(x) \neq 0\}$$

is a finite set. In particular, M_w is a finite set. We define the ring

$$O_w = \{x \in \mathbb{Q}(w) \mid V_\mathfrak{p}(x) \geq 0 \text{ for all prime ideals } \mathfrak{p} \notin M_w\}$$

Note that $O \subset O_w$ and $w, w^{-1} \in O_w$. The group of units in the ring O_w is given by

$$O_w^\times = \{x \in \mathbb{Q}(w) \mid V_\mathfrak{p}(x) = 0 \text{ for all prime ideals } \mathfrak{p} \notin M_w\}$$

3.2. **Order of w in a finite quotient.** We review some results in [15, Section 5.6].

Given any ideal I in O such that $M_I \cap M_w = \emptyset$, we define a nature number $t_w(I, s)$ for any $s \geq 1$ by the following:

$$t_w(I, s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \mod I^s O_w\}$$

In particular, let q be a prime number and $\mathfrak{q} \notin M_w$ be a prime ideal in O which contains q, we have

$$t_w(q, s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \mod q^s O_w\}$$

$$t_w(\mathfrak{q}, s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \mod \mathfrak{q}^s O_w\}$$

Note that if $qO = \mathfrak{q}_1^{\gamma_1} \mathfrak{q}_2^{\gamma_2} \cdots \mathfrak{q}_r^{\gamma_r}$, then $M_q = \{\mathfrak{q}_1, \mathfrak{q}_2, \cdots \mathfrak{q}_r\}$ and $t_w(q, s)$ is the least common multiple of $t_w(\mathfrak{q}_i^{\gamma_i}, s) = t_w(\mathfrak{q}_i, s\nu_i)$ for $i = 1, 2, \cdots, r$.

Lemma 3.1. [15, Lemma 5.29] *Let q be a prime number and $\mathfrak{q} \notin M_w$ be a prime ideal in O which contains q, then:*

  (i) *The ring $O_w/\mathfrak{q}O_w$ is a finite field of characteristic q which is isomorphic to $O/\mathfrak{q}O$.*

  (ii) *$t_w(\mathfrak{q}, 1)$ and q are coprime. In particular, $t_w(\mathfrak{q}, 1) \neq 0$.*

  (iii) *For every $s \in \mathbb{N}$ we have $t_w(\mathfrak{q}, s+1) = qt_w(\mathfrak{q}, s)$ or $t_w(\mathfrak{q}, s+1) = t_w(\mathfrak{q}, s)$.*

  (iv) *Let $(a, b) \in O_w/\mathfrak{q}^s O_w \rtimes_w \mathbb{Z}/t_w(\mathfrak{q}, s)$ such that $b \notin t_w(\mathfrak{q}, 1)\mathbb{Z}/t_w(\mathfrak{q}, s)$. Then (a, b) is conjugate to $(0, b)$.*

3.3. **Dress subgroups of finite quotients for $O_w \rtimes_w \mathbb{Z}$.** We define our new group Γ_w as the semi-product: $\Gamma_w = O_w \rtimes_w \mathbb{Z}$, where the multiplication in Γ_w is given by

$$(x_1, y_1)(x_2, y_2) = (x_1 + w^{y_1}x_2, y_1 + y_2).$$

for $(x_i, y_i) \in O_w \rtimes_w \mathbb{Z}$, $i = 1, 2$. Since $\mathbb{Z}[w, \frac{1}{w}] \subset O$, $\mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}$ is a subgroup of Γ_w. we want to study its finite quotient.

