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# $\Sigma_1$-Wellorders without Collapsing

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**Abstract** Given an uncountable cardinal $\kappa$ that satisfies $\kappa^{<\kappa} = \kappa$, we provide a forcing that is $<\kappa$-closed, has size $2^\kappa$ and is $\kappa^+$-cc (and thus in particular preserves all cofinalities), to introduce a $\Sigma_1$-definable wellorder (with parameters) of $\mathrm{H}(\kappa^+)$. This improves (and also simplifies the proof of) the main result of [HL], where such a wellorder is introduced by a forcing which potentially collapses cardinals, and where the additional requirement that $2^\kappa$ be regular is needed.

As an application, we use this to infer that $\Sigma_1$-definable wellorderings (using parameters) of $\mathrm{H}(\kappa^+)$ can be introduced for many different cardinals $\kappa$ simultaneously, while preserving a lot of ground model structure, improving results of [FH] and [FL].

Moreover the results of this paper answer [HL, Questions 5.2 and 5.3].

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## 1 Introduction

The work in this paper is strongly based on and improves the following:

**Theorem 1** *[HL, Theorem 1.1] Let $\kappa$ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$ and $2^\kappa$ is regular.*[1] *Then there is a partial order $\mathbb{P}$ with the following properties.*

*(i) $\mathbb{P}$ is $<\kappa$-closed and forcing with $\mathbb{P}$ preserves cofinalities less than or equal to $2^\kappa$ and the value of $2^\kappa$.*

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[1] Note that every cardinal $\kappa$ with $\kappa = \kappa^{<\kappa}$ is regular.

*(ii)* *If* $G$ *is* $\mathbb{P}$*-generic over the ground model* $\mathrm{V}$*, then there is a well-ordering of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *that is definable over the structure* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ *by a* $\Sigma_1$*-formula with parameters.*

However ([HL, Proposition 2.12]) forcing with $\mathbb{P}$ collapses $2^{<2^\kappa}$ to $2^\kappa$ (if they were different in the first place) and it is not known whether it preserves cofinalities greater than $2^{<2^\kappa}$. [2]

For $n < \omega$, we say that there *is a* $\mathbf{\Sigma}_n$*-definable wellordering of* $\mathrm{H}(\kappa^+)$ if there is a wellordering of $\mathrm{H}(\kappa^+)$ that is definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a $\Sigma_n$-formula with parameters (from $\mathrm{H}(\kappa^+)$). We say that $\kappa$ is *suitable* if $\kappa$ is an uncountable cardinal satisfying $\kappa = \kappa^{<\kappa}$.

We will present an improved forcing $\mathbb{P}^\kappa$ that witnesses Theorem 1 while preserving all cofinalities and all values of the continuum function (that is the function $\kappa \mapsto 2^\kappa$), that also does not rely on the assumption that $2^\kappa$ be regular and that moreover is uniformly definable in parameter $\kappa$, which will later be useful for constructing iterations of these forcings.

**Theorem 2** *If* $\kappa$ *is suitable, then there is a partial order* $\mathbb{P}^\kappa$ *that is uniformly definable in parameter* $\kappa$ *and has the following properties.*

*(i)* $\mathbb{P}^\kappa$ *is* $<\kappa$*-closed and has a dense subset of conditions below each of which it has size* $2^\kappa$ *and is* $\kappa^+$*-cc. In particular,* $\mathbb{P}^\kappa$ *preserves all cofinalities and the values of the continuum function.*
*(ii)* *If* $G$ *is* $\mathbb{P}^\kappa$*-generic over the ground model* $\mathrm{V}$*, then in* $\mathrm{V}[G]$ *there is a* $\mathbf{\Sigma}_1$*-definable well-ordering of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$*.* [3]

We then iterate this forcing to obtain the following, which improves (a consequence of) [FH, Corollary 23, Claim 24, Theorem 25 and Theorem 37], where this has been shown under the additional assumption of the GCH:

**Theorem 3** *Assume the* SCH*. There is a* ZFC*-preserving class sized reverse Easton iteration* $\mathbb{P}$ *such that if* $G$ *is* $\mathbb{P}$*-generic over* $\mathrm{V}$*, then the following hold.*

– *Forcing with* $\mathbb{P}$ *preserves all cofinalities and the value of* $2^\kappa$ *for every cardinal* $\kappa$*. In particular,* $\mathrm{V}$ *and* $\mathrm{V}[G]$ *have the same suitable cardinals.*
– *If* $\kappa$ *is suitable, there is a* $\mathbf{\Sigma}_1$*-definable wellorder of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *in* $\mathrm{V}[G]$*.*
– *Forcing with* $\mathbb{P}$ *allows for various forms of large cardinal preservation.*

The role of the SCH in the above (and also in the following) theorem is very similar to the situation in [FHL]. We refer the reader to the first chapter of that paper (or also to [FL]) for a more detailed discussion. Note that it is possible to force the SCH to hold using a class-sized iteration that preserves the cofinality of all regular cardinals $\kappa$ such that there is no singular strong limit

[2] The reason for the former is essentially that the forcing constructed in [HL] adds a generic object that has properties similar to a Cohen subset of $2^\kappa$.