Lemma 3.2. *Given any integer $N > 0$, there exists primes $q_1, q_2, q_3 > N$ such that M_w, M_{q_1}, M_{q_2}, M_{q_3} are pairwise disjoint and $1 + w \notin \mathfrak{q}_i O_w$ for any $\mathfrak{q}_i \in M_{q_i}(i = 1, 2, 3)$.*

Proof Given $\mathfrak{p} \in M_w$, then $O/\mathfrak{p}O$ is a finite field. Let M be maximal of the characteristic of these finite fields. Now choose q_1, q_2, q_3 to be different primes that are bigger than N and M, they then have the properties in the lemma. Note that there are only finite many prime ideals $\mathfrak{q}$ such that $1 + w$ lies in $\mathfrak{q}O_w$, thus the last condition can also be reached by choosing q_1, q_2, q_3 big. $\qquad\square$

Remark 3.3. *Note that by our choices $1 + w$ is invertible in $O_w/\mathfrak{q}_i^s$ for any i and $s \geq 1$. In particular $1 + w$ is invertible in $O_w/(q_1^s q_2^s q_3^s)$.*

Given any $N > 0$, let q_1, q_2, q_3 be three primes choosing as in Lemma 3.2. Recall $t_w(q_1q_2q_3, s)$ is defined to be the nature number such that

$$t_w(q_1q_2q_3, s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \mod q_1^s q_2^s q_3^s O_w\}$$

Given any $\mathfrak{q}_i \in M_{q_i}$ with $v_{\mathfrak{q}_i}(q_i) = v_i \geq 1$, we also consider the nature number $t_w(\mathfrak{q}_1^{\gamma_1} \mathfrak{q}_2^{\gamma_2} \mathfrak{q}_3^{\gamma_3}, s)$ defined by

$$t_w(\mathfrak{q}_1^{\gamma_1} \mathfrak{q}_2^{\gamma_2} \mathfrak{q}_3^{\gamma_3}, s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \mod \mathfrak{q}_1^{v_1 s} \mathfrak{q}_2^{v_2 s} \mathfrak{q}_3^{v_3 s} O_w\}$$

Note that by Lemma 3.1, $O_w/\mathfrak{q}_i O_w$ is a finite field of Characteristic q_i, hence $O_w/\mathfrak{q}_i O_w$ has order some power of q_i. By the Chinese Reminder Theorem, $O_w/(\mathfrak{q}_1^{v_1 s} \mathfrak{q}_2^{v_2 s} \mathfrak{q}_3^{v_3 s}) \cong O_w/\mathfrak{q}_1^{v_1 s} \oplus O_w/\mathfrak{q}_2^{v_2 s} \oplus O_w/\mathfrak{q}_3^{v_3 s}$ and $O_w/(q_1^s q_2^s q_3^s) \cong O_w/q_1^s \oplus O_w/q_2^s \oplus O_w/q_3^s$. Hence we have $t_w(q_1q_2q_3, s)$ is the least common multiple of $t_w(q_i, s)(i = 1, 2, 3)$ and $t_w(\mathfrak{q}_1^{\gamma_1} \mathfrak{q}_2^{\gamma_2} \mathfrak{q}_3^{\gamma_3}, s)$ is the least common multiple of $t_w(\mathfrak{q}_i, v_i s)(i = 1, 2, 3)$.

Given $n > 0$, we define a number $\gamma(s, n)$ depends on s and n by

$$(3.1) \qquad\qquad \gamma(s, n) = \prod_{i=1,2,3} t_w^{n+1}(\mathfrak{q}_i, s\nu_i)$$

Now we define finite quotients of $O_w \rtimes_w \mathbb{Z}$ via the following maps.

$$O_w \rtimes_w \mathbb{Z} \xrightarrow{\ \pi_1\ } \quad O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}$$

$$\xrightarrow{\ \pi_2\ } O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n) \xrightarrow{\ Pr\ } \mathbb{Z}/\gamma(s, n)$$

where π_1 is defined by quotienting out the normal subgroup $(\mathfrak{q}_1^s \mathfrak{q}_2^s \mathfrak{q}_3^s) O_w \rtimes_w \{0\}$, π_2 is defined by quotienting out the normal subgroup $\{0\} \rtimes_w \gamma(s, n)\mathbb{Z}$ and Pr is the projection of $\mathbb{Z}/(q_1^n q_2^n q_3^n) \rtimes_w \mathbb{Z}/\gamma(s, n)$ to the second factor.

We proceed to study Dress subgroups of $O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n)$.