[3] Note that the forcing $\mathbb{P}^\kappa$ below any condition in the above dense set thus witnesses the statement provided in the first sentence of the abstract of this paper.

cardinal $\lambda$ with $\lambda^+ < \kappa \leq 2^\lambda$ and the value of $2^\kappa$ for all cardinals $\kappa$ such that there is no singular strong limit cardinal $\lambda$ with $2^\lambda > \lambda^+$ and $\lambda^+ \leq \kappa \leq 2^\lambda$.

Finally, we show the following, the point of which lies in the preservation of supercompact cardinals - we do not know how to generally preserve those in the context of Theorem 3. If $X$ is a class of singular strong limit cardinals, SCH at $X$ abbreviates the statement that $2^\kappa = \kappa^+$ for every $\kappa \in X$.

**Theorem 4** *Assume the* SCH *holds at singular limits of inaccessibles. There is a* ZFC*-preserving class sized reverse Easton iteration* $\mathbb{P}$ *such that if* $G$ *is* $\mathbb{P}$*-generic over* V*, then the following hold.*

- *Forcing with* $\mathbb{P}$ *preserves all cofinalities and the value of* $2^\kappa$ *for all* $\kappa$*.*
- *Whenever* $\kappa$ *is inaccessible, there is a* $\mathbf{\Sigma}_1$*-definable wellorder of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *in* $\mathrm{V}[G]$*.*
- *Forcing with* $\mathbb{P}$ *allows for various forms of large cardinal preservation, in particular it preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals.*

This improves [FL, Theorem 1.4], where the above is obtained only with a $\mathbf{\Sigma}_2$-definable wellorder of $\mathrm{H}(\kappa^+)$ whenever $\kappa$ is inaccessible.[4]

## 2 The single step forcing

In this section, we will give a proof of Theorem 2. As in [HL], the coding device woven into the definition of the forcing that we will give below is Club Coding, as introduced in [AHL, Section 3].

*Proof (of Theorem 2)* Assume $\kappa$ is suitable and let $\lambda = 2^\kappa$. $\mathbb{P}^\kappa$ will be a three stage iteration. At the first stage, we force with the lottery sum of all injective $\lambda$-sequences of elements of ${}^\kappa 2$. A generic for this stage picks an injective sequence $\mathbf{w} = \langle w_\gamma \mid \gamma < \lambda \rangle$ of elements of ${}^\kappa 2$. This forcing does not add new sets and is trivial below any nontrivial condition.

Let $\prec\cdot,\cdot\succ : \mathrm{On} \times \mathrm{On} \longrightarrow \mathrm{On}$ denote the *Gödel pairing function.* Define $A = \{w_\delta \oplus w_\gamma \mid \delta < \gamma < \lambda\}$, where given $x, y \in {}^\kappa\kappa$, $x \oplus y \in {}^\kappa\kappa$ is defined by setting

$$(x \oplus y)\ (\alpha) := \begin{cases} x(\beta) & \text{if } \alpha = \prec 0, \beta \succ, \\ y(\beta) & \text{if } \alpha = \prec 1, \beta \succ, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha < \kappa$.

At the second stage, we want to apply a coding forcing which is $<\kappa$-closed and has a dense subset of conditions below each of which it is $\kappa^+$-cc and has size $2^\kappa$, that makes $A$ $\mathbf{\Sigma}_1$-definable over $\mathrm{H}(\kappa^+)$ and such that this is persistent

[4] In a different direction, [FL, Theorem 1.4] was also improved in [FHL], where the authors obtain a lightface $\Sigma_2$-definable wellorder of $\mathrm{H}(\kappa^+)$ whenever $\kappa$ is inaccessible.

under further $<\kappa$-closed forcing. This can for example be achieved by almost disjoint coding at $\kappa$ (see [HL2, Sections 2 and 4]). [5]

The core of our construction is to define the forcing used at the third stage. In the model obtained after the second stage above (which will be our *ground model* in the following), we define a sequence of forcings $\mathbb{P}^\kappa_{\mathbf{w}} = \langle \mathbb{P}^\kappa_\gamma \mid \gamma \leq \lambda \rangle$ and then force with $\mathbb{P}^\kappa_\lambda$.

We say that *a subset $X$ of $\kappa$ codes an element $z$ of* $\mathrm{H}(\kappa^+)$ if there is a bijection $b : \kappa \longrightarrow \mathrm{tc}(\{z\})$ such that

$$X \;=\; \{\prec 0, \prec\alpha,\beta\succ\succ \mid \alpha,\beta<\kappa,\ b(\alpha)\in b(\beta)\} \;\cup\; \{\prec 1,\alpha\succ \mid \alpha<\kappa,\ b(\alpha)\in z\}.$$

Note that $z$ and $b$ are uniquely determined by $X$.

If $\alpha, \beta < \kappa$, then we define $c(\alpha,\beta) \in {}^\kappa 2$ by setting

$$c(\alpha,\beta)\ (\gamma) := \begin{cases} 1 \text{ if } \gamma \in \{\prec 0,\alpha\succ, \prec 1,\beta\succ\}, \\ 0 \text{ otherwise,} \end{cases}$$

for all $\gamma < \kappa$.