Lemma 3.4. *Given a positive integer n, there exist primes $q_1, q_2, q_3 > n$ and a positive integer $s > n$ such that for any $\mathfrak{q}_i \in M_{q_i}(i = 1, 2, 3)$ one of the following is true for each Dress subgroup D of $O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n)$:*

- *(i) The index $[\mathbb{Z}/\gamma(s, n), Pr(D)] \geq n$;*
- *(ii) $D \cap Q_i = \{0\}$ for some $i = 1, 2$ or 3 where Q_i denotes the q_i-Sylow subgroup of $O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \{0\}$.*

Proof   Let q_1, q_2, q_3 be the three primes determined by Lemma 3.2 with $N = n$. We now choose s big enough, so that $w^z - 1 \notin \mathfrak{q}_i^s$ for any $1 \leq z \leq t_w(\mathfrak{q}_i, 1)$ for $i = 1, 2, 3$. This implies $\frac{t_w(\mathfrak{q}_i, s)}{t_w(\mathfrak{q}_i, 1)} > 1$. Since $\frac{t_w(\mathfrak{q}_i, s)}{t_w(\mathfrak{q}_i, 1)}$ is a power of q_i by Lemma 3.1 (iii) and $q_i > n$, we have $\frac{t_w(\mathfrak{q}_i, s)}{t_w(\mathfrak{q}_i, 1)} > n$. Applying Lemma 3.1 (iii) again, we have $\frac{t_w(\mathfrak{q}_i, \nu_i s)}{t_w(\mathfrak{q}_i, 1)} > n$ since $\nu_i \geq 1$ for $i = 1, 2, 3$. In particular q_i appears in the prime factorization of $t_w(\mathfrak{q}_1^{\nu_1} \mathfrak{q}_2^{\nu_2} \mathfrak{q}_3^{\nu_3}, s)$.

Now let D be a Dress subgroup with $D_1 \lhd D_2 \lhd D$ such that D_1 is a p_1-group, D_2/D_1 is cyclic and D/D_2 is a p_2-group. By Lemma 2.2, we can assume $(p_1, |D_2/D_1|) = (p_2, |D_2/D_1|) = 1$. Since we know $\gamma(s, n)$'s prime decomposition contains all the prime factor q_i and $q_i > n$, without loss of generality, we can assume $p_1 \neq q_3$ and $p_2 \neq q_3$. In fact, we can assume $p_1 = q_1, p_2 = q_2$ otherwise property (i) in the Lemma holds.

Assume now $D_2/D_1 \cong C$, then $D_2 \cong D_1 \rtimes C$, where C is a finite cyclic group. Notice that

$$D \cap Q_3 = C \cap Q_3$$

Now let the generator of C to be $(a, b) \in O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n)$. Then

$$b \in \left(\frac{t_w^{n+1}(\mathfrak{q}_1, \nu_1 s)}{t_w^{n+1}(\mathfrak{q}_1, 1)} \cdot \frac{t_w^{n+1}(\mathfrak{q}_2, \nu_2 s)}{t_w^{n+1}(\mathfrak{q}_2, 1)} \right) \mathbb{Z}/\gamma(s, n)$$

and the order of b divides $t_w^{n+1}(\mathfrak{q}_1, 1) t_w^{n+1}(\mathfrak{q}_2, 1) t_w^{n+1}(\mathfrak{q}_3, \nu_3 s)$. Let the order of $b \in \mathbb{Z}/\gamma(s, n)$ to be k, then $D \cap Q_3$ is a cyclic group generated by $(a, b)^k$. If we assume property (i) in the Lemma does not hold, then

(3.2) $$(t_w^{n+1}(\mathfrak{q}_1, 1) t_w^{n+1}(\mathfrak{q}_2, 1) t_w^{n+1}(\mathfrak{q}_3, \nu_3 s)) : k < n$$

Note that this implies the order of b must divides $t_w(\mathfrak{q}_3, 1)$. In fact, if this is not true, then there exist a prime p such that p^l divides $t_w(\mathfrak{q}_3, 1)$, but p^l does not divide k for some $l \geq 1$. This implies $p^{(n+1)l}$ divides $t_w^{n+1}(\mathfrak{q}_3, 1)$, but p^l does not divide k, we have

$$t_w^{n+1}(\mathfrak{q}_3, 1) : k \geq p^{nl} > n$$

which is a contradiction to the inequality 3.2. Since $q_3 > n$, we have $t_w(\mathfrak{q}_3, \nu_3 s)$ also divides k otherwise propery (i) in the lemma holds.