We construct the sequence $\mathbb{P}^\kappa_{\mathbf{w}} = \langle \mathbb{P}^\kappa_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders – with the property that $\mathbb{P}^\kappa_\delta$ is a complete subforcing of $\mathbb{P}^\kappa_\gamma$ whenever $\delta \leq \gamma \leq \lambda$ and that each $\mathbb{P}^\kappa_\delta$ preserves both $\kappa$ and the value of $2^\kappa$ – inductively. Fix $\gamma \leq \lambda$ and assume that we have already constructed $\mathbb{P}^\kappa_\delta$ with the above properties for every $\delta < \gamma$.

**Definition 1** We call a tuple

$$p \;=\; \langle \mathbf{N}_p, s_p, t_p, \mathbf{c}_p \rangle$$

a $\mathbb{P}^\kappa_\gamma$*-candidate* if either $p = \mathbf{1}_{\mathbb{P}^\kappa_\gamma} := \langle \emptyset,\emptyset,\emptyset,\emptyset\rangle$ or the following statements hold for some ordinals $\beta_p < \kappa$ and $\gamma_p \leq \gamma$.

(i) $\mathbf{N}_p = \langle N_{p,\delta} \mid \delta < \gamma_p \rangle$ and each $N_{p,\delta} = \langle N_{p,\delta}(i) \mid i < 2^\kappa \rangle$ is a sequence of $\mathbb{P}^\kappa_\delta$-names such that $\mathbf{1}_{\mathbb{P}^\kappa_\delta}$ forces $\langle N_{p,\delta}(i) \mid i < 2^\kappa \rangle$ to be a sequence of subsets of $\kappa$, each of which we view as a code for an element of $\mathrm{H}(\kappa^+)$, so that the corresponding sequence of elements of $\mathrm{H}(\kappa^+)$ gives an injective enumeration of $\mathrm{H}(\kappa^+)$, of order-type $2^\kappa$. Moreover we require that if $\xi < \delta$ then for every $i < 2^\kappa$ there is $j < 2^\kappa$ such that $N_{p,\xi}(i) = N_{p,\delta}(j)$. [6]
(ii) $s_p : \beta_p + 1 \longrightarrow {}^{<\kappa}2$.
(iii) $t_p : \beta_p + 1 \longrightarrow 2$.
(iv) $\mathbf{c}_p = \langle c_{p,x} \mid x \in a_p \rangle$ satisfies the following properties.
  (a) $a_p$ is a subset of $\{w_\delta \oplus c(\alpha, i) \mid \delta < \gamma_p,\ \alpha < \kappa,\ i < 2\}$ of cardinality less than $\kappa$.

[5] Note however that as defined in [HL2, Section 2], the almost disjoint coding forcing at $\kappa$ is not uniformly definable in parameter $\kappa$, as it depends on a prior choice of an enumeration of ${}^{<\kappa}\kappa$. However uniformity can easily be achieved by first choosing such an enumeration generically. The resulting coding forcing will then be as desired, the required dense set of conditions being those that decide for such an enumeration.

[6] We emphasize here that this last statement is about names and not their evaluations.

(b) If $x \in a_p$, then $c_{p,x}$ is a closed subset of $\beta_p + 1$ and the implication

$$s_p(\alpha) \subseteq x \longrightarrow t_p(\alpha) = 1$$

holds for every $\alpha \in c_{p,x}$.

For such $p$, we let $\mathbf{x}_p = \langle x_{p,\delta} \mid \delta < \gamma_p \rangle$ where (inductively) for every $\delta < \gamma_p$, if $\delta = \prec\delta_0, \delta_1\succ$, $x_{p,\delta} = N_{p,\delta_0}(\xi)$ with $\xi$ minimal so that $x_{p,\delta}$ is different from $x_{p,\epsilon}$ for $\epsilon < \delta$. [7]

Given a $\mathbb{P}^\kappa_\gamma$-candidate $p$ and $\delta \leq \gamma$, we define $p \restriction \delta$ to be the tuple

$$\langle \mathbf{N}_p \restriction \delta, s_p, t_p, \langle c_{p,x} \mid x \in a_p \restriction \delta \rangle \rangle,$$

where $a_p \restriction \delta = a_p \cap \{w_{\bar\delta} \oplus c(\alpha, i) \mid \bar\delta < \delta,\ \alpha < \kappa,\ i < 2\}$.

**Definition 2** A $\mathbb{P}^\kappa_\gamma$-candidate $p$ is a condition in $\mathbb{P}^\kappa_\gamma$ if the following statement holds for all $\delta < \gamma_p$, $\alpha < \kappa$ and $i < 2$ with $w_\delta \oplus c(\alpha, i) \in a_p$.

(v) If $p \restriction \delta$ is a condition in $\mathbb{P}^\kappa_\delta$, then

$$p \restriction \delta \Vdash_{\mathbb{P}^\kappa_\delta} \text{“}i = 1 \longleftrightarrow \check\alpha \in x_{p,\delta}\text{”}.$$ [8]

Given conditions $p$ and $q$ in $\mathbb{P}^\kappa_\gamma$, we define $p \leq_{\mathbb{P}^\kappa_\gamma} q$ to hold if $\mathbf{N}_q = \mathbf{N}_p \restriction \gamma_q$,[9] $s_q = s_p \restriction (\beta_q + 1)$, $t_q = t_p \restriction (\beta_q + 1)$, $a_q \subseteq a_p$ and $c_{q,x} = c_{p,x} \restriction (\beta_q + 1)$ for every $x \in a_q$.