Now we can assume b's order $k = k' t_w(\mathfrak{q}_3, \nu_3 s)$, then we have the following calculation in our group $O_w/(\mathfrak{q}_3^s \mathfrak{q}_2^s \mathfrak{q}_3^s) \rtimes_w \mathbb{Z}/\gamma(s, n)$:

$$(a, b)^k = ((a, b)^{t_w(\mathfrak{q}_3, \nu_3 s)})^{k'} = ((1 + w + \cdots w^{t_w(\mathfrak{q}_3, \nu_3 s) - 1}) b, b)^{k'}$$

Now by our choices of q_i and Remark 3.3, we have $1 + w$ is invertible in $O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s})$, thus

$$(a, b)^k = ((1 + w)^{-1}(1 - w^{t_w(\mathfrak{q}_3, \nu_3 s)}) b, b)^{k'}$$

Now $w^{t_w(\mathfrak{q}_3, \nu_3 s)} \equiv 1 \mod \mathfrak{q}_3^{\nu_3 s}$ in O_w, we have $(1 + w)^{-1}(1 - w^{t_w(\mathfrak{q}_3, \nu_3 s)}) b \in \mathfrak{q}_3^{\nu_3 s} O_w$ and $(a, b)^k \cap Q_3 = \{0\}$.

$\square$

Similar to Lemma 3.1 (iv), we have the following

Lemma 3.5. *Let q_1, q_2, q_3 be primes such that $M_w, M_{q_1}, M_{q_2}, M_{q_3}$ are pairwise disjoint and $(a, b) \in O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n)$. If $b \notin t_w(\mathfrak{q}_i, 1) \mathbb{Z}/\gamma(s, n)$ for $i = 1, 2, 3$, then (a, b) is conjugate to $(0, b)$.*

Proof Note first that $t_w(\mathfrak{q}, \nu_i s)$ divides $\gamma(s, n)$. Now if $b \notin t_w(\mathfrak{q}_i, 1) \mathbb{Z}/\gamma(s, n)$, by the proof of [15, Lemma 5.29(5)], we have $w^b - 1$ is a unit in $O_w/\mathfrak{q}_i^{\nu_i s}$ for $i = 1, 2, 3$. Hence $w^b - 1$ is a unit in $O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s})$. Now set $x := (w^b - 1)^{-1} a \in O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s})$. Then $(x, 0)(a, b)(x, 0)^{-1} = (0, b)$. $\square$

We continue to refine Lemma 3.4 to the following using Lemma 3.5.

Proposition 3.6. *Given a positive integer n, there exists primes $q_1, q_2, q_3 > n$ and a positive integer $s > n$ such that for any $\mathfrak{q}_i \in M_{q_i}$ one of the following is true for each Dress subgroup D of $O_w/(\mathfrak{q}_1^s \mathfrak{q}_2^s \mathfrak{q}_3^s) \rtimes_w \mathbb{Z}/\gamma(s, n)$:*

(i) The index $[\mathbb{Z}/\gamma(s, n), Pr(D)] \geq n$;

(ii) there exists $g \in O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n)$ such that $gDg^{-1} \subset \mathfrak{q}_i^{\nu_i s} O_w/(\mathfrak{q}_1^{\nu_1 s} \mathfrak{q}_2^{\nu_2 s} \mathfrak{q}_3^{\nu_3 s}) \rtimes_w \mathbb{Z}/\gamma(s, n)$ for some $i = 1, 2$ or 3.

Proof By Lemma 3.4, we can choose nice primes q_1, q_2, q_3 and s big such that, if the index $[\mathbb{Z}/\gamma(s,n), Pr(D)] < n$ then $D \cap Q_i \subset \mathfrak{q}_i^{\nu_i s} Q_i$ for some $i = 1, 2$ or 3 where Q_i denotes the q_i-Sylow subgroup of $O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\}$. We can further assume q_i big such that q_i lies in none of the prime ideals which appear in the prime factorization of the fractional ideal $(w^l - 1)O$ with $1 \le l \le n$ for $i = 1, 2, 3$. This guarantees that $t_w(\mathfrak{q}_i, 1) > n$ for $\mathfrak{q}_i \in M_{q_i}$.

Now Consider the projection pr restricted to D, then $pr(D)$ is a cyclic subgroup of $\mathbb{Z}/\gamma(s,n)$ and the kernel $K = D \cap O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\}$. We have the following short exact sequence:

$$(3.3) \qquad\qquad 1 \to K \to D \to Pr(D) \to 1.$$

Choose any $(a, b) \in D$ such that b generates $Pr(D)$. Assume the Dress subgroup D does not satisfy property (i) of our Proposition, then

Claim: The subgroup generated by (a, b) in D is isomorphic to $Pr(D)$ via Pr. In particular, D can be generated by elements in K and (a, b).

Let the order of b to be k. If property (i) does not hold, then $\frac{\gamma(s,n)}{k} \le n$. Similar to the arguments in the proof of Lemma 3.4 (after inequality 3.2), we have $t_w(\mathfrak{q}_i, \nu_i s)$ divides k for any $i = 1, 2, 3$. Since $1 - w$ is invertible in $O_w/\mathfrak{q}_i^{\nu_i s}$, if we let $k = t_w(\mathfrak{q}_i, \nu_i s) k_i$, then

$$(a, b)^k = ((a, b)^{t_w(\mathfrak{q}_i, \nu_i s)})^{k_i} = ((1 - w)^{-1}(1 - w^{t_w(\mathfrak{q}_i, \nu_i s)}), t_w(\mathfrak{q}_i, \nu_i s) b)^{k_i}$$

Now $w^{t_w(\mathfrak{q}_i, \nu_i s)} = 1 \in O_w/\mathfrak{q}_i^{\nu_i s}$ for $i = 1, 2, 3$. We have

$$(a, b)^k = (0, 0) \in O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \mathbb{Z}/\gamma(s,n).$$

This proves the claim. Now $b \notin t_w(\mathfrak{q}_i, 1)\mathbb{Z}/\gamma(s,n)$ for any i otherwise the index $[\mathbb{Z}/\gamma(s,n), Pr(D)] \ge n$ since by our choices $t_w(\mathfrak{q}_1, 1) > n$. This means we can conjugate (a, b) to $(0, b)$ by Lemma 3.5 via $g = (w^b - 1)^{-1} a, 0)$. Since $gKg^{-1} \in O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\}$, we have gDg^{-1} is generated by elements in $O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\}$ and $(0, b)$. Now $gDg^{-1} \cap Q_i = gKg^{-1} \cap Q_i = \{0\}$ for some $i = 1, 2$ or 3 where Q_i denotes the q_i-Sylow subgroup of $O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\}$, hence $gDg^{-1} \cap O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\} \subset \mathfrak{q}_i^{\nu_i s} O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_3^{\gamma_3 s}) \rtimes_w \{0\}$ and we have now Property (ii) in the theorem holds.

$\square$

Remark 3.7. *We summarize choices we need to make for q_1, q_2, q_3 and s such that Proposition 3.6 holds:*

(a) *q_1, q_2, q_3 are different primes bigger than n;*

(b) *M_w, M_{q_1}, M_{q_2} and M_{q_3} are pairwise disjoint;*

(c) *$1 + w \notin \mathfrak{q}_i O_w$ for any $\mathfrak{q}_i \in M_{q_i} (i = 1, 2, 3)$.*

(d) *q_i lies in none of the prime ideals which appear in the prime factorization of the fractional ideal $(w^l - 1)O$ with $1 \le l \le n$ for $i = 1, 2, 3$;*