**Proposition 1** *If $p \in \mathbb{P}^\kappa_\gamma$ and $\delta < \gamma$, then $p \restriction \delta \in \mathbb{P}^\kappa_\delta$.*

*Proof* Straightforward by induction on $\delta < \gamma$, noting that if $\bar\delta < \delta < \gamma$, then $(p \restriction \delta) \restriction \bar\delta = p \restriction \bar\delta$. □

**Lemma 1** *If $\delta < \gamma$, then $\mathbb{P}^\kappa_\delta$ is a complete subforcing of $\mathbb{P}^\kappa_\gamma$.*

*Proof* Obviously, $\mathbb{P}^\kappa_\delta \subseteq \mathbb{P}^\kappa_\gamma$ and the extension relation on $\mathbb{P}^\kappa_\delta$ is just the restriction to $\mathbb{P}^\kappa_\delta$ of the extension relation on $\mathbb{P}^\kappa_\gamma$. Now assume that $A$ is a maximal antichain of $\mathbb{P}^\kappa_\delta$ and $p \in \mathbb{P}^\kappa_\gamma$; we want to show that $p$ is compatible (in $\mathbb{P}^\kappa_\gamma$) with some element of $A$ and thus $A$ is a maximal antichain of $\mathbb{P}^\kappa_\gamma$. We may assume (by possibly strengthening $p$) that $\gamma_p = \gamma$. $p \restriction \delta$ is compatible (in $\mathbb{P}^\kappa_\delta$) with some element of $A$, as witnessed by $q \in \mathbb{P}^\kappa_\delta$ which is stronger than both. It is now easy to check that $p$ and $q$ are compatible in $\mathbb{P}^\kappa_\gamma$ - this is witnessed by $r = \langle \mathbf{N}_p, s_q, t_q, \langle c^*_x \mid x \in a_p \cup a_q \rangle \rangle$, where $c^*_x = c_{q,x}$ for $x \in a_q$ and $c^*_x = c_{p,x}$ otherwise. □

[7] By Clause (i) this implies that $\mathbf{1}_{\mathbb{P}^\kappa_\delta}$ forces that $x_{p,\delta}$ is different from $x_{p,\epsilon}$ for $\epsilon < \delta$. Moreover let us remark here that this way of choosing $\mathbf{x}_p$ using $\mathbf{N}_p$ is the key difference (and improvement) with respect to [HL].

[8] The idea behind this construction is that the set $a_p$ collects information about the interpretations of the $x_{p,\delta}$ that is already decided by the condition $p$. This will allow us to use the coding (Club Coding) that is woven into our forcing construction (see clause (iv) (b) in Definition 1) to add a subset of $\kappa$ that in the end codes $\bigcup_{p \in G} a_p$ and thus also $\langle x^G_{p,\delta} \mid \delta < \lambda \rangle$ whenever $G$ is $\mathbb{P}^\kappa_\lambda$-generic.

[9] Note again that this equality refers to the actual (sequences of) names.

**Lemma 2** *For $\gamma \leq \lambda$, $\mathbb{P}^\kappa_\gamma$ has a dense subset of conditions below each of which it satisfies the $\kappa^+$-chain condition and has size at most $2^\kappa$.*

*Proof* Let $p \in \mathbb{P}^\kappa_\gamma$ be such that $\gamma_p = \gamma$. Obviously, $\mathbb{P}^\kappa_\gamma$ has size at most $2^\kappa$ below $p$. We show that $\mathbb{P}^\kappa_\gamma$ is $\kappa^+$-cc below $p$. Assume $A = \langle p_\xi \mid \xi < \kappa^+\rangle$ is an antichain of $\mathbb{P}^\kappa_\gamma$ below $p$, with $p_\xi = \langle \mathbf{N}_p, s_\xi, t_\xi, \mathbf{c}_\xi\rangle$ and $\mathbf{c}_\xi = \langle c_{\xi,x} \mid x \in a_\xi\rangle$ for every $\xi < \kappa^+$. By a standard $\Delta$-system argument, there is $B \subseteq \kappa^+$ of size $\kappa^+$ and a set $r$ such that whenever $\xi_0 < \xi_1 < \kappa^+$ and $\xi_0$ and $\xi_1$ are both in $B$, then $a_{\xi_0} \cap a_{\xi_1} = r$. We may assume that $B = \kappa^+$. Thinning out once again, we may also assume that $\langle s_\xi, t_\xi, \langle c_{\xi,x} \mid x \in r\rangle\rangle$ is the same for every $\xi \in \kappa^+$. Now any two conditions in $A$ (assuming it has been thinned out as above) are in fact compatible: Given $\xi_0 < \xi_1 < \kappa^+$, this is witnessed by $q = \langle \mathbf{N}_p, s_{\xi_0}, t_{\xi_0}, \mathbf{c}_{\xi_0} \cup \mathbf{c}_{\xi_1}\rangle$, a contradiction. □

For any notion of forcing $P$ and any $D \subseteq P$, we say that $D$ is *directed* if any two conditions in $D$ have a common strengthening in $D$. Given a cardinal $\kappa$, we say that $P$ is *$<\kappa$-directed closed* if whenever $D \subseteq P$ is directed and of size less than $\kappa$, then there is a condition $p \in P$ that is stronger than all conditions in $D$.