(e) *Choose s big such that $w^z - 1 \notin \mathfrak{q}_i^s$ for any $1 \leq z \leq t_w(\mathfrak{q}_i, 1)$ for $i = 1, 2, 3$;*

Remark 3.8. *For any $(a, b) \in D$ such that b generates $Pr(D)$, the proof shows the g in (ii) can be chosen to be $(w^b - 1)^{-1}a, 0)$.*

We proceed to prove our main theorem in this section. Similar to $\gamma(s, n)$, we define a number $\bar{\gamma}(s, n)$ depends on s and n by

$$(3.4) \qquad\qquad \bar{\gamma}(s, n) = \prod_{i=1,2,3} t_w^{n+1}(q_i, s)$$

And we define another finite quotient of $O_w \rtimes_w \mathbb{Z}$ via the following maps.

$$O_w \rtimes_w \mathbb{Z} \xrightarrow{\bar{\pi}_1} O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z} \xrightarrow{\bar{\pi}_2} O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n) \xrightarrow{\bar{Pr}} \mathbb{Z}/\bar{\gamma}(s, n)$$

where $\bar{\pi}_1$ is defined by quotienting out the normal subgroup $(q_1^s q_2^s q_3^s)O_w \rtimes_w \{0\}$, $\bar{\pi}_2$ is defined by quotienting out the normal subgroup $\{0\} \rtimes_w \bar{\gamma}(s, n)\mathbb{Z}$ and $\bar{Pr}$ is the projection of $\mathbb{Z}/(q_1^n q_2^n q_3^n) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n)$ to the second factor.

The following theorem is parallel to [15, Corollary 5.31].

Theorem 3.9. *Given a positive integer n, there exists primes $q_1, q_2, q_3 > n$ and a positive integer $s > n$ such that one of the following is true for each Dress subgroup D of $O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n)$:*

(i) The index $[\mathbb{Z}/\bar{\gamma}(s, n), \bar{Pr}(D)] \geq n$;

(ii) there exists $g \in O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n)$ such that $gDg^{-1} \subset q_i^s O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n)$ for some $i = 1, 2$ or 3.

Proof   We choose q_1, q_2, q_3 under the assumptions of Remark 3.7 with condition (e) strengthened to the following: Choose s big such that $w^z - 1 \notin \mathfrak{q}_i^s$ for $\mathfrak{q}_i \in M_{q_i}$ and any $1 \leq z \leq t_w(\mathfrak{q}_i, 1)$ for $i = 1, 2, 3$. Then Proposition 3.6 holds for any such q_1, q_2, q_3 and any $\mathfrak{q}_i \in M_{q_i} (i = 1, 2, 3)$. Now let q_i's prime ideal factorization be the following:

$$q_i O = \mathfrak{q}_{i1}^{\nu_{i1}} \mathfrak{q}_{i2}^{\nu_{i2}} \cdots \mathfrak{q}_{ir_i}^{\nu_{ir_i}}$$

By the Chinese Remainder Theorem and the fact that M_w, M_{q_1}, M_{q_2} and M_{q_3} are pairwise disjoint, we have

$$O_w/(q_1^s q_2^s q_3^s) \cong \bigoplus_{1 \leq i \leq 3} O_w/q_i^s \cong \bigoplus_{1 \leq i \leq 3, 1 \leq j \leq r_i} O_w/\mathfrak{q}_{ij}^{\nu_{ij}}$$

Now let D be a Dress subgroup with $D_1 \lhd D_2 \lhd D$ such that D_1 is a p_1-group, D_2/D_1 is cyclic and D/D_2 is a p_2-group. We assume property (i) in the theorem does not hold. Similar to the proof of Lemma 3.4, without loss generality, we can assume $p_1 = q_1, p_2 = q_2$.