**Lemma 3** *If $\gamma \leq \lambda$, then $\mathbb{P}^\kappa_\gamma$ is $<\kappa$-directed closed.*

*Proof* Let $D$ be a directed subset of $\mathbb{P}^\kappa_\gamma$ of size less than $\kappa$. Let $\bar{\gamma} = \bigcup_{d\in D} \gamma_d$, $\mathbf{N} = \bigcup_{d\in D} \mathbf{N}_d$, $a = \bigcup_{d\in D} a_d$ and let $c_x = \bigcup\{c_{d,x} \mid d \in D \wedge x \in a_d\}$ for each $x \in a$.

First assume that there is $e \in D$ such that $\beta_d \leq \beta_e$ for all $d \in D$. Then the tuple $p_* = \langle \mathbf{N}, s_e, t_e, \langle c_x \mid x \in a\rangle\rangle$ is a $\mathbb{P}^\kappa_\gamma$-candidate. To show that $p_*$ is a condition in $\mathbb{P}^\kappa_\gamma$, fix $\delta < \bar{\gamma}$, $\beta < \kappa$ and $i < 2$ with $x = w_\delta \oplus c(\beta, i) \in a$. Then there is $d \in D$ with $x \in a_d$. If $p_* \restriction \delta$ is a condition in $\mathbb{P}^\kappa_\delta$, then it is stronger than $d \restriction \delta$, and hence it forces the desired statement of (v) in Definition 2. This shows that $p_*$ is a condition in $\mathbb{P}^\kappa_\gamma$, and obviously $p_* \leq_{\mathbb{P}^\kappa_\gamma} d$ for every $d \in D$.

Now assume that for every $d \in D$ there is $e \in D$ with $\beta_d < \beta_e$. Define

- $\beta = \sup_{d\in D} \beta_d$.
- $s = \{\langle \beta, \emptyset\rangle\} \cup \bigcup_{d\in D} s_d$.
- $t = \{\langle \beta, 1\rangle\} \cup \bigcup_{d\in D} t_d$.
- $p_* = \langle \mathbf{N}, s, t, \langle c_x \cup \{\beta\} \mid x \in a\rangle\rangle$.

This ensures that $p_*$ is a $\mathbb{P}^\kappa_\gamma$-candidate and the same argument as above shows that $p_*$ is actually a condition in $\mathbb{P}^\kappa_\gamma$ with $p_* \leq_{\mathbb{P}^\kappa_\gamma} d$ for all $d \in D$. □

Since the above implies that each $\mathbb{P}^\kappa_\gamma$ preserves both $\kappa$ and the value of $2^\kappa$, this completes the construction of the sequence $\mathbb{P}^\kappa_{\mathbf{w}}$ of partial orders.

If $G$ is $\mathbb{P}^\kappa_\lambda$-generic and $\gamma < 2^\kappa$, we define $x^G_\gamma := x^G_{p,\gamma}$ for any $p \in G$ with $\gamma_p > \gamma$.

**Lemma 4** *If $G$ is $\mathbb{P}^\kappa_\lambda$-generic, then for every $x \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, there is a unique $\gamma < \lambda$ such that $x^G_\gamma$ codes $x$.*

*Proof* Assume $x \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ is given, let $p \in G$ be such that $\gamma_p = \lambda$ and let $\dot{x}$ be a nice name for a subset of $\kappa$, coding $x$ in $\mathrm{V}[G]$, in the forcing $\mathbb{P}^\kappa_\lambda$ below $p$. By the $\kappa^+$-cc of $\mathbb{P}^\kappa_\lambda$ below $p$ together with $\mathrm{cof}(\lambda) > \kappa$, there is $\delta < \lambda$ such that we can identify $\dot{x}$ with a $\mathbb{P}^\kappa_\delta$-name $\dot{y}$ below $p \upharpoonright \delta$, so that whenever $H$ is $\mathbb{P}^\kappa_\lambda$-generic with $p \in H$, $\dot{x}^H = \dot{y}^H$. $\dot{y}$ is obtained by replacing, within every condition $q$ that appears in the name $\dot{x}$, $\mathbf{N}_q$ by $\mathbf{N}_q \upharpoonright \delta$. By our choice of $N_{p,\delta}$, there is a unique $\xi < \lambda$ such that $N_{p,\delta}(\xi)^G = \dot{y}^G$. By our choice of the $x_{p,\gamma}$'s, we can now find a unique $\gamma < \lambda$ so that $x_{p,\gamma} = N_{p,\delta}(\xi)$ and hence $x^G_\gamma = \dot{y}^G$. □

It remains to show that if $G$ is $\mathbb{P}^\kappa_\lambda$-generic, then there is a $\mathbf{\Sigma}_1$-definable wellordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. This is mostly done as in [HL], with appropriate notational changes. We provide a self-contained proof in the following for the convenience of the reader.

**Lemma 5** *If $G$ is $\mathbb{P}^\kappa_\lambda$-generic over* V*, then the set*

$$D(G) \;=\; \{w_\delta \oplus c(\alpha,i) \mid \delta<\lambda, \alpha<\kappa, i<2, (i=1 \;\longleftrightarrow\; \alpha \in x^G_\delta)\}$$

*is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a $\Sigma_1$-formula with parameters.*

*Proof* Let $G$ be $\mathbb{P}^\kappa_\lambda$-generic over V. We prove a number of claims whose combination will imply the statement of the lemma.

*Claim* If $x = w_\delta \oplus c(\alpha,i) \in D(G)$, then there is $p \in G$ with $x \in a_p$.

*Proof* There is $q \in G$ witnessing that $x \in D(G)$ and such that $q \upharpoonright \delta \Vdash_{\mathbb{P}^\kappa_\delta}$ "$i = 1 \longleftrightarrow \check{\alpha} \in x_{q,\delta}$". We may assume that $x \notin a_q$. Fix $p_0 \in \mathbb{P}^\kappa_\lambda$ with $p_0 \leq_{\mathbb{P}^\kappa_\lambda} q$ and $x \notin a_{p_0}$. If we define

$$p \;=\; \langle \mathbf{N}_{p_0}, s_{p_0}, t_{p_0}, \{\langle x, \emptyset\rangle\} \cup \langle c_{p_0,y} \mid y \in a_{p_0}\rangle\rangle,$$

then $p \in \mathbb{P}^\kappa_\lambda$ is stronger than $p_0$. Hence the set of all conditions $p$ in $\mathbb{P}^\kappa_\lambda$ with $x \in a_p$ is dense below $q \in G$. □

*Claim* $\kappa = \sup\{\beta_p \mid p \in G\}$ and $\kappa = \sup\{\sup(c_{p,x}) \mid p \in G,\ x \in a_p\}$ whenever $x \in D(G)$.

*Proof* Fix a condition $q$ in $\mathbb{P}^\kappa_\lambda$ with $x \in a_q$ and fix $\beta_q < \beta < \kappa$. Define

- $s \;=\; s_q \;\cup\; \{\langle \alpha, \emptyset\rangle \mid \beta_q < \alpha \leq \beta\}$.
- $t \;=\; t_q \;\cup\; \{\langle \alpha, 1\rangle \mid \beta_q < \alpha \leq \beta\}$.
- $p \;=\; \langle \mathbf{N}_q, s, t, \langle c_{q,x} \;\cup\; (\beta_q, \beta] \mid x \in a_q\rangle\rangle$.

Then $p$ is a condition in $\mathbb{P}^\kappa_\lambda$ with $p \leq_{\mathbb{P}^\kappa_\lambda} q$, $\beta_p = \beta$ and $\sup(c_{p,x}) = \beta$. □

We fix $\mathbb{P}^\kappa_\lambda$-names $\dot{s}$ and $\dot{t}$ in V such that $\dot{s}^H = \bigcup\{s_p \mid p \in H\} : \kappa \longrightarrow {}^{<\kappa}2$ and $\dot{t}^H = \bigcup\{t_p \mid p \in H\} : \kappa \longrightarrow 2$ holds whenever $H$ is $\mathbb{P}^\kappa_\lambda$-generic over V. The following is now immediate.

*Claim* If $x \in D(G)$, then $C^x_G = \bigcup\{c_{p,x} \mid p \in G,\ x \in a_p\}$ is a club subset of $\kappa$ such that the implication

$$\dot{s}^G(\alpha) \subseteq x \ \longrightarrow\ \dot{t}^G(\alpha) = 1 \tag{1}$$

holds for all $\alpha \in C^x_G$. □

*Claim* Assume $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$ is such that (1) holds for every element $\alpha$ of some club subset $C$ of $\kappa$. Then $x \in D(G)$.

*Proof* Let $\dot{a}$ be the canonical $\mathbb{P}^\kappa_\lambda$-name such that $\dot{a}^H = \bigcup\{a_p \mid p \in H\}$ holds whenever $H$ is $\mathbb{P}^\kappa_\lambda$-generic over V. Assume, towards a contradiction, that $x$ is not an element of $\dot{a}^G$. Then we can find $q \in G$ and $\mathbb{P}^\kappa_\lambda$-names $\dot{C}$ and $\dot{x}$ such that $x = \dot{x}^G$ and

$$q \Vdash_{\mathbb{P}_\lambda} \text{“}\dot{x} \in {}^{\check{\kappa}}2 \setminus \dot{a} \ \wedge\ \dot{C} \subseteq \check{\kappa} \textit{ club} \ \wedge\ \forall \alpha \in \dot{C}\ [\dot{s}(\alpha) \subseteq \dot{x} \longrightarrow \dot{t}(\alpha) = 1]\text{”}$$

Fix a condition $p_0$ in $\mathbb{P}^\kappa_\lambda$ that is stronger than $q$. By using the above assumptions, we can recursively construct

- a descending sequence $\langle p_n \mid n < \omega\rangle$ of conditions in $\mathbb{P}^\kappa_\lambda$,
- strictly increasing sequences $\langle \alpha_n \mid n < \omega\rangle$ and $\langle \beta_n \mid n < \omega\rangle$ of ordinals less than $\kappa$, and
- a sequence $\langle s_n \mid n < \omega\rangle$ of elements of ${}^{<\kappa}2$

that satisfy the following statements for all $n < \omega$.