Now Consider the projection $\bar{pr}$ restricted to D, then $\bar{pr}(D)$ is a cyclic subgroup of $\mathbb{Z}/\bar{\gamma}(s,n)$ and the kernel $\bar{K} = D \cap O_w/(q_1^s q_2^s q_3^s) \rtimes_w \{0\}$. We have the following short exact sequence:

$$(3.5) \qquad\qquad 1 \to \bar{K} \to D \to \bar{Pr}(D) \to 1.$$

Now choose $(a,b) \in D$ such that b generates $\bar{Pr}(D)$. Note that $w^b - 1 \notin \mathfrak{q}_i O_w$ for any $\mathfrak{q}_i \in M_{q_i}$, otherwise b divides $t(\mathfrak{q}_{ij}, 1)$ and we have property (i) holds. This means $w^b - 1$ is invertible in $O_w/(q_1^s q_2^s q_3^s)$. We claim now, if we let $g = ((w^b - 1)^{-1}a, 0)$, then $gDg^{-1} \subset q_3^s O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s,n)$.

In fact, fix any two prime ideals $\mathfrak{q}_1 \in M_{q_1}$, $\mathfrak{q}_2 \in M_{q_2}$. For each $1 \le j \le r_3$, we define projections via the following maps

$$O_w/(q_1^s q_2^s q_3^s) \rtimes \mathbb{Z}/\bar{\gamma}(s,n) \xrightarrow{\;\rho_{1j}\;} O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_{3j}^{\gamma_{3j} s}) \rtimes_w \bar{\gamma}(s,n)$$

$$\xrightarrow{\;\rho_{2j}\;} O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_{3j}^{\gamma_{3j} s}) \rtimes_w \mathbb{Z}/\gamma_j(s,n) \xrightarrow{\;\bar{Pr}_j\;} \mathbb{Z}/\gamma_j(s,n)$$

where $\gamma_j(s,n)$ is the $\gamma(s,n)$ corresponding to the prime ideals $\mathfrak{q}_1, \mathfrak{q}_2$ and $\mathfrak{q}_{3j}$ defined by equation 3.1. We set $\rho_j = \rho_{2j} \circ \rho_{1j}$ and $\rho_j(a,b) = (a_j, b_j) \in O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_{ij}^{\gamma_{3j} s}) \rtimes_w \mathbb{Z}/\gamma_j(s,n)$.

Now a quotient of a Dress group is still a Dress group, thus $\rho_j(D)$ is a Dress subgroup of $O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_{3j}^{\gamma_{3j} s}) \rtimes_w \mathbb{Z}/\gamma_j(s,n)$. Consider the projection $\bar{Pr}_j$ restricted to $\rho_j(D)$, then since we assume property (i) in the theorem does not hold, we have the index $[\mathbb{Z}/\gamma_j(s,n), \rho_j(D)] \le n$. By the claim in the proof of Proposition 3.6, we have $\bar{pr}_j(\rho_j(D))$ is a cyclic subgroup of $\mathbb{Z}/\gamma_j(s,n)$ generated by b_j and (a_j, b_j) generate a cyclic subgroup in $\rho_j(D)$ isomorphic to $\bar{pr}_j(\rho_j(D))$ via $\bar{pr}_j$.

Now by Proposition 3.6 and Remark 3.8, we have

$$((w^{b_j} - 1)^{-1}a_j, 0)\rho_j(D)((w^{b_j} - 1)^{-1}a_j, 0)^{-1} \in \mathfrak{q}_3^{\gamma_{3j} s} O_w/(\mathfrak{q}_1^{\gamma_1 s} \mathfrak{q}_2^{\gamma_2 s} \mathfrak{q}_{3j}^{\gamma_{3j} s}) \rtimes_w \mathbb{Z}/\gamma_j(s,n)$$

And since $\rho_j(((w^b - 1)^{-1}a, 0)) = ((w^{b_j} - 1)^{-1}a_j, 0)$, we have for any $1 \le j \le r_3$

$$((w^b - 1)^{-1}a, 0)D((w^b - 1)^{-1}a, 0)^{-1} \subset \mathfrak{q}_{3j}^{\gamma_{3j} s} O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s,n)$$

Therefore,

$$((w^b - 1)^{-1}a, 0)D((w^b - 1)^{-1}a, 0)^{-1} \subset q_3^s O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s,n)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 3.10. *Since the map $\bar{\pi} : O_w \rtimes_w \mathbb{Z} \to O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s,n)$ is surjective, our g can be lifted to an element in $O_w \rtimes_w \mathbb{Z}$.*

4. Proof of the main theorem

We prove our main theorem in this section.