(i) $\beta_{p_n} < \alpha_n \leq \beta_n < \beta_{p_{n+1}}$.
(ii) $s_n \neq y \restriction \alpha_n$ for all $y \in a_{p_n}$.
(iii) $p_{n+1} \Vdash_{\mathbb{P}^\kappa_\lambda}$ “$\dot{x} \restriction \check{\alpha}_n = \check{s}_n \ \wedge\ \check{\beta}_n = \min(\dot{C} \setminus \check{\alpha}_n)$”.

Next, we define

- $\mathbf{N} = \bigcup\{\mathbf{N}_{p_n} \mid n < \omega\}$.
- $\beta = \sup_{n<\omega} \alpha_n = \sup_{n<\omega} \beta_n$.
- $s_\omega = \bigcup\{s_n \mid n < \omega\}$.
- $s = \{\langle \beta, s_\omega\rangle\} \cup \bigcup\{s_{p_n} \mid n < \omega\}$.
- $t = \{\langle \beta, 0\rangle\} \cup \bigcup\{t_{p_n} \mid n < \omega\}$.
- $a = \bigcup\{a_{p_n} \mid n < \omega\}$ and $c_y = \{\beta\} \cup \bigcup\{c_{p_n,y} \mid n < \omega,\ y \in a_{p_n}\}$ for every element $y$ of $a$.

Since $s_\omega \nsubseteq y$ for every $y \in a$, the tuple $p = \langle \mathbf{N}, s, t, \langle c_y \mid y \in a\rangle\rangle$ is a condition in $\mathbb{P}^\kappa_\lambda$ that is stronger than $p_0$. Our construction ensures

$$p \Vdash_{\mathbb{P}^\kappa_\lambda} \text{“}\check{\beta} \in \dot{C} \ \wedge\ \dot{s}(\check{\beta}) = \check{s} \subseteq \dot{x} \ \wedge\ \dot{t}(\check{\beta}) = 0\text{”},$$

a contradiction. Hence we can conclude that $x \in \dot{a}^G$.

The above computations show that there are $p \in G$, $\delta < \gamma_p$, $\alpha < \kappa$ and $i < 2$ with $x = w_\delta \oplus c(\alpha, i) \in a_p$. Since $p \restriction \delta \in G \cap \mathbb{P}^\kappa_\delta$, Definition 2 implies that $i = 1$ iff $\alpha \in x^G_\delta$. Hence $x \in D(G)$. □

The above statements allow us to conclude that

$$D(G) \;=\; \{x \in ({}^{\kappa}2)^{\mathrm{V}[G]} \mid \exists C \subseteq \kappa \; club \; \forall \alpha \in C \; [\dot{s}^G(\alpha) \subseteq x \;\longrightarrow\; \dot{t}^G(\alpha) = 1]\}$$

and this equality yields a $\Sigma_1$-definition of $D(G)$ over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ using the parameters $\dot{s}^G$ and $\dot{t}^G$. □

**Lemma 6** *Let $G$ be $\mathbb{P}^{\kappa}_{\lambda}$-generic. Then in* $\mathrm{V}[G]$*, there is a $\mathbf{\Sigma}_1$-definable well-order of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.

*Proof* Define $W = \{w_\delta \mid \delta < \lambda\}$. Then our assumptions imply that $W$ is also $\mathbf{\Sigma}_1$-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$. Note that

$$x^G_\delta \;=\; \{\alpha < \kappa \mid w_\delta \oplus c(\alpha, 1) \in D(G)\} \;=\; \{\alpha < \kappa \mid w_\delta \oplus c(\alpha, 0) \notin D(G)\}.$$

We define $P$ to be the set of all pairs $\langle z, w \rangle$ such that $z \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, $w \in W$ and there is a subset $y$ of $\kappa$ coding $z$ and satisfying

$$[\alpha \in y \longrightarrow w \oplus c(\alpha, 1) \in D(G)] \;\wedge\; [\alpha \notin y \longrightarrow w \oplus c(\alpha, 0) \in D(G)]. \qquad (2)$$

Lemma 5 implies that $P$ is $\mathbf{\Sigma}_1$-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$.

*Claim* Let $z \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and let $\delta_z$ be the unique ordinal (given by Lemma 4) such that $x^G_{\delta_z}$ codes $z$. Then $w_{\delta_z}$ is the unique element of $W$ with $\langle z, w_{\delta_z} \rangle \in P$.

*Proof* $x^G_{\delta_z}$ witnesses that $\langle z, w_{\delta_z} \rangle \in P$. Now assume, towards a contradiction, that there is $\delta < \lambda$ with $\delta \neq \delta_z$ and $\langle z, w_\delta \rangle \in P$. Let $y \subseteq \kappa$ code $z$ and satisfy (2). Then $y = x^G_\delta$ and thus cannot code $z$, a contradiction. □

Let $\prec_{\mathbf{w}}$ be the relation (wellorder) on $W$ defined by letting $w \prec_{\mathbf{w}} \bar{w}$ iff $w = w_\alpha$, $\bar{w} = w_\beta$ and $\alpha < \beta$. Note that $w \prec_{\mathbf{w}} \bar{w}$ iff $w \oplus \bar{w} \in A$, hence $\prec_{\mathbf{w}}$ is $\mathbf{\Sigma}_1$-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$.