Lemma 4.1. *The full Isomorphism Conjecture in A-theroy holds for solvable groups iff it holds for $G_w = \mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z}$ where w is any non-zero algebraic number.*

Proof The proof follows exactly the same as [15, Proposition 3.3]. In fact, the proof there only uses inheritance properties of the Isomorphism Conjecture in K- and L-theory and the fact that the conjecture holds for hyerbolic and CAT(0) groups. By [8, Theorem 1.1], these inheritance properties and results also hold for the Isomorphism Conjecure in A-theory. □

Proposition 4.2. *Let $w \in \bar{\mathbb{Q}}^{\times}$ be a non-zero algebraic number of infinite order. Then the group G_w is a DFHJ group with respect to the family of subgroups that satisfy the full Isomorphism Conjecture in A-theory.*

Proof The proof again follows almost exactly the same as in [15, Proposition 5.33]. Notice first that the only difference between the definition of Dress-Farrell-Hsiang-Jones group (Definition 2.12) and Farrell-Hsiang-Jones group [15, Definition 4.1] is that one has to replace the hyper-elementary subgroup to Dress-subgroup. This issue is addressed already in Theorem 3.9. We sketch now how should the proof actually go based on the proof of [15, Proposition 5.33].

We choose N to be the number appearing in [15, Proposition 5.26]. Given $n > 0$, let $\bar{n} = 4nm_2$ and $s > \max\{1, -\log_2 B\}$, where m_2 is defined in the beginning of proof of [15, Proposition 5.33] and B is defined in [15, Proposition 5.26].

Now choose q_1, q_2, q_3 and s as in Remark 3.7 for $n = \bar{n}$ and define α_n as the composition of the following maps

$$\mathbb{Z}[w, \frac{1}{w}] \rtimes_w \mathbb{Z} \hookrightarrow O_w \rtimes_w \mathbb{Z} \to O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, \bar{n})$$

in particular $F_n = O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, \bar{n})$, where $\bar{\gamma}(s, \bar{n})$ was defined by Equation 3.4 and quotient map are defined right below that. By Theorem 3.9, we have for every Dress subgroup D in F_n

- Case (1), there exists $g \in O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n)$ such that $gDg^{-1} \subset q_i^s O_w/(q_1^s q_2^s q_3^s) \rtimes_w \mathbb{Z}/\bar{\gamma}(s, n)$ for some $i = 1, 2$ *or* 3, note that $q_i^{-s} \le 2^{-s} \le B$ and g can be lifted to an element in $O_w \rtimes_w \mathbb{Z}$ (Remark 3.10).
- Case (2), The index $[\mathbb{Z}/\bar{\gamma}(s, n), \bar{P}r(D)] \ge \bar{n} = 2nm_2$;

The rest of the proof follows as in [15, Proposition 5.33] without change. In particular, for case (1), we choose the corrsponding space $X_{n,D}$, number $\Lambda_{n,D}$ and simplicial complex $E_{n,D}$ via [15, Proposition 5.26]. For case (2), we choose $X_{n,D}$ to be a point and $E_{n,D}$ to be the real line with elements $l\mathbb{Z}$ as vertices, where $l = [\mathbb{Z}/\bar{\gamma}(s, n), \bar{P}r(D)]$. □

Now when w is a non-zero algebraic number of infinite order, by Lemma 2.14, any finite wreath product of G_w is also a DFHJ group with respect to the family of subgroups that already satisfy the full Isomorphism Conjecture. By Theorem 2.15 and the transitivity principle [14, Proposition 11.2], we have G_w satisfies the full Isomorphism Conjecture in A-theory. On the other hand, if w is a non-zero algebraic number of finite order, then G_w is virtually abelian and hence satisfies the full Isomorphism Conjecture by [14, Theorem 1.1]. Thus by Lemma 4.1, we have the full Isomorphism Conjecture in A-theory holds for all solvable groups.

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