Define $\prec_*$ to be the set of all pairs $\langle z, \bar{z} \rangle$ in $\mathrm{H}(\kappa^+)$ such that

$$\exists w, \bar{w} \in W \; [\langle z, w \rangle \in P \;\wedge\; \langle \bar{z}, \bar{w} \rangle \in P \;\wedge\; w \prec_{\mathbf{w}} \bar{w}].$$

$\mathbf{\Sigma}_1$-definability of $W$, $P$ and $\prec_{\mathbf{w}}$ implies that $\prec_*$ is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a $\Sigma_1$-formula with parameters. Given $z_0, z_1 \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and $\delta_0, \delta_1 < \lambda$ such that $\delta_i$ is the unique ordinal with the property that $x^G_{\delta_i}$ codes $z_i$, we have $z_0 \prec_* z_1$ if and only if $\delta_0 < \delta_1$. This shows that $\prec_*$ is a well-ordering of $\mathrm{H}(\kappa^+)$. □

□

## 3 Iterating the single step forcings

In this section, we provide proofs for Theorem 3 and Theorem 4. The natural candidates for forcings to witness those theorems are of course reverse Easton iterations of forcings of the form $\mathbb{P}^\kappa$ for suitable $\kappa$. For any iteration $\mathbb{P}$, we will denote the notion of forcing invoked at stage $\kappa$ as $\mathbb{P}(\kappa)$ and the iteration below $\kappa$ as $\mathbb{P}_{<\kappa}$.

*Proof (of Theorem 3)* Let $\mathbb{P}$ be the reverse Easton iteration in which we let $\mathbb{P}(\kappa) = \mathbb{P}^\kappa$ (as defined in the respective intermediate model) if $\kappa$ is suitable and let $\mathbb{P}(\kappa)$ be the trivial forcing otherwise. $\mathbb{P}$ preserves ZFC, all cofinalities and the value of $2^\kappa$ for every $\kappa$ by the arguments of [AHL, Section 6]. In particular, forcing with $\mathbb{P}$ does not alter suitability of $\kappa$ for any $\kappa$. Thus it is easily seen that $\mathbb{P}$ is as desired, i.e. in any $\mathbb{P}$-generic extension, we have that whenever $\kappa$ is suitable, there is a $\mathbf{\Sigma}_1$-definable wellorder of $\mathrm{H}(\kappa^+)$.

As a sample result for large cardinal preservation, assume that $\lambda$ is $\omega$-superstrong. It follows essentially as in [FHL, Claim 23] that there is a $\mathbb{P}$-generic extension of the universe that preserves the $\omega$-superstrength of $\lambda$. □

*Proof (of Theorem 4)* Let $\mathbb{P}$ be the reverse Easton iteration so that $\mathbb{P}(\kappa) = \mathbb{P}^\kappa$ (as defined in the respective intermediate model) whenever $\kappa$ is inaccessible and let $\mathbb{P}(\kappa)$ be the trivial forcing otherwise. The arguments that $\mathbb{P}$ preserves ZFC, all cofinalities and the value of $2^\kappa$ for every $\kappa$ are very similar to those of [AHL, Section 6]. Obviously thus, forcing with $\mathbb{P}$ preserves all inaccessible cardinals and introduces the desired wellorders. Also, forcing with $\mathbb{P}$ preserves all supercompact cardinals and for any $\omega$-superstrong cardinal $\kappa$ there is a $\mathbb{P}$-generic extension in which the $\omega$-superstrength of $\kappa$ is preserved. In fact, if $\gamma$ is a cardinal that satisfies $2^\gamma = \gamma^+$ and $2^\nu \leq \gamma$, where we let $\nu = \sup\{\alpha \leq \gamma \mid \alpha \text{ is inaccessible}\}$, and if $\kappa$ is $\gamma$-supercompact with $\gamma = \gamma^{<\kappa}$, then forcing with $\mathbb{P}$ preserves the $\gamma$-supercompactness of $\kappa$.

Preservation of (degrees of) supercompactness is shown essentially as in [FL, Section 5], and as a further sample result on large cardinal preservation, preservation of $\omega$-superstrength is essentially as in [FHL, Claim 23]. [10] □

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## References

AHL. David Asperó, Peter Holy, and Philipp Lücke. Forcing lightface definable well-orders without GCH. Accepted for the Annals of Pure and Applied Logic.

FH. Sy-David Friedman and Peter Holy. Condensation and large cardinals. Fundamenta Mathematicae, 215(2):133–166, 2011.

[10] Note that the techniques for supercompactness preservation from [FL, Section 5] need sufficient gaps between cardinals where we apply nontrivial forcing. This is why we restrict ourselves to introducing $\mathbf{\Sigma}_1$-definable wellorders of $\mathrm{H}(\kappa^+)$ only for inaccessible $\kappa$ here.

FHL. Sy-David Friedman, Peter Holy, and Philipp Lücke. Large cardinals and lightface definable wellorders, without the GCH. Accepted for the Journal of Symbolic Logic.

FL. Sy-David Friedman and Philipp Lücke. Large cardinals and definable well-orders, without the GCH. Accepted for the Annals of Pure and Applied Logic.

HL. Peter Holy and Philipp Lücke. Locally $\Sigma_1$-definable well-orders of $H(\kappa^+)$. Fundamenta Mathematicae, 226(3):221–236, 2014.

HL2. Peter Holy and Philipp Lücke. Simplest possible locally definable Wellorders. Submitted.